Faraday's law in textbooks

Faraday's law of electromagnetic induction:

(1) $$\oint_L \mathbf{E}\cdot\mathrm{d}\,\boldsymbol{l} = -\frac{\mathrm{d}\,\Phi_B}{\mathrm{d}\,t} \, ,$$

is considered as an experimental fact and as such is generally accepted as one of the cornerstones of electromagnetism. We will prove here that under a lean condition it is equivalent to the Hertzian form of the first Maxwell's equation:

(2) $$\mathrm{rot}\,\mathbf{E} = -\frac{\mathrm{d}\,\mathbf{B}}{\mathrm{d}\,t} \, .$$

However, as a rule in textbooks of electromagnetism, it is used as starting point for deduction of the Heaviside's form of the first Maxwell's equation:

(3) $$\mathrm{rot}\,\mathbf{E} = -\frac{\partial\,\mathbf{B}}{\partial\,t} \, ,$$

claiming that they are equivalent. We will show here that this equation needs a more strict condition, namely static loops and corresponding surfaces to achieve the equivalence.

To try to resolve the problem, we will perform "reverse engineering", i.e. we will integrate equations (2) and (3) and compare the result with equation (1). At the second part of this paper, I will cite some passages from textbooks of electromagnetism relating to the issue, and point where the mistake is performed. I must emphasize that my intention is not to point personally to anybody's mistakes rather to try to break some, what I am convinced, globally accepted misconceptions.

Comparing equations

We will firstly disintegrate equation (1) using an identity from mathematical analysis (Smirnov, [18], vol. 2, p. 346) stating that the rate of change of vector field flux $\mathbf{B}(\mathbf{r}, t)$ over a moving surface $S_L(t)$, is given by the expression:

(4) $$\frac{\mathrm{d}\,\Phi_B}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}\,t}\int_{S_L(t)}\mathbf{B}\cdot\mathrm{d}\mathbf{S} = \int_{S_L(t)}\left(\frac{\partial}{\partial t}\mathbf{B} - \mathrm{rot}(\mathbf{v}_d\times\mathbf{B}) + \mathbf{v}_d\cdot\mathrm{div}\,\mathbf{B}\right)\cdot\mathrm{d}\,\mathbf{S} \, .$$

Using the vector identity:

(5) $$\mathrm{rot}(\mathbf{v}_d\times\mathbf{a}) = \mathbf{v}_d\cdot(\mathrm{div}\,\mathbf{a} - \mathrm{grad}\,\mathbf{a}) - \mathbf{a}\cdot(\mathrm{div}\,\mathbf{v}_d - \mathrm{grad}\,\mathbf{v}_d) \, ,$$

we get:

(6) $$\frac{\mathrm{d}\,\Phi_B}{\mathrm{d}t} = \int_{S_L(t)}\left((\frac{\partial\,\mathbf{B}}{\partial\,t} + \mathbf{v}_d\cdot\nabla\,\mathbf{B}) + (\mathrm{div}\,\mathbf{v}_d - \mathrm{grad}\,\mathbf{v}_d)\cdot\mathbf{B}\right)\cdot\mathrm{d}\,\mathbf{S} \, .$$

By this way, equation (1) finally becomes:

(7) $$\oint_L \mathbf{E}\cdot\mathrm{d}\,\boldsymbol{l} = \int_{S_L(t)}\left((\frac{\partial\,\mathbf{B}}{\partial\,t} + \mathbf{v}_d\cdot\nabla\,\mathbf{B}) + (\mathrm{div}\,\mathbf{v}_d - \mathrm{grad}\,\mathbf{v}_d)\cdot\mathbf{B}\right)\cdot\mathrm{d}\,\mathbf{S} \, .$$

Now, we will integrate equation (2) over a moving surface $S_L(t)$:

(8) $\int\limits_{S_L} \text{rot}\,\mathbf{E}\cdot\mathbf{S} = \oint\limits_{L} \mathbf{E}\cdot\boldsymbol{l} = -\int\limits_{S_L} \dfrac{d\,\mathbf{B}}{d\,t}\cdot\mathbf{S}$,

where we used Stokes' theorem on the left side. Now, using the expression for total derivative:

(9) $\dfrac{d}{d\,t} = \dfrac{\partial}{\partial\,t} + v_x\dfrac{\partial}{\partial\,x} + v_y\dfrac{\partial}{\partial\,y} + v_z\dfrac{\partial}{\partial\,z} = \dfrac{\partial}{\partial\,t} + \mathbf{v}\cdot\nabla$,

we get:

(10) $\oint\limits_{L} \mathbf{E}\cdot d\,\boldsymbol{l} = \int\limits_{S_L(t)} \left(\dfrac{\partial\,\mathbf{B}}{\partial\,t} + \mathbf{v}_d\cdot\nabla\,\mathbf{B}\right)\cdot d\mathbf{S}$.

