Is there a bug in classical electromagnetism?

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Abstract: In literally every textbooks of classical electromagnetism there are some inconsistencies relating to the definitions of magnetic vector **H** and electric displacement vector **D**. In this work, the author shows the inconsistency of the equation:

(1) $$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M} \, ,$$

for magnetic materials relating to the Maxwell's second equation, as well as the inconsitency of the equation:

(2) $$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} \, ,$$

for dielectric materials relating to the Maxwell's third equation. More precisely, it is shown that these two equation holds only inside the materials, and that the generalisation to the whole space has yielded some erroneous results of rather theoretical then practical character.

Magnetic vector H

Let us denote by $\mathbf{H}_a$ magnetic vector, which satisfies the integral form of the second Maxwell equation for stationary currents, (i.e. d**D**/dt):

(a) $$\oint_C \mathbf{H}_a \cdot \mathrm{d}\boldsymbol{l} = \sum \mathrm{I} \, ,$$

where C represents closed loop through which pass macroscopic currents ΣI. Let $\mathbf{H}_b$ denotes the corresponding vector obtained from the expression:

(b) $$\mathbf{H}_b = \frac{\mathbf{B}}{\mu_0} - \mathbf{M} \, ,$$

where **B** denotes magnetic field vector and **M** magnetization vector inside the ferromagnetic core.

If there is no ferromagnetic core (**M**≡0), these two expressions coincide, so that it may be written:

(3) $$\oint_C \frac{\mathbf{B}}{\mu_0} \cdot \mathrm{d}\boldsymbol{l} = \sum \mathrm{I} \, ,$$

Further, let us take, for an example, solenoid of finite dimensions and with ferromagnetic core as it is represented in Fig. 1.

In this example magnetic field originates from macroscopic currents (NI), as well as from resultant Amperian current (I_A) at the surface of the ferromagnetic core. So, if we treat the Amperian current as a real one, ferromagnetic core may be neglected and the second Maxwell equation (i.e. the Ampere's law) gets the form:

(4) $$\oint_C \frac{\mathbf{B}}{\mu_0}\cdot \mathrm{d}\boldsymbol{l} = \mathrm{N I} + \mathrm{I_A}\ .$$

Let us denote by $\mathbf{B}_\mathrm{I}$ that part magnetic field vector which originates from macroscopic currents N*I* only, and by $\mathbf{B}_\mathbf{A}$ the corresponding vector which originates only from resultant Amperian current $\mathrm{I_A}$ (see Fig. 1):

(5) $$\oint_C \frac{\mathbf{B}_\mathrm{I}}{\mu_0}\cdot \mathrm{d}\boldsymbol{l} = \mathrm{N I}\ ,$$

(6) $$\oint_C \frac{\mathbf{B}_\mathrm{A}}{\mu_0}\cdot \mathrm{d}\boldsymbol{l} = \mathrm{I_A}\ .$$

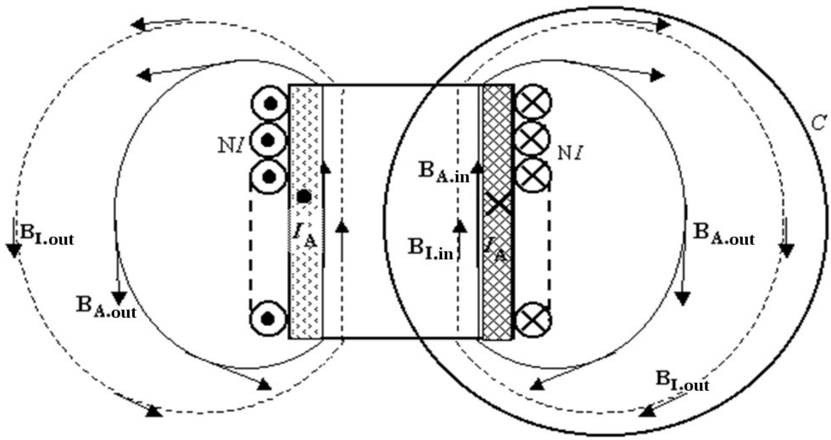

Fig. 1 – Pictorial representation of components of the magnetic field vector **B** for a short solenoid with ferromagnetic core. "A" in index denotes the component which originates from resultant Amperian current I_A, while "I" in index denotes the component which originates from macroscopic currents N*I*. "in" in index relates to corresponding components inside the core, while "out" relates to components outside the core.

