Given: For Generic form of the thermal conductivity to be $k(T) = a + bT + cT^2$, evaluate the residual and Jacobian as in (14) and (26), Comment on the Matrix which arises from the new quadratic term when forming the Jacobian.

Solution:

$$k(T) = a + bT + cT^2$$

$$L(T) = -\frac{d}{dx}\left(k(T)\frac{dT_{(x)}}{dx}\right) - S = 0 \quad \Omega_e\ (L, R)$$

Boundary Conditions

$T(X_L) = T_L \quad \partial\Omega_1 \in L$

$T(X_R) = T_R \quad \partial\Omega_2 \in R$

Assume Series Expansion approximate form

$$T^2(x) = T^{N,2} = \sum_{\alpha=1}^{n} \Psi_\alpha T_\alpha^2 = \Psi_\alpha(x) T_\alpha^2 \quad \text{for } 1 \le \alpha N \qquad (1)$$

Forming the Galerkin Weak Statement

$$GWS^N = \int_\Omega \Psi_{\beta(x)} L(T^{N,2})\, dx = 0 \quad 1 \le \beta \le N \qquad (2)$$

Applying the Operator

$$GWS^N = \int_\Omega \Psi_{\beta(x)}\left(-\frac{d}{dx}\left(k(T^{N,2})\frac{dT^{N,2}}{dx}\right) - S\right) dx = 0 \quad 1 \le \beta \le N \qquad (3)$$

Integrating by Parts and Expanding

$$\int_0^L u\,dv = -\int_0^L v\,du + uv\Big|_0^L \qquad u = \Psi_{\beta(x)} \quad du = \frac{d\Psi_{\beta(x)}}{dx} \quad dv = -\frac{d}{dx}\left(K(T^{N,2})\frac{dT^{N,2}}{dx}\right)$$

$$v = -K(T^{N,2})\frac{dT^{N,2}}{dx}$$

$$GWS^N = \int_\Omega \frac{d\Psi_\beta}{dx} k(T^{N,2}) \frac{dT^{N,2}}{dx} dx - \int_\Omega \Psi_\beta S\, dx - \Psi_\beta k(T)\frac{dT^{N,2}}{dx}\Big|_{\partial\Omega} = 0 \quad 1 \le \beta \le N \quad (4)$$

Substitute Variation in for k

$$GWS^N = \int_\Omega \frac{d\Psi_\beta}{dx}(a + bT + cT^2)\frac{dT^{N,2}}{dx} dx - \int_\Omega \Psi_\beta S\, dx - \Psi_\beta k(T)\frac{dT^N}{dx}\Big|_{\partial\Omega} = 0 \quad 1 \le \beta \le N \quad (5)$$

Substituting Series Expansion

$$GWS^N = a\int_\Omega \frac{d\Psi_\beta}{dx}\frac{d\Psi_\alpha}{dx} dx\, T_\alpha + bT_\alpha \int_\Omega \Psi_\alpha \frac{d\Psi_\beta}{dx}\frac{d\Psi_\alpha}{dx} dx\, T_\alpha + cT_\alpha^2 \int_\Omega \Psi_\alpha \frac{d\Psi_\beta}{dx}\frac{d\Psi_\alpha}{dx} dx\, T_\alpha - \int_\Omega \Psi_\beta S\, dx - \Psi_\beta k(T)\frac{dT^N}{dx}\Big| = 0 \quad (6)$$

Discretizing the Domain

$$GWS^h = S_e\left(a\int_{\Omega}\frac{d\{N_k\}}{dx}\frac{d\{N_k\}^T}{dx}d\bar{x}\{T\}_e + b\{T\}_e^T\int_{\Omega}\{N_k\}\frac{d\{N_k\}}{dx}\frac{d\{N_k\}^T}{dx}d\bar{x}\{T\}_e + c\{T^2\}_e^T\int\{N_k\}\frac{d\{N_k\}}{dx}\frac{d\{N_k\}^T}{dx}d\bar{x}\{T\}_e\right.$$

$$\left. - \int_{\Omega}\{N_k\}\{N_k\}^T dx\{S\}_e = \{0\}_e\right) \qquad (7)$$

0

Use Newton Algorithm to iterate

$$\{Q\}^{p+1} = \{Q\}^p + \{\delta Q\}^{p+1}$$

$$[JAC]^p\{\delta Q\}^{p+1} = -\{FQ\}^p$$

$\{FQ\}^p$ is entire weak statement evaluated using the previous values of $\{Q\}^p$

