3. Maxwell-Hertz's Equations

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Maxwell's equations are derived in a natural and logically consistent manner from Lorentz equations. It appears that the obtained equations have Hertz's invariant form of Maxwell's equations. We will name those equations either Maxwell-Hertz's equations or Hertz's equations, to avoid ambiguity. Visually, they differ slightly from ordinary Maxwell's equations (i.e. the Heaviside's form of the Maxwell's equations), partial time derivatives should be changed by total time derivatives, but the consequences are tremendous. Namely, Maxwell-Hertz's equations are invariant to Galilean transformations, therefore there is no need for Einstein's special relativity, as far as the first postulate of relativity is concerned.

We will firstly derive differential form of Maxwell-Hertz's equations, and after that, the corresponding integral forms. Moreover, from integral forms it is possible to get a unique complex four-dimensional equation of electromagnetic field.

3.1 Some auxiliary relations

We will firstly derive some useful relations, which will much alleviate and clarify

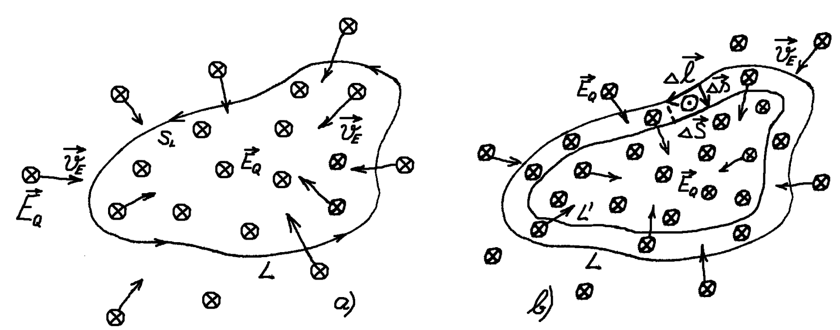

Figure III-1. Illustration of gradual change of electrostatic field flux. In case of flux growth, after an interval Δt, the lines will shrink from L to L' moving by velocity $\vec{v}_E$.

the whole procedure of derivation of Hertz's equations.

3.1.1. Changes of elementary fluxes

We will start with the derivation by emphasizing the difference between two kinds of change of electrostatic field flux.

In the first case, a point charge q moves without passing through elementary surface ΔS (Fig. III-1). The change of flux can be made as low as it is necessary, depending on time interval Δt in which the change occur. The surface segment detects this change as a shrink or stretch of the field lines, depending on if the charge moves toward or outward the surface element. This kind of change we will name gradual change.

In the other case, a point charge moves through the surface element ΔS, and the change of flux is about 200%, doesn't matter how small time interval Δt we take (Fig. III-2). The surface segment detects this change as a change of direction of the field lines. It is, therefore, named radical change of flux.

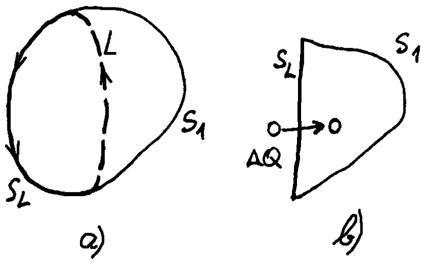

Figure III-2. Illustration for calculation of radical change of electrostatic flux through surface S_L. S_1 is an auxiliary surface, which tied to S_L makes closed surface $S_L \cup S_1$.

3.1.2. Main auxiliary equation

We will, now, analyze the change of flux $\Delta\Phi_E$ of the electrostatic field $\mathbf{E}_Q$ on a fixed unclosed surface S_L, bordered by a fixed oriented line L. In the following derivations we will assume that the density of field lines (i.e. the number of lines per square unit) is proportional to intensity of the corresponding field.

In case of gradual change of flux (denoted by ΔΦ'), we will take in mind that the change corresponds to change of number of field lines that pass through the surface. In case of flux growth, the lines will shrink. As it is represented in Fig. III-1-b, the lines bordered by L will shrink to line L' after corresponding time interval Δt. It means that the change of flux corresponds to the number of lines between L and L'. If the field lines moves by velocity $\mathbf{v}_E$, the flux change in element $\Delta\boldsymbol{l}$ of line L is given by:

(3-1) $$\Delta\Phi'(\mathbf{r}) = \mathbf{E}_Q(\mathbf{r})\cdot\Delta\boldsymbol{l}\times\Delta\mathbf{s} = \mathbf{E}_Q(\mathbf{r})\cdot\Delta\boldsymbol{l}(\mathbf{r})\times\mathbf{v}_E(\mathbf{r})\Delta t \;.$$

The rate of flux change for the whole line L is, therefore:

$$\frac{\partial\Phi'}{\partial t} = \oint_L \mathbf{E}_Q\cdot\mathbf{v}_E\times d\boldsymbol{l}\,,$$

which can be written as:

(3-2) $$\frac{\partial\Phi'}{\partial t} = \oint_L \mathbf{v}_E\times\mathbf{E}_Q\cdot d\boldsymbol{l} = \oint_{S_L} \mathrm{rot}(\mathbf{v}_E\times\mathbf{E}_Q)\cdot d\boldsymbol{S}\,,$$

according to Stokes's theorem.

To get the radical flux change ΔΦ", we will attach to the surface S_L, a new surface S_1, in such a way that those two surfaces make a new closed surface $S_L \cup S_1$, into which enters a point charge ΔQ that is the cause of the radical flux change (Fig. III-2.).