Furthermore, let us denote by 'in' the inside part of the vectors, and by "out" the outside part of the vectors, i.e.:

(7) $$\begin{aligned} &\mathbf{B}_{\mathrm{I.in}} = \mathbf{B}_\mathrm{I},\ \text{and}\ \mathbf{B}_{\mathrm{A.in}} = \mathbf{B}_\mathrm{A},\ \text{inside the feromagnetic core, and} \\ &\mathbf{B}_{\mathrm{I.in}} = \mathbf{B}_{\mathrm{A.in}} = 0,\qquad \text{outside the feromagnetic core,} \end{aligned}$$

(8) $$\begin{aligned} &\mathbf{B}_{\mathrm{I.out}} = \mathbf{B}_{\mathrm{A.out}} = 0,\qquad \text{inside the feromagnetic core, and} \\ &\mathbf{B}_{\mathrm{I.out}} = \mathbf{B}_\mathrm{I},\ \text{and}\ \mathbf{B}_{\mathrm{A.out}} = \mathbf{B}_\mathrm{A},\ \text{outside the feromagnetic core,} \end{aligned}$$

which is visually represented in the Fig. 1. It is obvious that:

(9) $$\mathbf{B}_\mathrm{I} = \mathbf{B}_{\mathrm{I.in}} + \mathbf{B}_{\mathrm{I.out}},$$

(10) $\mathbf{B}_{A} = \mathbf{B}_{A.in} + \mathbf{B}_{A.out}$
and
(11) $\mathbf{B} = \mathbf{B}_{I} + \mathbf{B}_{A}$.

On basis of these notations, it is obvious that:

(12) $$\mathbf{M} = \frac{\mathbf{B}_{A.in}}{\mu_0},$$

(13) $$\mathbf{H}_a = \frac{\mathbf{B}_I}{\mu_0}, \text{ and}$$

(14) $$\mathbf{H}_b = \frac{\mathbf{B}}{\mu_0} - \mathbf{M} = \frac{\mathbf{B}_I + \mathbf{B}_{A.out}}{\mu_0}.$$

As it may be seen the two vectors $\mathbf{H}_a$ and $\mathbf{H}_b$ are not identical. They agree only inside the ferromagnetic core. It means that one must opt for one among the expressions (a) and (b) as definition of vector **H**. As for me, it is reasonable to take (a) or (13) being that it is in accordance with Amper's law.

Electric displacement vector D

Literally the same treatment may be proceeded with the trinity of vectors **E**, **D** and **P**, because of their strong analogy with **B**, **H** and **M**.

Let us denote by $\mathbf{D}_a$ electric displacement vector, which satisfies the integral form of the Maxwell's third equation:

(a') $$\oint_S \mathbf{D}_a \cdot d\mathbf{S} = \sum Q,$$

where S represents closed surface in which is found free charge ΣQ. Let $\mathbf{D}_b$ denotes the corresponding vector obtained from the expression:

(b') $$\mathbf{D}_b = \varepsilon_0 \mathbf{E} + \mathbf{P},$$

where **E** denotes electric field vector and **P** electric polarization vector inside the dielectric.

When there is no dielectric medium (i.e. $\mathbf{P} \equiv 0$), these two vectors coincide (i.e. $\mathbf{D}_a = \mathbf{D}_b = \mathbf{D}$), so that it may be written:

(17) $$\oint_S \varepsilon_0 \mathbf{E} \cdot d\mathbf{S} = \sum Q$$

Further, let us take, for an example plate condenser of finite dimensions and with dielectric medium as it is represented in Fig. 2.

In this example electric field originates from free charges ($+Q$ and $-Q$), as well as from resultant polarization charges ($-Q_P$ and $+Q_P$) at the surface on the plate side of the dielectric. So, if we simply treat the resultant polarization charge as a free charge, we may neglect dielectric and the Maxwell's third equation, (i.e. Gauss' law) gets the form:

(18) $$\oint_S \varepsilon_0 \mathbf{E} \cdot d\mathbf{S} = Q - Q_P,$$

where S denotes the surface closed around the positive side of the condenser (see Fig. 2).