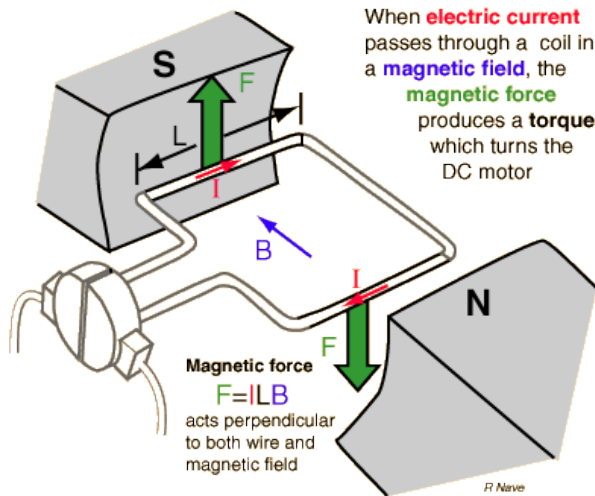The equation (3), on the other hand, after integration becomes:

(11) $\oint\limits_{L} \mathbf{E}\cdot d\,\boldsymbol{l} = \int\limits_{S_L(t)} \dfrac{\partial\,\mathbf{B}}{\partial\,t}\cdot d\,\mathbf{S}$.

Now we can compare obtained results. It is evident that left sides of all three equations (7), (10) and (11) are the same, while there is a gradation of differences on the right side. On the first look, it is obvious that (10) resembles more to (7) then (11), for the last lacks the expression $\mathbf{v}_d\cdot\nabla\,\mathbf{B}$. By this way, we may say that the equivalence of the first Maxwell's equation (3 and 11) with the Faraday's law (eq. 1 and 7) is confined to stationary surfaces (i.e. $\mathbf{v}_d(\mathbf{r},t)|_S = 0$), so that it is suitable for analysis of stationary devices such as transformers. However, it is useless for the consideration of machines with moving fields, such as electro-motors and electro-generators.

Furthermore, being that the velocity $\mathbf{v}_d$ represents the velocity of surface elements $\Delta S_L(t)$ at $\mathbf{r}\in S_L(t)$, that means that it must be treated as a vector field on the surface S_L (i.e. that $\text{div}\,\mathbf{v}_d$ and $\text{grad}\,\mathbf{v}_d$ must be taken into the consideration). The expression $\text{grad}\,\mathbf{v}_d$ in (7) represents *deformation* of the moving surface $S_L(t)$, while the component $\text{div}\,\mathbf{v}_d$ represents special type of deformation named *cubical dilatation*. Due to the fact that every deformation consists of *cubical dilatation* and *equivolume deformation*[10], the condition that:

(12) $(\text{div}\,\mathbf{v}_d)\,\mathbf{B} - \text{grad}\,\mathbf{v}\cdot\mathbf{B} = 0$,

should be related to *equivolume deformation* solely (here I need help from a good mathematician who could articulate that condition). In general case, the shape of the surface $S_L(t)$ may be changed during the time *t* into whatever you wish. The above condition means that the identity of equations (1) and (2) (i.e. (7) and (10)) is reduced to moving surfaces that does not undergo that (for me not yet strictly specified) kind of deformations.

When **electric current** passes through a coil in a **magnetic field**, the **magnetic force** produces a **torque** which turns the DC motor

Magnetic force
F=ILB
acts perpendicular to both wire and magnetic field

(http://hyperphysics.phy-astr.gsu.edu/Hbase/magnetic/motdc.html).
Fig. 1. Simple electro-motor. No one uses Maxwell's equations to explain the functioning of electric motors and generators, for they are useless for that purpose. Instead, Lorentz and magnetic force (i.e. Faraday's law) are used

Faraday's law and first Maxwell's equation in textbooks

Let's now analyze some passage from textbooks relating to the issue. I will mention some textbooks I found on Internet (thanks to Dubinushka...), even though the way of deduction follows the main idea that Faraday's law and first Maxwell's equation should be identical at any cost.