$$FQ = S_e\left(a\int\frac{d\{N_k\}}{dx}\frac{d\{N_k\}^T}{dx}d\bar{x}\{T\}_e^p + b\{T\}_e^{p,T}\int_{\Omega}\{N_k\}\frac{d\{N_k\}}{dx}\frac{d\{N_k\}^T}{dx}d\bar{x}\{T\}_e^p + c\{T\}_e^{2,T}\int_{\Omega}\{N_k\}\frac{d\{N_k\}}{dx}\frac{d\{N_k\}^T}{dx}d\bar{x}\{T\}_e^p - \int_{\Omega}\{N_k\}\{N_k\}^T d\bar{x}\{S\}_e\right)$$

Differentiate entire weak statement with respect to the unknowns to form JAC

Source term first

$$\frac{\partial}{\partial\{T\}_e}\left(-\int_{\Omega}\{N_k\}\{N_k\}^T d\bar{x}\{S\}_e\right) = -\int_{\Omega_e}\{N_k\}\{N_k\}^T d\bar{x}\,\frac{\partial\{S\}_e}{\partial\{T\}_e}$$

$$\frac{\partial\{S\}_e}{\partial\{T\}_e} = \frac{\partial\begin{Bmatrix}S_L\\S_R\end{Bmatrix}_e}{\partial\begin{Bmatrix}T_L\\T_R\end{Bmatrix}_e} = \begin{bmatrix}\frac{\partial S_L}{\partial T_L} & \frac{\partial S_L}{\partial T_L}\\ \frac{\partial S_R}{\partial T_L} & \frac{\partial S_R}{\partial T_R}\end{bmatrix} = \begin{bmatrix}0&0\\0&0\end{bmatrix}_e \Rightarrow S_e\left(-\int_{\Omega}\{N_k\}\{N_k\}^T d\bar{x}[0]_e\right.$$

Linear conductivity term next

$$\frac{\partial}{\partial\{T\}_e}\left(a\int_{\Omega}\frac{d\{N_k\}}{dx}\frac{d\{N_k\}^T}{dx}d\bar{x}\{T\}_e\right) = a\int_{\Omega_e}\frac{d\{N_k\}}{dx}\frac{d\{N_k\}^T}{dx}d\bar{x}\,\frac{\partial\{T\}_e}{\partial\{T\}_e}$$

$$\frac{\partial\{T\}_e}{\partial\{T\}_e} = \frac{\partial\begin{Bmatrix}T_L\\T_R\end{Bmatrix}_e}{\partial\begin{Bmatrix}T_L\\T_R\end{Bmatrix}_e} = \begin{bmatrix}\frac{\partial T_L}{\partial T_L} & \frac{\partial T_L}{\partial T_R}\\ \frac{\partial T_R}{\partial T_L} & \frac{\partial T_R}{\partial T_R}\end{bmatrix} = \begin{bmatrix}1&0\\0&1\end{bmatrix}_e = S_e\left(a\int_{\Omega_e}\frac{d\{N_k\}}{dx}\frac{d\{N_k\}^T}{dx}d\bar{x}[1]_e\right.$$

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$2\begin{bmatrix}T&0\\0&T\end{bmatrix}$

1^{st} Non-Linear Term Differentiating via the product rule

$$\frac{\partial}{\partial\{T\}_e}\left(b\{T\}_e^T\int_{\Omega_e}\{N_k\}\frac{d\{N_k\}}{dx}\frac{d\{N_k\}^T}{dx}d\bar{x}\{T\}_e\right) = b\{T\}_e^T\int_{\Omega_e}\{N_k\}\frac{d\{N_k\}}{dx}\frac{d\{N_k\}^T}{dx}d\bar{x}\frac{\partial\{T\}_e}{\partial\{T\}_e} + b\{T\}_e^T\int_{\Omega_e}\frac{d\{N_k\}}{dx}\frac{d\{N_k\}}{d\bar{x}}\{N_k\}^T d\bar{x}\frac{\partial\{T\}_e}{\partial\{T\}_e}$$

Assembling yields:

$$S_e\left(b\{T\}_e^T\int_{\Omega_e}\{N_k\}\frac{d\{N_k\}}{dx}\frac{d\{N_k\}^T}{dx}d\bar{x}\,[1]_e + b\{T\}_e^T\int_{\Omega_e}\frac{d\{N_k\}}{dx}\frac{d\{N_k\}}{dx}\{N_k\}^T d\bar{x}\,[1]_e\right)$$