If a charge ΔQ pass through the surface S_L and enters into $S_L \cup S_1$, the change of flux for $S_L \cup S_1$ is according to electrostatic Gauss' theorem:

(3-3) $$\Delta\Phi''_{S_L \cup S_1} = \frac{\Delta Q}{\varepsilon_0} .$$

Being that this flux change is gradual for S_1 and radical for S_L, choosing as small time interval Δt as necessary, we may write:

(3-4) $$\Delta\Phi''_{S_L \cup S_1} = \Delta\Phi''_{S_L} .$$

By this way, the rate of radical flux change through S_L is:

(3-5) $$\frac{\partial\Phi''}{\partial t} = \frac{\mathrm{d}Q}{\varepsilon_0 \mathrm{d}t} = \int_{S_L} \boldsymbol{j}\cdot\mathrm{d}\boldsymbol{S} ,$$

where the minus sign comes because of opposite directions of moving charge and the surface $S_L \cup S_1$. On the other way, from the expression for total flux:

$$\Phi = \int_{S_L} \mathbf{E}_\mathrm{Q}\cdot\mathrm{d}\boldsymbol{S} ,$$

we have:

(3-6) $$\frac{\partial\Phi}{\partial t} = \int_{S_L} \frac{\partial\mathbf{E}_\mathrm{Q}}{\partial t}\cdot\mathrm{d}\boldsymbol{S} ,$$

for the surface S_L is fixed. Being that:

(3-7) $$\frac{\partial\Phi}{\partial t} = \frac{\partial\Phi'}{\partial t} + \frac{\partial\Phi''}{\partial t} ,$$

we have:

$$\int_{S_L} \frac{\partial\mathbf{E}_\mathrm{Q}}{\partial t}\cdot\mathrm{d}\boldsymbol{S} = \int_{S_L} \mathrm{rot}(\mathbf{v}_\mathrm{E}\times\mathbf{E}_\mathrm{Q})\cdot\mathrm{d}\boldsymbol{S} = -\frac{1}{\varepsilon_0}\int_{S_L} \boldsymbol{j}\cdot\mathrm{d}\boldsymbol{S} ,$$

or

(3-8) $$\int_{S_L} \left(\mathrm{rot}(\mathbf{v}_\mathrm{E}\times\mathbf{E}_\mathrm{Q}) - \frac{\partial\mathbf{E}_\mathrm{Q}}{\partial t} - \frac{1}{\varepsilon_0}\boldsymbol{j}\right)\cdot\mathrm{d}\boldsymbol{S} = 0 .$$

From the arbitrariness of the surface S_L, it follows that:

$$\mathrm{rot}(\mathbf{v}_\mathrm{E}\times\mathbf{E}_\mathrm{Q}) - \frac{\partial\mathbf{E}_\mathrm{Q}}{\partial t} - \frac{1}{\varepsilon_0}\boldsymbol{j} = 0 ,$$

which can be written as:

(3-9) $$\mathrm{rot}(\mathbf{v}_\mathrm{E}\times\mathbf{E}_\mathrm{Q}) = \frac{\partial\mathbf{E}_\mathrm{Q}}{\partial t} + \frac{1}{\varepsilon_0}\boldsymbol{j} .$$

The obtained equation is the purpose of our introductory preparation for derivation of Maxwell-Hertz's equations. (Actually, I found this expression in [Barnes, Thomas G., *Foundations of Electricity and Magnetism, 3rd Ed.*, Ch. 14. Emission theory of electromagnetism], but it was not derived in satisfactory manner.)

3.1.3. Equation of continuity

Applying div operator on identity (3-9), we get:

$$\operatorname{div}\operatorname{rot}(\mathbf{v}_E \times \mathbf{E}_Q) = \operatorname{div}\left(\frac{\partial \mathbf{E}_Q}{\partial t} + \frac{1}{\varepsilon_0}\boldsymbol{j}\right),$$

Due to identities div rot = 0, and $\operatorname{div}\mathbf{E}_Q = \dfrac{\rho}{\varepsilon_0}$, we get:

(3-9b) $$\frac{\partial \rho}{\partial t} = -\operatorname{div}\boldsymbol{j}\ .$$

This is the current density form of the equation of continuity. Due to identity:

(1-2) $$\boldsymbol{j} = \mathbf{v}_Q \cdot \rho\ ,$$

it may be rewritten as:

(3-9c) $$\frac{\partial \rho}{\partial t} = -\operatorname{div}(\mathbf{v}_Q \cdot \rho)\ .$$

This is the classical form of the continuity equation from fluid mechanics. Now, we can get yet another classical identity. Combining it with the operator formula:

(3-9e) $$\frac{d}{dt} = \frac{\partial}{\partial t} + \left(\mathbf{v}_D \cdot \frac{\partial}{\partial \mathbf{r}}\right)\ (=)\ \frac{\partial}{\partial t} + (\mathbf{v}_D \cdot \nabla)\ ,$$

we get:

$$\frac{d\rho}{dt} - \mathbf{v}_Q \cdot \nabla \rho + \operatorname{div}(\mathbf{v}_Q\, \rho) = 0\ .$$

Now due to vector field identity:

