1. Electrostatics

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Electrostatics is branch of electromagnetism that deals with electric phenomena not associated with electric charge in motion.

Fundamental terms of electrostatics are electric charge, Coulomb's force, electrostatic force, electrostatic field and potential, as well as the corresponding laws and theorems. All these terms follow in a natural way from Coulomb's experiments with charged particles.

1.1. Electric charge

Researchers in past centuries showed that electric phenomena originate from existence of electric charges (or simply “charges”). It was shown, too, that there are two types of charges, which are named simply: positive and negative. In the beginning of the twentieth century it was noticed that there are two subatomic elementary charged particles, named electron and proton (as well as one uncharged, named neutron, which is of no interest in electromagnetism). Electron (e) and proton (p) have the same quantity of electric charge, proton being positive and electron negative. In other words, every electric charge Q in nature originates from imbalance of number of protons and electrons:

(1-1) $$Q = \mathrm{N_p} \cdot p + \mathrm{N_e} \cdot e \; .$$

However, due to the fact that protons are almost 2000 times heavier than electrons, it follows that most of electric phenomena originate from imbalance of number of electrons, which are easier to move.

If charges are distributed in small particles, they are named point charges and usually denoted by letter q or Q. If charges are distributed continuously, usually it is defined (volume) charge density by the expression:

(1-2) $$\rho(\mathbf{r}) = \frac{\mathrm{d}Q}{\mathrm{dV}} \; .$$

If the charge moves through space and time by the velocity $\mathbf{v}_\mathrm{Q}$, it is defined current density by the expression:

(1-3) $$\boldsymbol{j}(\mathbf{r}) = \rho(\mathbf{r})\,\mathbf{v}_\mathrm{Q}(\mathbf{r}) \; .$$

Relative current density for a detector that moves by the velocity $\mathbf{v}$, is given by the expression:

(1-4) $$\boldsymbol{j}(\mathbf{r}) = \rho(\mathbf{r})(\mathbf{v}_\mathrm{Q}(\mathbf{r}) - \mathbf{v}(\mathbf{r})) \; .$$

Electric current i through surface S is then:

(1-5) $$i = \int_\mathrm{S} \boldsymbol{j}(\mathbf{r}) \cdot \mathrm{d}\mathbf{S} = \int_\mathrm{S} \rho(\mathbf{r})(\mathbf{v}_\mathrm{Q}(\mathbf{r}) - \mathbf{v}(\mathbf{r})) \cdot \mathrm{d}\mathbf{S} \; ,$$

where **v**(**r**) in the right expression denotes the possible velocity of the surface element ΔS(**r**, t). If the current density ***j*** is perpendicular to the surface S, having the same value on it, the current is:

(1-6) $$i = \boldsymbol{j}\cdot\mathbf{S} = \rho\,\mathrm{v}\cdot\mathrm{S} = \frac{\Delta Q}{\Delta \mathrm{V}}\,\frac{\Delta l}{\Delta \mathrm{t}}\,\mathrm{S} = \frac{\Delta Q}{\Delta \mathrm{t}}\ .$$

1.2. Electrostatic force

Encouraged by the Newton's low of gravitation, French military officer Charles-Augustin de Coulomb performed some experiments with charged pellets in second half of the XVIII century and concluded that if position of a small particle with a quantity of charge *q* is defined by the vector $\mathbf{r}_q$ and the position of another particle with a quantity of charge *Q* is defined by the vector $\mathbf{r}_Q$, than on the particle *q* acts force given by the formula:

(1-7) $$\mathbf{F} = \frac{1}{4\pi\varepsilon_0}\cdot\frac{Q\cdot q}{\left|\mathbf{r}_q - \mathbf{r}_Q\right|^3}\cdot(\mathbf{r}_q - \mathbf{r}_Q)\ ,$$

where ε_0 = 8.854 10^{-12} C^2/Nm^2 is dielectric constant of vacuum.

This formula and the corresponding force are usually named Coulomb's law and Coulomb's force respectively.

Furthermore, if there are many charged particles Q_1, Q_2, Q_3, … in the vicinity of the particle *q*, than the force which acts on that particle is given by the sum of particular Coulomb's forces:

(1-8) $$\mathbf{F} = \sum_k \mathbf{F}_k = \frac{q}{4\pi\varepsilon_0}\sum_k \frac{Q_k}{\left|\mathbf{r}_q - \mathbf{r}_{Qk}\right|^3}\cdot(\mathbf{r}_q - \mathbf{r}_{Qk})\ .$$

That force is usually named electrostatic force. In case of continuous distribution of electric charge, characterized by density:

(1-2) $$\rho(\mathbf{r}) = \frac{\mathrm{d}Q}{\mathrm{dV}}\ ,$$

electrostatic force on *q* that originates from the volume V_Q is given by:

(1-9) $$\mathbf{F}(\mathbf{r}_q) = \frac{q(\mathbf{r}_q)}{4\pi\varepsilon_0}\sum_k \frac{\Delta Q_k}{\Delta \mathrm{V}_k}\cdot\frac{(\mathbf{r}_q - \mathbf{r}_{Qk})}{\left|\mathbf{r}_q - \mathbf{r}_{Qk}\right|^3}\cdot\Delta \mathrm{V}_k = \frac{q(\mathbf{r}_q)}{4\pi\varepsilon_0}\int_{\mathrm{VQ}} \rho\,\frac{(\mathbf{r}_q - \mathbf{r}_Q)}{\left|\mathbf{r}_q - \mathbf{r}_Q\right|^3}\,\mathrm{dV}(\mathbf{r}_Q).$$

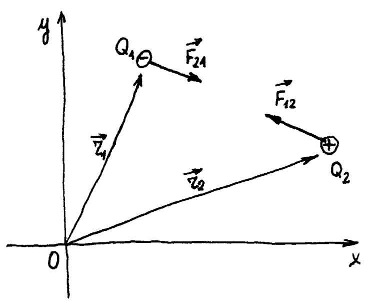

Figure I-1. Vector representation of Coulomb force in Descartes' coordinate system.