Differentiating 2^{nd} Non-Linear Term

$$c\{T^2\}_e^T\int\{N_k\}\frac{d\{N_k\}}{dx}\frac{d\{N_k\}^T}{dx}d\bar{x}\,\{T\}_e = c\int_{\Omega_e}\{T^2\}_e^T\{N_k\}\frac{d\{N_k\}}{dx}\frac{d\{N_k\}^T}{dx}\{T\}_e\,d\bar{x}$$

$$= c\int_{\Omega_e}\frac{d\{N_k\}^T}{dx}\{T_e^2\}\frac{d\{N_k\}}{dx}\{T\}_e^T\{N_k\}\,d\bar{x}$$

$$= c\int_{\Omega_e}\{T^2\}_e^T\frac{d\{N_k\}}{dx}\frac{d\{N_k\}}{dx}\{N_k\}^T d\bar{x}\,\{T\}_e$$

$$= c\{T^2\}_e^T\int_{\Omega_e}\frac{d\{N_k\}}{dx}\frac{d\{N_k\}}{dx}\{N_k\}^T d\bar{x}\,\{T\}_e$$

$$\frac{\partial}{\partial\{T\}_e}\left(c\{T^2\}_e^T\int_{\Omega_e}\{N_k\}\frac{d\{N_k\}}{dx}\frac{d\{N_k\}^T}{dx}d\bar{x}\{T\}_e\right) = c\{T^2\}_e^T\int_{\Omega_e}\{N_k\}\frac{d\{N_k\}}{dx}\frac{d\{N_k\}^T}{dx}d\bar{x}\frac{\partial\{T\}_e}{\partial\{T\}_e}$$

$$+ c\{T^2\}_e^T\int_{\Omega_e}\frac{d\{N_k\}}{dx}\frac{d\{N_k\}}{dx}\{N_k\}^T d\bar{x}\frac{\partial\{T^2\}_e}{\partial\{T\}_e}$$

$$\frac{\partial\{T^2\}_e}{\partial\{T\}_e} = \frac{\partial\begin{Bmatrix}T_L^2\\T_R^2\end{Bmatrix}}{\partial\begin{Bmatrix}T_L\\T_R\end{Bmatrix}} = \begin{bmatrix}\frac{\partial T_L^2}{\partial T_L} & \frac{\partial T_L^2}{\partial T_R}\\ \frac{\partial T_R^2}{\partial T_L} & \frac{\partial T_R^2}{\partial T_R}\end{bmatrix} = 2\begin{bmatrix}T & 0\\0 & T\end{bmatrix}.$$

Assembling 2^{nd} Non-Linear Term

$$S_e\left(\frac{\partial}{\partial\{T\}_e}\left(c\{T^2\}_e^T\int_{\Omega}\{N_k\}\frac{d\{N_k\}}{dx}\frac{d\{N_k\}^T}{dx}d\bar{x}\{T\}_e\right)\right) = S_e\left(c\{T^2\}_e^T\int_{\Omega_e}\{N_k\}\frac{d\{N_k\}}{dx}\frac{d\{N_k\}^T}{dx}d\bar{x}\,[T]_e + c\{T^2\}_e^T\int_{\Omega_e}\frac{d\{N_k\}}{dx}\frac{d\{N_k\}}{dx}\{N_k\}^T d\bar{x}\,[T]_e\right)$$

Evaluating the Jacobian with the previous values of $\{T\}_e^p$

$$[JAC]^p = S_e\left(a\int_{\Omega_e} \frac{d\{N_k\}}{dx}\frac{d\{N_k\}}{dx}\,d\bar{x}\,[1]_e\right) + S_e\left(b\{T\}_e^{p,T}\int_{\Omega_e}\{N_k\}\frac{d\{N_k\}}{dx}\frac{d\{N_k\}^T}{dx}\,d\bar{x}\,[1]_e\right)$$

$$+ S_e\left(b\{T\}_e^{p,T}\int_{\Omega_e}\frac{d\{N_k\}}{dx}\frac{d\{N_k\}}{dx}\{N_k\}^T dx\,[1]_e\right) + S_e\left(c\{T^2\}_e^{p,T}\int_{\Omega_e}\{N_k\}\frac{d\{N_k\}}{dx}\cdot\frac{d\{N_k\}^T}{dx}\,d\bar{x}\,[T]_e\right)$$

$$+ S_e\left(2c\{T\}_e^{p,T}\int\frac{d\{N_k\}}{dx}\frac{d\{N_k\}}{dx}\{N_k\}^T dx\,[T]_e\right)$$