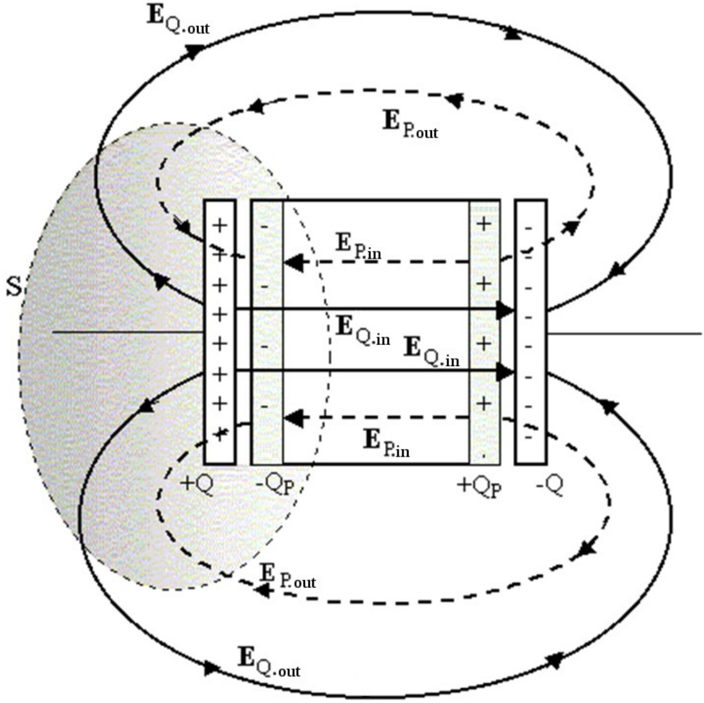

Fig. 2 – Pictorial representation of components of the electric field vector $\vec{\mathbf{E}}$ for a finite plate condenser with dielectric medium. P in index denotes the component which originates from resultant polarization charge Q_P, while Q in index denotes the component which originates from free charge Q on plates. (.in) in index relates to corresponding components inside the condenser, while (.out) relates to components outside the condenser.

Let us denote by $\mathbf{E}_Q$ that part of electric field of the condenser which originates only from free charge on the plates Q, and by $\mathbf{E}_P$ the corresponding vector that originates only from resultant polarization charge Q_P:

(19) $$\oint_S \varepsilon_0 \mathbf{E}_Q \cdot \mathrm{d}\mathbf{S} = +Q ,$$

(20) $$\oint_S \varepsilon_0 \mathbf{E}_P \cdot \mathrm{d}\mathbf{S} = Q_P .$$

Furthermore, let us denote (just as in the case of magnetic induction):

(21) $$\begin{array}{ll} \mathbf{E}_{Q.in} = \mathbf{E}_Q \text{, and } \mathbf{E}_{P.in} = \mathbf{E}_P, & \text{inside the condenser, and} \\ \mathbf{E}_{Q.in} = \mathbf{E}_{P.in} = 0, & \text{outside the condenser,} \end{array}$$

(22) $\mathbf{E}_{\text{Q.out}} = \mathbf{E}_{\text{P.out}} = 0,$ inside the condenser, and
$\mathbf{E}_{\text{Q.out}} = \mathbf{E}_{\text{Q}}$, and $\mathbf{E}_{\text{P.out}} = \mathbf{E}_{\text{P}}$, outside the condenser,

which is visualy represented in the Fig. 2. It is obvious that:

(23) $\mathbf{E}_{\text{Q}} = \mathbf{E}_{\text{Q.in}} + \mathbf{E}_{\text{Q.out}}$,

(24) $\mathbf{E}_{\text{P}} = \mathbf{E}_{\text{P.in}} + \mathbf{E}_{\text{P.out}}$

and

(25) $\mathbf{E} = \mathbf{E}_{\text{Q}} + \mathbf{E}_{\text{P}}$.

On the basis of the accepted notations, it is obvious that:

(26) $\mathbf{P} = -\varepsilon_0 \mathbf{E}_{\text{P.in}}$,

(27) $\mathbf{D}_{\text{a}} = \varepsilon_0 \mathbf{E}_{\text{Q}}$, and

(28) $\mathbf{D}_{\text{b}} = \varepsilon_0 \mathbf{E} + \mathbf{P} = \varepsilon_0 (\mathbf{E}_{\text{Q}} + \mathbf{E}_{\text{P.out}})$.

Just as in the case of the magnetic field vector, the two vectors $\mathbf{D}_a$ and $\mathbf{D}_b$ are not identical. It means that the equation

(b') $\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$

is not valid, or rather it is valid only inside the dielectric. The only valid explicit definition for vector **D** can be the expression (27), or implicitly the third Maxwell equation (b′).

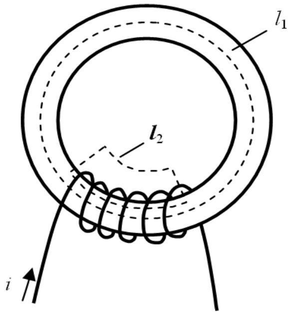

Fig. 3. An illustration of computation of vector field **H**, where one gets two quite different values depending on path of integration.

Consequences

Whatever definition of vector fields **H** or **D** we chose, it is obvious that Maxwell's equations for media lose their sense.

There are other consequences, too, of above inconsistencies. On the first place is the so-called magnetic circuit theory, then boundary relations etc.

Being based on these inconsistencies, it may be said that magnetic circuit theory is a nonsense theory. There are ample of examples that support this conclusion. We will mention a simple case of a coil around toroidal magnetic core (Fig. 3).