For example in *Introduction to Electrodynamics* [5], from Griffiths D. J., p. 302 it looks like this:

Faraday had an ingenious inspiration:

> **A changing magnetic field induces an electric field.**

It is this "induced" electric field that accounts for the emf in Experiment 2.6 Indeed, if (as Faraday found empirically) the emf is again equal to the rate of change of the flux,

$$\mathscr{E} = \oint_L \mathbf{E} \cdot d\boldsymbol{l} = -\frac{d\Phi}{dt} \tag{7.14}$$

then E is related to the change in B by the equation

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{a} \tag{7.15}$$

This is Faraday's law, in integral form. We can convert it to differential form by applying Stokes' theorem:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{7.16}$$

In other words the writer states that equations (7.14) /i.e. 7/ and (7.15) /i.e. 11/ are identical, what is incorrect.

In the book *Electromagnetic Field Theory* [19] from Bo Thidé we find the same assertion:

It has been established experimentally that a nonconservative EMF field is produced in a closed circuit *C* if the magnetic flux through this circuit varies with time. This is formulated in *Faraday's law* which, in Maxwell's generalised form, reads

$$\mathscr{E}(t,\mathbf{x}) = \oint_C \mathrm{d}\mathbf{l}\cdot\mathbf{E}(t,\mathbf{x}) = -\frac{\mathrm{d}}{\mathrm{d}t}\Phi_m(t,\mathbf{x})$$
$$= -\frac{\mathrm{d}}{\mathrm{d}t}\int_S \mathrm{d}\mathbf{S}\cdot\mathbf{B}(t,\mathbf{x}) = \int_S \mathrm{d}\mathbf{S}\cdot\frac{\partial}{\partial t}\mathbf{B}(t,\mathbf{x}) \tag{1.31}$$

where Φ_m is the *magnetic flux* and *S* is the surface encircled by *C* which can be interpreted as a generic stationary 'loop' and not necessarily as a conducting circuit.

...

Any change of the magnetic flux Φ_m will induce an EMF. Let us therefore consider the case, illustrated if Figure 1.4, that the 'loop' is moved in such a way that it links a magnetic field which varies during the movement. The *convective derivative* is evaluated according to the well-known operator formula

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla \tag{1.33}$$

which follows immediately from the rules of differentiation of an arbitrary differentiable function $f(t;\mathbf{x}(t))$. Applying this rule to Faraday's law, Equation (1.31) on the preceding page, we obtain

$$\mathscr{E}(t,\mathbf{x}) = -\frac{\mathrm{d}}{\mathrm{d}t}\int_S \mathrm{d}\mathbf{S}\cdot\mathbf{B} = \int_S \mathrm{d}\mathbf{S}\cdot\frac{\partial}{\partial t}\mathbf{B} - \int_S \mathrm{d}\mathbf{S}\cdot(\mathbf{v}\cdot\nabla)\mathbf{B} \tag{1.34}$$

...(end of citation)

What follows is another complicate "derivation" even though equation (1.34) could be simply rewritten:

$$\mathscr{E}(t,\mathbf{x}) = -\frac{\mathrm{d}}{\mathrm{d}t}\int_S \mathrm{d}\mathbf{S}\cdot\mathbf{B} = \int_S \mathrm{d}\mathbf{S}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{B} \tag{1.34'}$$

that agrees with (2), and directly contradicts the last equal sign in (1.31). However, combining the first equality from (1.31) with (1.34') one just confirms the equivalence between equations (2) and (3). Similar, but more "sophisticated" manipulation with dif-

ferent types of electric fields we find in *Classical Electrodynamics* [6] from Jackson, J. D.:

> Let us now consider Faraday's law for a moving circuit and see the consequences of Galilean invariance. Expressing (6.3) in terms of the integrals over **E'** and **B**, we have

$$\oint_C \mathbf{E}' \cdot d\mathbf{l} = -k \frac{d}{dt} \int_S \mathbf{B} \cdot \mathbf{n} \, da \tag{6.4}$$

> The induced electromotive force is proportional to the total time derivative of the flux—the flux can be changed by changing the magnetic induction or by changing the shape or orientation or position of the circuit. In form (6.4) we have a far-reaching generalization of Faraday's law. The circuit C can be thought of as any closed geometrical path in space, not necessarily coincident with an electric circuit. Then (6.4) becomes a relation between the fields themselves. It is important to note, however, that the electric field, E' is the electric field at *d*l in the coordinate system or medium in which *d*l is at rest, since it is that field that causes current to flow if a circuit is actually present.