(3-9f) $$\operatorname{div}(\mathbf{v}_Q\, \rho) = \rho\ \operatorname{div}\mathbf{v}_Q + \mathbf{v}_Q \cdot \nabla \rho\ ,$$

we get:

(3-9g) $$\frac{d\rho}{dt} + \rho \cdot \operatorname{div}\mathbf{v}_Q = 0\ .$$

The expression $\operatorname{div}\mathbf{v}_Q$ may be different from zero in case that $\mathbf{v}_Q$ represents a vector field. However, assuming that charge is incompressible, i.e.:

(3-9h) $$\frac{d\rho}{dt} = 0\ ,$$

we get finally:

(3-9i) $$\operatorname{div}\mathbf{v}_Q = 0\ .$$

This is ordinary identity from continuum mechanics. It is logical if we assume that the divergence simply means the expansion or contraction of the corresponding physical quantity.

3.2. Differential form of Maxwell-Hertz's equations

3.2.1. Second Maxwell-Hertz's equation

Static magnetic field

Combining the equation (3-9) with the equation for magnetic field:

(*Lorentz equations* 2-34) $\mathbf{B}_\mathrm{Q} = \frac{1}{c^2}(\mathbf{v}_\mathrm{E} - \mathbf{v}_\mathrm{d}) \times \mathbf{E}_\mathrm{Q}$,

we get:

$$\mathrm{rot}\,\mathbf{B}_\mathrm{Q} = \mu_0\,\boldsymbol{j}(\mathbf{r}) + \frac{\partial\,\mathbf{E}_\mathrm{Q}}{c^2\,\partial\,\mathrm{t}} - \mathrm{rot}\,\frac{\mathbf{v}_\mathrm{d} \times \mathbf{E}_\mathrm{Q}}{c^2} \ .$$

Using vector identity [1]:

(3-10) $\mathrm{rot}\,(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \cdot (\mathrm{div}\,\mathbf{b} - \mathrm{grad}\,\mathbf{b}) - \mathbf{b} \cdot (\mathrm{div}\,\mathbf{a} - \mathrm{grad}\,\mathbf{a})$,

the above expression can be further simplified:

(3-11) $\mathrm{rot}\,\mathbf{B}_\mathrm{Q} = (\mu_0\,\boldsymbol{j} - \frac{\mathbf{v}_\mathrm{d}}{c^2} \cdot \mathrm{div}\,\mathbf{E}_\mathrm{Q}) + (\frac{\partial\,\mathbf{E}_\mathrm{Q}}{c^2\,\partial\,\mathrm{t}} + \frac{\mathbf{v}_\mathrm{d} \cdot \nabla}{c^2}\,\mathbf{E}_\mathrm{Q})$,

where is taken in mind that detector velocity $\mathbf{v}_\mathrm{d}$ is not a vector field.

From the well-known operator formula:

(3-12) $\frac{\mathrm{d}}{\mathrm{d\,t}} = \frac{\partial}{\partial\,\mathrm{t}} + (\mathbf{v}_\mathrm{Q} \cdot \nabla)$,

follows the final expression:

(3-13) $\mathrm{rot}\,\mathbf{B}_\mathrm{Q}(\mathbf{r},\mathrm{t}) = \mu_0\,\boldsymbol{j}_\mathrm{r}(\mathbf{r},\mathrm{t}) + \frac{\mathrm{d}\,\mathbf{E}_\mathrm{Q}(\mathbf{r},\mathrm{t})}{c^2\mathrm{d\,t}}$,

where:

(3-14) $\boldsymbol{j}_\mathrm{r} = (\mathbf{v}_\mathrm{Q} - \mathbf{v}_\mathrm{d})\,\rho = \boldsymbol{j} - \varepsilon_0\,\mathbf{v}_\mathrm{d}\,\mathrm{div}\,\mathbf{E}_\mathrm{Q}$,

denotes relative current density. The obtained expression relates to static magnetic field being that it originates from (moving) electrostatic field.

Induced magnetic field

Being that electric field can be divided on electrostatic and induced:

(3-15) $\mathbf{E} = \mathbf{E}_\mathrm{Q} + \mathbf{E}_\mathrm{B}$,

it arises a hypothesis that even the change of induced electric field makes magnetic field in same manner:

(3-16) $\mathrm{rot}\,\mathbf{B}_\mathrm{B}(\mathbf{r},\mathrm{t}) = \frac{\mathrm{d}\,\mathbf{E}_\mathrm{B}(\mathbf{r},\mathrm{t})}{c^2\mathrm{d\,t}}$,

which we will name induced magnetic field. Actually, this is the only logical leap in this paper.

Total magnetic field

Adding equations (3-13) and (3-16) we obtain complete differential form of the (Hertz's form of the) second Maxwell equation:

(3-17) $$\mathrm{rot}\,\mathbf{B}(\mathbf{r},t) = \mu_0\,\boldsymbol{j}_r(\mathbf{r},t) + \frac{\mathrm{d}\,\mathbf{E}(\mathbf{r},t)}{c^2\,\mathrm{d}t}\ ,$$

while the expressions (3-13) and (3-16) are named partial forms of the second Maxwell-Hertz's equation.