According to magnetic circuit theory, vector **H** is uniformly allocated inside the core. So, integrating along line l_1, we get:

(31) $$H = \frac{Ni}{l_1}.$$

On the other hand, if we take integral along line l_2, we get:

(32) $$H = \frac{Ni}{l_2},$$

which is obviously a contradiction.

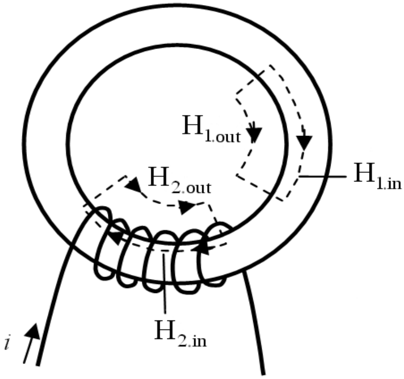

Fig. 4. Vectors $\mathbf{H}_{1.out}$ and $\mathbf{H}_{2.out}$ along concentric circles outside magnetic core have opposite directions.

Moreover, there is another contradiction in this example. Applying Ampere's law (second Maxwell's equation) to loops from Fig. 4, one gets that vectors $\mathbf{H}_{1.out}$ and $\mathbf{H}_{2.out}$ along concentric circles outside magnetic core have opposite directions. being that in vacuum $\mathbf{B} = \mu_0\mathbf{H}$, follows that even magnetic field vector **B** has opposite directions in the same circles (Fig. 5).

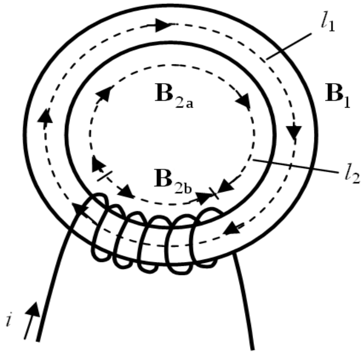

Fig. 5. One of the consequences of the "magnetic circuit theory" is the "sourceness" of the magnetic field vector, i.e. that a mere coil creates a pair of magnetic monopoles.

Another example of the so called "magnetic circuit theory" is ″strange″ distribution of vector field **H** inside and around a permanent magnet or of electric displacement vector **D** inside an electret, which may be found in textbooks of electromagnetism (e.g. [14], Fig. 10.24).

How it might happen

In order to show how the mistake crippled into electromagnetism, we will now analyze a passage from one of the most representative textbooks on electromagnetism from. Mr. D.J. Griffiths [5]. On page 175 we find:

4.3 The Electric Displacement

4.3.1 Gauss's Law in the Presence of Dielectrics

In Sect. 4.2 we found that the effect of polarization is to produce accumulations of bound charge, $\rho_b = -\nabla\cdot\mathbf{P}$ within the dielectric and $\sigma_b = \mathbf{P}\cdot\mathbf{n}$ on the surface. The field due to polarization of the medium is just the field of this bound charge. We are now ready to put it all together: the field attributable to bound charge plus the field due to everything else (which, for want of a better term, we call **free charge**). The free charge might consist of electrons on a conductor or ions embedded in the dielectric material or whatever; any charge, in other words, that is not a result of polarization. Within the dielectric, then, the total charge density can be written:

$$\rho = \rho_b + \rho_f \tag{4.20}$$

and Gauss's law reads

$$\boxed{\varepsilon_0\,\nabla\cdot\mathbf{E} = \rho = \rho_b + \rho_f = -\nabla\cdot\mathbf{P} + \rho_f\,,}$$

where E is now the total field, not just that portion generated by polarization.

It is convenient to combine the two divergence terms:

$$\nabla\cdot(\varepsilon_0\,\mathbf{E} + \mathbf{P}) = \rho_f\,.$$

The expression in parentheses, designated by the letter D,

$$\boxed{\mathbf{D} =\cdot\ \varepsilon_0\,\mathbf{E} + \mathbf{P},} \tag{4.21}$$

is known as the **electric displacement**. In terms of D, Gauss's law reads

$$\boxed{\nabla\cdot\mathbf{D} = \rho_f\,.} \tag{4.22}$$

(Colored frames put by B.P.)

As we see the first expression ($\rho_b = -\nabla\cdot\mathbf{P}$) is confined to dielectric, because $\mathbf{P}$=0 outside dielectric. But, in general case, ρ_b is the source of electric field outside dielectric as well as inside dielectric, just as ρ_f is. The expressions (4.21) and (4.22) have sense only in case when the net electric field that originates from ρ_b is zero outside dielectric (e.g. ideal capacitor). Therefore, in general case, the expression (4.22) is incorrect, providing that (4.21) is correct (and vice versa).

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