> If the circuit C is moving with a velocity v in some direction, as shown in Fig. 6.2, the total time derivative in (6.4) must take into account this motion. The flux through the circuit may change because (a) the flux changes with time at a point, or (b) the translation of the circuit changes the location of the boundary. It is easy to show that the result for the total time derivative of flux through the moving circuit is

$$\frac{d}{dt} \int_S \mathbf{B} \cdot \mathbf{n} \, da = \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n} \, da + \oint_C (\mathbf{B} \times \mathbf{v}) \cdot d\mathbf{l} \tag{6.5}$$

> Equation (6.4) can now be written in the form,

$$\oint_C [\mathbf{E}' - k(\mathbf{B} \times \mathbf{v})] \cdot d\mathbf{l} = -k \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n} \, da \tag{6.6}$$

> This is an equivalent statement of Faraday's law applied to the moving circuit C. But we can choose to interpret it differently. We can think of the circuit C and surface S as instantaneously at a certain position in space in the laboratory. Applying Faraday's law (6.4) to that fixed circuit, we find

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -k \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{n} \, da \tag{-6.7}$$

> where E is now the electric field in the laboratory.

(end of citation)

Mr. Jackson is right admitting that electric field **E'** from Faraday's law (6.4) is not the same electric field **E** from the first Maxwell's equation (6.6). Furthermore, he admits that Lorentz force (as constituent $\mathbf{B} \times \mathbf{v}$) is actually part of the Faraday's law.

In *Electromagnetics* [13] from Rothwell and Cloud one may find that equation (4) is simply named Faraday's law.

> In Minkowski's form of Maxwell's equations, the mediating field is the electromagnetic field consisting of the set of four vector fields **E**(**r**, t), **D**(**r**, t), **B**(**r**, t),

and $\mathbf{H}(\mathbf{r}, t)$. The field equations are the four partial differential equations referred to as the Maxwell Minkowski equations:

$$\nabla \times \mathbf{E}(\mathbf{r},t) = -\frac{\partial}{\partial t}\mathbf{B}(\mathbf{r},t) \tag{2.1}$$

$$\nabla \times \mathbf{H}(\mathbf{r},t) = \boldsymbol{J}(\mathbf{r},t) - \frac{\partial}{\partial t}\mathbf{D}(\mathbf{r},t) \tag{2.2}$$

$$\nabla \cdot \mathbf{D}(\mathbf{r},t) = \rho(\mathbf{r},t) \tag{2.3}$$

$$\nabla \times \mathbf{B}(\mathbf{r},t) = 0 \ , \tag{2.4}$$

along with the continuity equation

$$\nabla \cdot \boldsymbol{J}(\mathbf{r},t) = \frac{\partial}{\partial t}\rho(\mathbf{r},t) \tag{2.5}$$

Here (2.1) is called *Faraday's law*, (2.2) is called *Ampere's law*, (2.3) is called *Gauss's law*, and (2.4) is called the *magnetic Gauss's law*. For brevity we shall often leave the dependence on $\mathbf{r}$ and t implicit, and refer to the Maxwell- Minkowski equations as simply the "Maxwell equations," or "Maxwell's equations."
(end of citation)

Commentary

As it is already mentioned, it is obvious even for undergraduate students that Faraday's law can not be equated with the first Maxwell's equation. However, a kind of hypnosis by authority is taking place. It seems that there is a strong psycho-logical pyramidal pressure that creates a system of prohibitions upon subordinates, which may be an interesting theme for psychoanalysts. To my opinion, it is the same kind of pressure that was in the Middle Ages, though in milder manner. Then, the disobedient would have risk to be burnt on stake, but today, you risk not to pass exam, lose job, or to be ridiculed from the side of obedient ones. It seems that the civilization has evolved technically retaining old psychological patterns.

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