3.2.2. First Maxwell-Hertz's equation

Similarly to the previous case, we will observe the change of magnetic field flux $\Delta\Phi_B$ over fixed surface S_L, bordered by closed line L. Supposing that there are no magnetic monopoles, we may conclude that (similarly to Eq. 3-9):

(3-18) $$\mathrm{rot}\,(\mathbf{v}_B \times \mathrm{v}\,\mathbf{B}) = \frac{\partial\mathbf{B}}{\partial t}\ ,$$

From the first Lorentz equation:

(*Lorentz equations* 2-19) $$\mathbf{E}_B = (\mathbf{v}_d - \mathbf{v}_B) \times \mathbf{B}\ .$$

we have:

(3-19) $$\mathbf{v}_B \times \mathbf{B} = \mathbf{v}_d \times \mathbf{B} - \mathbf{E}_B\ .$$

Applying **rot** operator and mixing with (3-16), we get:

(3-20) $$\mathrm{rot}\,\mathbf{E}_B = -\frac{\partial\mathbf{B}}{\partial t} + \mathrm{rot}(\mathbf{v}_d \times \mathbf{B})\ .$$

By the same reasoning as for the second Maxwell-Hertz's equation (3-10/13), we get:

(3-21) $$\mathrm{rot}\,\mathbf{E}_B = -\frac{\mathrm{d}\,\mathbf{B}}{\mathrm{d}t}\ .$$

Assuming that for electrostatic field:

(3-22) $$\mathrm{rot}\,\mathbf{E}_Q = 0\ ,$$

we get the final expression for the first Maxwell-Hertz's equation:

(3-23) $$\mathrm{rot}\,\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} + \mathrm{rot}(\mathbf{v}_d \times \mathbf{B})\ ,\ \text{or}$$

(3-24) $$\mathrm{rot}\,\mathbf{E} = -\frac{\mathrm{d}\,\mathbf{B}}{\mathrm{d}t}\ .$$

The expressions (3-21) and (3-22) are named partial forms of the first Maxwell-Hertz's equation.

3.2.3. Third Maxwell-Hertz's equation

The third Hertz's equation for electrostatic field is a generalization of the Gauss law:

(*Electrostatics* 1-22) $$\mathrm{div}\,\mathbf{E}_Q(\mathbf{r}) = -\Delta\varphi(\mathbf{r}) = \frac{\rho}{\varepsilon_0}\ .$$

From the fact that induced electric field does not have sources, it follows that:

(3-25) $$\mathrm{div}\,\mathbf{E}_B = 0\ .$$

Summing these two equations, we get the differential form of the third Maxwell-Hertz's equation:

(3-26) $\operatorname{div} \mathbf{E}(\mathbf{r}) = \dfrac{\rho}{\varepsilon_0}$.

3.2.4. Fourth Maxwell-Hertz's equation

The fourth Maxwell-Hertz's equation is direct consequence of sourceless character of magnetic field and Helmholtz' theorem for vector fields:

(3-27) $\operatorname{div} \mathbf{B}(\mathbf{r}, \mathrm{t}) = 0$.

It is obvious that, such character have static and (hypothetical) induced magnetic fields:

(3-28) $\operatorname{div} \mathbf{B}_{\mathrm{Q}}(\mathbf{r}, \mathrm{t}) = 0$,

(3-29) $\operatorname{div} \mathbf{B}_{\mathrm{B}}(\mathbf{r}, \mathrm{t}) = 0$.

3.3. Integral form of Maxwell-Hertz's equations

Integral form of Maxwell's (Hertz's) equations are useful for computing real electromagnetic fields. Before their derivation, we will prove a helping lemma.

For a vector field $\mathbf{f}(\mathbf{r},t)=0$ and a moving surface $S_L(t)$ (bordered by a moving loop L(t)) that does not undergo equivolume deformation it is valid the expression:

(3-30) $$\frac{d}{dt}\int_{S_L}\mathbf{f}(\mathbf{r},t)\cdot d\mathbf{S} = \int_{S_L}\frac{d}{dt}\mathbf{f}(\mathbf{r},t)\cdot d\mathbf{S} \; .$$

Proof:

It is shown in mathematical analysis (Smirnov, [10], vol. 2, p. 346) that the rate of change of vector field flux $\mathbf{a}(\mathbf{r}, t)$ over a moving surface S_L, is given by expression:

(3-31) $$\frac{d\Phi_a}{dt} = \frac{d}{dt}\int_{S_L(t)}\mathbf{a}(\mathbf{r},t)\cdot d\mathbf{S} = \int_{S_L(t)}\left(\frac{\partial}{\partial t}\mathbf{a}(\mathbf{r},t) - \mathrm{rot}(\mathbf{v}_d\times\mathbf{a}) + \mathbf{v}_d\cdot\mathrm{div}\,\mathbf{a}\right)\cdot d\mathbf{S} \; ,$$

where $\mathbf{v}_d(\mathbf{r},t)$ now denotes velocity of the surface element $\Delta S_L(\mathbf{r},t)$ at $\mathbf{r}\in S_L(t)$. For that reason, it should be treated as a surface vector field. We will now use the identity:

(3-10) $$\mathrm{rot}(\mathbf{v}_d\times\mathbf{a}) = \mathbf{v}_d\cdot(\mathrm{div}\,\mathbf{a} - \mathrm{grad}\,\mathbf{a}) - \mathbf{a}\cdot(\mathrm{div}\,\mathbf{v}_d - \mathrm{grad}\,\mathbf{v}_d) \; .$$

The above mentioned condition (absence of equivolume deformation) means just that:

(3-32) $$\int_{S_L(t)}\mathbf{a}\cdot(\mathrm{div}\,\mathbf{v}_d - \mathrm{grad}\,\mathbf{v}_d)\cdot d\mathbf{S} = 0 \; .$$

The expression $\mathrm{grad}\,\mathbf{v}_d$ represents deformation of the surface, while the component $\mathrm{div}\,\mathbf{v}_d$ represents special type of deformation named cubical dilatation. Due to the fact that every deformation consists of cubical dilatation and equivolume deformation, the expression (3-32a) should be valid for equivolume deformation (here I need help from a good mathematician who could articulate that condition). By this way, the above expression becomes:

(3-33) $$\frac{d\Phi_a}{dt} = \frac{d}{dt}\int_{S_L(t)}\mathbf{a}\cdot d\mathbf{S} = \int_{S_L(t)}\left(\frac{\partial}{\partial t}\mathbf{a} + \mathbf{v}_d\cdot\mathrm{div}\,\mathbf{a}\right)\cdot d\mathbf{S} = \int_{S_L(t)}\frac{d\mathbf{a}}{dt}\cdot d\mathbf{S} \; ,$$

where we used the operator formula for total derivative (3-12).

Q.E.D.

Warning: The equation (3-31) may be true even if the condition that the surface $S_L(t)$ doesn't undergo equivolume deformation is not satisfied. However, that condition is sufficient, not necessary. In other words, it is possible that $S_L(t)$ does undergo equivolume deformation and that after multiplication by $\mathbf{a}(\mathbf{r},t)$ and subsequent integration that part vanish. This is the most strict condition I know at this moment. In previous editions of this book, I mistakenly omitted this important condition.

3.3.1. Second Maxwell-Hertz's equation

Integrating the equation (3-17) over a moving equivolumely undeformed surface $S_L(t)$, bordered by a closed line L(t), we get:

(3-34) $$\int_{S_L} \mathrm{rot}\,\mathbf{B}(\mathbf{r},t)\cdot d\mathbf{S} = \mu_0 \int_{S_L} \boldsymbol{j}_r(\mathbf{r},t)\cdot d\mathbf{S} + \frac{1}{c^2}\int_{S_L} \frac{d\mathbf{E}(\mathbf{r},t)}{dt}\cdot d\mathbf{S},$$

where $\boldsymbol{j}_r$ is given by the expression (3-14). Left side of the expression is according to Stoke's theorem:

(3-35) $$\int_{S_L} \mathrm{rot}\,\mathbf{B}\cdot d\mathbf{S} = \oint_L \mathbf{B}\cdot d\boldsymbol{l}.$$

First integral from the right side is electric current:

(3-36) $$i = \int_{S_L} \boldsymbol{j}_r\cdot d\mathbf{S} = \int_{S_L} \rho(\mathbf{v}_Q - \mathbf{v}_d)\cdot d\mathbf{S} = \int_{S_L} \left(\boldsymbol{j} - \frac{\mathbf{v}_d \,\mathrm{div}\,\mathbf{E}}{\varepsilon_0}\right)\cdot d\mathbf{S},$$

while the second is according to the helping lemma:

(3-37) $$\int_{S_L} \frac{d\mathbf{E}}{dt}\cdot d\mathbf{S} = \frac{d}{dt}\int_{S_L} \mathbf{E}\cdot d\mathbf{S} = \frac{d\Phi_E}{dt}.$$

By this way, we have the final expression, i.e. the integral form of the second Hertz's equation:

(3-38) $$\oint_L \mathbf{B}\cdot d\boldsymbol{l} = \mu_0 i + \frac{d\Phi_E}{c^2 dt}.$$

Comparing equations (3-13) and (3-16) with (3-17), we may conclude that their integral form should be similar:

(3-39) $$\oint_L \mathbf{B}_Q\cdot d\boldsymbol{l} = \mu_0 i + \frac{d\Phi_{EQ}}{c^2 dt},$$

and

(3-40) $$\oint_L \mathbf{B}_B\cdot d\boldsymbol{l} = \mu_0 i + \frac{d\Phi_{EB}}{c^2 dt},$$

where Φ_{EQ} and Φ_{EB} denote flux of electrostatic and induced electric fields, respectively.

3.3.2. First Maxwell-Hertz's equation

Being that the differential form of the first Hertz's equations (3-21), (3-22) and (3-24) have similar form as the second (3-13), (3-16) and (3-17), it is obvious that their integral forms are:

(3-41) $$\oint_L \mathbf{E}\cdot d\boldsymbol{l} = -\frac{d\Phi_B}{dt},$$

(3-42) $$\oint_L \mathbf{E}_Q\cdot d\boldsymbol{l} = 0,$$

and

(3-43) $$\oint_L \mathbf{E}_B\cdot d\boldsymbol{l} = -\frac{d\Phi_B}{dt}.$$

It is obvious that such obtained first Hertz's equation coincides with Faraday's law. It is a great verification of the whole method exposed in this work.

3.3.3. Third Maxwell-Hertz's equation

Combining electrostatic Gauss' law (1-23):

(3-44) $$\Phi|_S = \oint_S \mathbf{E}_Q \cdot \mathrm{d}\,\mathbf{S} = \int_V \nabla \cdot \mathbf{E}_Q \,\mathrm{d}\,V = \frac{Q|_S}{\varepsilon_0} ,$$

with the sourcelessness of the induced electric field:

(3-45) $$\oint_S \mathbf{E}_B \cdot \mathrm{d}\,\mathbf{S} = 0 ,$$

we get complete integral form of the third Hertz's equation:

(3-46) $$\oint_S \mathbf{E} \cdot \mathrm{d}\,\mathbf{S} = \frac{Q|_S}{\varepsilon_0} .$$

3.3.4. Fourth Maxwell-Hertz's equation

From the equations (3-28), (3-29) and (3-30) it is obviously:

(3-47) $$\oint_S \mathbf{B}_Q \cdot \mathrm{d}\,\mathbf{S} = 0 ,$$

(3-48) $$\oint_S \mathbf{B}_B \cdot \mathrm{d}\,\mathbf{S} = 0 ,$$

and

(3-49) $$\oint_S \mathbf{B} \cdot \mathrm{d}\,\mathbf{S} = 0 .$$

3.3.5. Complete set of Maxwell-Hertz's equations

Number of equation	Differential form of equation	Integral form of equation
First Partial forms	$\mathrm{rot}\,\mathbf{E}_\mathrm{B} = -\dfrac{\mathrm{d}\,\mathbf{B}}{\mathrm{d}\,t}$	$\oint_L \mathbf{E}_\mathrm{B}\cdot \mathrm{d}\boldsymbol{l} = -\dfrac{\mathrm{d}\Phi_\mathrm{B}}{\mathrm{d}\,t}$
	$\mathrm{rot}\,\mathbf{E}_\mathrm{Q} = 0$	$\oint_L \mathbf{E}_\mathrm{Q}\cdot \mathrm{d}\boldsymbol{l} = 0$
Complete form	$\mathrm{rot}\,\mathbf{E} = -\dfrac{\mathrm{d}\,\mathbf{B}}{\mathrm{d}\,t}$	$\oint_L \mathbf{E}\cdot \mathrm{d}\boldsymbol{l} = -\dfrac{\mathrm{d}\,\Phi_\mathrm{B}}{\mathrm{d}\,t}$
Second Partial forms	$\mathrm{rot}\,\mathbf{B}_\mathrm{Q} = \mu_0\,\boldsymbol{j}_\mathrm{r} + \dfrac{\mathrm{d}\,\mathbf{E}_\mathrm{Q}}{c^2\,\mathrm{d}\,t}$	$\oint_L \mathbf{B}_\mathrm{Q}\cdot \mathrm{d}\boldsymbol{l} = \mu_0\,i + \dfrac{\mathrm{d}\,\Phi_\mathrm{EQ}}{c^2\,\mathrm{d}\,t}$
	$\mathrm{rot}\,\vec{\mathbf{B}}_\mathrm{B} = \dfrac{\mathrm{d}\,\vec{\mathbf{E}}_\mathrm{B}}{\mathrm{c}^2\,\mathrm{d}\,t}$	$\oint_L \mathbf{B}_\mathrm{B}\cdot \mathrm{d}\boldsymbol{l} = \mu_0\,i + \dfrac{\mathrm{d}\,\Phi_\mathrm{EB}}{c^2\,\mathrm{d}\,t}$
Complete form	$\mathrm{rot}\,\mathbf{B} = \mu_0\,\boldsymbol{j}_\mathrm{r} + \dfrac{\mathrm{d}\,\mathbf{E}}{c^2\,\mathrm{d}\,t}$	$\oint_L \mathbf{B}\cdot \mathrm{d}\boldsymbol{l} = \mu_0\,i + \dfrac{\mathrm{d}\,\Phi_\mathrm{E}}{c^2\,\mathrm{d}\,t}$
Third Partial forms	$\mathrm{div}\,\mathbf{E}_\mathrm{Q} = \dfrac{\rho}{\varepsilon_0}$	$\oint_S \mathbf{E}_\mathrm{Q}\cdot \mathrm{d}\,\mathbf{S} = \dfrac{Q\vert_\mathrm{S}}{\varepsilon_0}$
	$\mathrm{div}\,\mathbf{E}_\mathrm{B} = 0$	$\oint_S \mathbf{E}_\mathrm{B}\cdot \mathrm{d}\mathbf{S} = 0$
Complete form	$\mathrm{div}\,\mathbf{E} = \dfrac{\rho}{\varepsilon_0}$	$\oint_S \mathbf{E}\cdot \mathrm{d}\,\mathbf{S} = \dfrac{Q\vert_\mathrm{S}}{\varepsilon_0}$
Forth Partial forms	$\mathrm{div}\,\mathbf{B}_\mathrm{Q} = 0$	$\oint_S \mathbf{B}_\mathrm{Q}\cdot \mathrm{d}\,\mathbf{S} = 0$
	$\mathrm{div}\,\mathbf{B}_\mathrm{B} = 0$	$\oint_S \mathbf{B}_\mathrm{B}\cdot \mathrm{d}\mathbf{S} = 0$
Complete form	$\mathrm{div}\,\mathbf{B} = 0$	$\oint_S \mathbf{B}\cdot \mathrm{d}\,\mathbf{S} = 0$

Table 1 The complete system of Maxwell-Hertz's equations.

In Table 1a a systematical presentation of Maxwell-Hertz's equations is given . As it may be seen, every equation has three forms: two partial (one static and one induced) and the third (named: complete), which is simply the sum of partial ones. Electric and magnetic fields in these equations have equivalent role, allowing thus inclusion of magnetic monopoles, i.e. "they allow nonzero divergence of **B**" [8]).

3.4. Unique equation of electromagnetic field

Starting with Maxwell-Hertz's equations, one may obtain a unique equation of electromagnetic field, in a relatively simple way.

We will start with integral form of the first Hertz's equation:

(3-41) $$\oint_L \mathbf{E}\cdot \mathrm{d}\boldsymbol{l} = -\frac{\mathrm{d}\Phi_B}{\mathrm{d}t},$$

that can be written as:

$$\oint_L \mathbf{E}\cdot \mathrm{d}\boldsymbol{l} = -\frac{\mathrm{d}(\int_{S_L} \mathbf{B}\cdot \mathrm{d}\mathbf{S})}{\mathrm{d}t},$$

or:

$$(\oint_L \mathbf{E}\cdot \mathrm{d}\boldsymbol{l})\mathrm{d}t + \mathrm{d}(\int_{S_L} \mathbf{B}\cdot \mathrm{d}\mathbf{S}) = 0 .$$

Integrating from t_0 to t, we get:

$$\int_{t_0}^{t}\oint_L \mathbf{E}\cdot \mathrm{d}\boldsymbol{l}\mathrm{d}t + \int_{S_L} \mathbf{B}\cdot \mathrm{d}\mathbf{S} - \int_{S_{L0}} \mathbf{B}\cdot \mathrm{d}\mathbf{S} = 0 ,$$

and normalizing the time t by tc and magnetic field by $\mathbf{B}c$, this equation becomes:

(3-50) $$\int_{t_0}^{t}\oint_L \mathbf{E}\cdot \mathrm{d}\boldsymbol{l}\mathrm{d}(ct) + \int_{S_L} \mathbf{B}c\cdot \mathrm{d}\mathbf{S} - \int_{S_{L0}} \mathbf{B}c\cdot \mathrm{d}\mathbf{S} = 0 .$$

We may notice that the region of integration of the above expression is a two dimensional tube in four-dimensional space-time. The bases of this tube are the surfaces S_{L0} and S_L, while its envelope is the surface bordered by the lines L_0 and L spreading from S_{L0} to S_L in time from t_0 to t. This closed surface is the border of the corresponding three-dimensional surface.

***

Now, we will treat the integral form of the second Maxwell-Hertz's equation:

(3-38) $$\oint_L \mathbf{B}\cdot \mathrm{d}\boldsymbol{l} = \mu_0 i + \frac{\mathrm{d}\Phi_E}{c^2\mathrm{d}t},$$

which can be rewritten in the form:

$$\oint_L \mathbf{B}\cdot \mathrm{d}\boldsymbol{l} = \mu_0\frac{\mathrm{d}Q}{\mathrm{d}t} + \frac{\mathrm{d}(\int_{S_L} \mathbf{E}\cdot \mathrm{d}\mathbf{S})}{c^2\mathrm{d}t} .$$

Dealing in similar manner as in the previous case, we get:

(3-51) $$\int_{t_0}^{t}\oint_L \mathbf{B}c\cdot \mathrm{d}\boldsymbol{l}\mathrm{d}(ct) - \int_{S_L} \mathbf{E}\cdot \mathrm{d}\mathbf{S} + \int_{S_{L0}} \mathbf{E}\cdot \mathrm{d}\mathbf{S} = \frac{Q-Q_0}{\varepsilon_0} .$$

The border of integration on the left side of the equation is, also, a closed two-dimensional surface in four-dimensional space-time enveloping corresponding three-dimensional surface. *Q* is the charge that left this three-dimensional surface (at the moment t), while Q_0 is the charge that entered it (at the moment t_0). It means that $Q - Q_0$ represents the total charge inside the whole two-dimensional surface, and we will denote it by Q_S:

$Q_S = Q_0 - Q$.

By this way, the above equation becomes:

(3-52) $$\int_{S_L} \mathbf{E}\cdot d\mathbf{S} - \int_{S_{L0}} \mathbf{E}\cdot d\mathbf{S} - \int_{t_0}^{t}\oint_{L} \mathbf{B}c\cdot d\boldsymbol{l}\, d(ct) = \frac{Q|_S}{\varepsilon_0} .$$

***

In third and fourth Hertz's equation:

(3-53) $$\oint_S \mathbf{E}\cdot d\mathbf{S} = \frac{Q|_S}{\varepsilon_0}$$, and

(3-54) $$\oint_S \mathbf{B}c\cdot d\mathbf{S} = 0 ,$$

$Q|_S$ denotes total charge inside closed (two dimensional) surface S.

All these equations may be unified in a single equation introducing some new complex quantities. We will firstly denote by σ a closed two dimensional surface in four-dimensional space-time:

(3-55) $$\sigma = \begin{cases} \text{either } S, \\ \text{or} \\ S_{LO} \cup S_L \cup j\cdot L \times [t_0,\ t] \end{cases}$$

where S denotes a closed (two dimensional) surface in space, while the union below denotes a closed space-time two dimensional surface. Six component vector element of such surface is given by the expression:

(3-56) $$d\vec{\boldsymbol{\sigma}} = d\mathbf{S} + j\, d\boldsymbol{l}\, d(ct) ,$$

where j denotes imaginary unit ($j = \sqrt{-1}$).

Six-component vector of electromagnetic field $\vec{\boldsymbol{\varepsilon}}$ is given by the expression:

(3-57) $$\vec{\boldsymbol{\varepsilon}} = \mathbf{E} + j\mathbf{B}c .$$

By this way, all four Maxwell-Hertz's equations may be expressed by a general equation of electromagnetic field:

(3-58) $$\oint_S \vec{\boldsymbol{\varepsilon}}\cdot d\vec{\boldsymbol{\sigma}} = \frac{Q|_\sigma}{\varepsilon_0} ,$$

where Q_σ denotes total charge inside the closed surface σ in space- time. That charge is real quantity, but in case of existence of magnetic monopoles, it may be even complex.

3.5. Some additional remarks

3.5.1. Discrepancy with ordinary Maxwell’s equations

It is obvious that ordinary Maxwell’s equations, which may be found in textbooks of electromagnetism, are in discrepancy with equations obtained here. We will prove that one of these theories is incorrect.

First and second ordinary Maxwell’s equations in vacuum have the following form:

(3-59) $\mathrm{rot}\,\mathbf{E}' = -\dfrac{\partial \mathbf{B}'}{\partial t}$, and

(3-60) $\mathrm{rot}\,\mathbf{B}' = \mu_0 \boldsymbol{j} + \dfrac{\partial \mathbf{E}'}{c^2 \partial t}$.

while these equations obtained here have the forms (3-24) and (3-17):

(3-61) $\mathrm{rot}\,\mathbf{E} = -\dfrac{\mathrm{d}\,\mathbf{B}}{\mathrm{d}\,t}\left(= -\dfrac{\partial \mathbf{B}}{\partial t} + \mathrm{rot}(\mathbf{v}_\mathrm{d} \times \mathbf{B})\right)$,

(3-62) $\mathrm{rot}\,\mathbf{B} = \mu_0 \boldsymbol{j}_\mathrm{r} + \dfrac{\mathrm{d}\,\mathbf{E}}{c^2 \mathrm{d}\,t}\left(= \mu_0 \boldsymbol{j} + \dfrac{\partial \mathbf{E}}{c^2 \partial t} - \mathrm{rot}\,\dfrac{\mathbf{v}_\mathrm{d} \times \mathbf{E}}{c^2}\right)$

Supposing that the corresponding fields on their right sides are the same (e.g. in case of moving bar magnet as a source of magnet field and a moving charged object as a source of electric field), we get:

(3-63) $\mathbf{E} = \mathbf{E}' + \mathbf{v}_\mathrm{d} \times \mathbf{B}$, and

(3-64) $\mathbf{B} = \mathbf{B}' - \dfrac{\mathbf{v}_\mathrm{d}}{c^2} \times \mathbf{E}$,

(although, it should stay rotor before these expressions, to be more precise). The above expressions just show that Heaviside’s electric and magnetic field is different from corresponding Hertz’ fields.

3.5.2. A particle moving in electric field

As an example, it may be taken a magnetic particle moving by constant velocity $\mathbf{v}_\mathrm{d}$, inside a static condenser with constant electrostatic field $\mathbf{E}_\mathrm{Q}$. In such case, Maxwell's equations do not give an answer, being that they are differential equations. On the other hand, generalized Lorentz equations can be useful, namely, according to (2-31 and 30):

(3-65) $\mathbf{B} = -\dfrac{\mathbf{v}_\mathrm{p}}{c^2} \times \mathbf{E}_\mathrm{Q}$.

The particle will detect the same field in case of static particle inside a condenser moving by velocity $\mathbf{v}_\mathrm{E} = -\mathbf{v}_\mathrm{p}$.

However, applying Lorentz transformation to ordinary Maxwell’s equations we get a strange result. Let **B'** denotes magnetic field, which the moving particle sense inside static condenser in the reference system bind to condenser, while, **B”** denotes the same field from reference system tied to the particle. According to Lorentz transformations for electromagnetic field we get:

(3-66) $$\mathbf{B}'' = \mathbf{B}^{*} - \frac{\beta}{c^2}\mathbf{v}_{\mathrm{p}}\times\mathbf{E}_{\mathrm{Q}} \ ,$$

where:

$$\mathbf{B}^{*} = \beta\,\mathbf{B}' - (\beta-1)\,\frac{\mathbf{v}_{\mathrm{p}}\cdot\mathbf{B}'}{2}\,\mathbf{v}_{\mathrm{p}} \ ,$$

and

$$\beta = \frac{1}{\sqrt{1-\mathrm{v}_{\mathrm{p}}^{2}/c^{2}}} \ .$$

If we take that the velocity of particle is too small comparing to the velocity of light (ie. $\beta\approx 1$), we get:

(3-67) $$\mathbf{B}'' = \mathbf{B}' - \frac{\mathbf{v}_{\mathrm{p}}}{c^2}\times\mathbf{E}_{\mathrm{Q}} \ .$$

Does that mean that if we put a magnetic needle in place of particle, the needle would turn depending on reference system? That was what I thought until I read a paper [8] of Mr. Thomas Phipps. It means that in relativistic electrodynamics the detector is tied to reference system, while in Hertzian electrodynamics the detector is tied to moving particle. In other words, if you wish to know what intensity of field does the particle feel, in relativistic electrodynamics, you must tie the reference system to the particle.

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