1.3. Electrostatic field

Dividing the expression (1-7) by q, we get the expression:

(1-10) $$\mathbf{E}_Q(\mathbf{r}) = \frac{\mathbf{F}}{q} = \frac{1}{4\pi\varepsilon_0}\sum_k Q_k \frac{(\mathbf{r}-\mathbf{r}_{Qk})}{|\mathbf{r}-\mathbf{r}_{Qk}|^3},$$

which is independent from the charge q and which characterizes charges Q_k (k=1, 2, 3, …), at every point in the space. This quantity is named electrostatic field. The lines, who electrostatic field is tangent in every point to, are named electrostatic lines (of forces). Analyzing the above expressions it may be concluded that that electrostatic field and its lines of forces have direction from positive to negative charge.

In case of continuous distribution of electric charge, the electrostatic field that originates from the volume V_Q is given by:

(1-11) $$\mathbf{E}(\mathbf{r}) = \frac{\mathbf{F}}{q} == \frac{1}{4\pi\varepsilon_0}\int_{VQ} \rho \frac{(\mathbf{r}-\mathbf{r}_Q)}{|\mathbf{r}-\mathbf{r}_Q|^3} dV(\mathbf{r}_Q).$$

For the reason of inverse square law of the Coulomb force (expr. 1-7) and the corresponding electrostatic field (1-10), it is supposed that every charged particle emits tiny particles, which are named virtual photons.

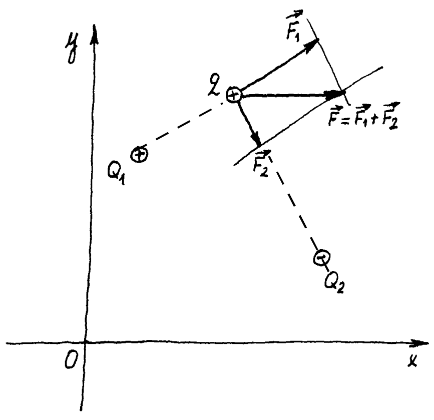

Figure I-2. A pictorial representation of vector sum of forces.

1.4. Electrostatic potential

From the differential vector identity:

(1-12) $$\frac{\partial\left(\frac{1}{\mathbf{r}}\right)}{\partial \mathbf{r}} = -\frac{\mathbf{r}}{|\mathbf{r}|^3} ,$$

where:

(1-13) $$\frac{\partial}{\partial \mathbf{r}} = \mathrm{grad} = \nabla = \left(\frac{\partial}{\partial \mathrm{x}}, \frac{\partial}{\partial \mathrm{y}}, \frac{\partial}{\partial \mathrm{z}}\right) ,$$

it follows that the electrostatic field $\mathbf{E}_Q(\mathbf{r})$ can be expressed through the expression:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{VQ} \rho(\mathbf{r}_Q)\frac{\partial\left(\frac{1}{\mathbf{r}-\mathbf{r}_Q}\right)}{\partial \mathbf{r}}\mathrm{dV}(\mathbf{r}_Q) = \frac{\partial\left(\frac{1}{4\pi\varepsilon_0}\int_{VQ}\frac{\rho(\mathbf{r}_Q)}{\mathbf{r}-\mathbf{r}_Q}\mathrm{dV}(\mathbf{r}_Q)\right)}{\partial \mathbf{r}} .$$

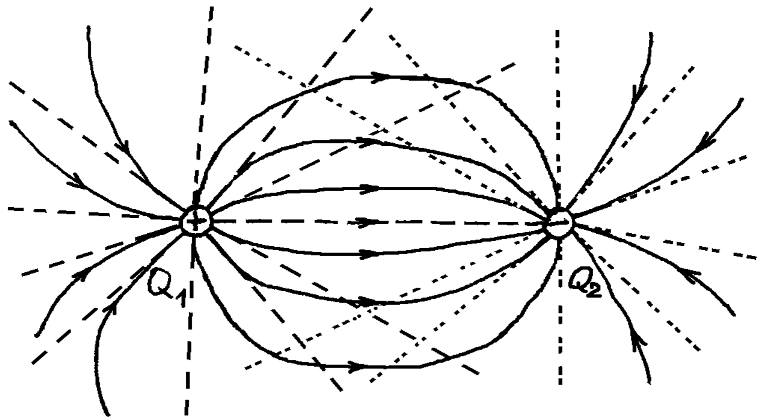

Figure I-3. Lines of (hypothetic) virtual photons moving (drawn by dashed lines) relates to the lines of fields of particular charges, as difference from the lines of electrostatic field (drawn by full lines), which are obtained as their resultant.

So, we can introduce a new physical quantity:

(1-14) $$\varphi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_{VQ}\frac{\rho(\mathbf{r}_Q)}{\mathbf{r}-\mathbf{r}_Q}\mathrm{dV}(\mathbf{r}_Q) ,$$

which is named *electrostatic potential*. Finally we get the relationship:

(1-15) $$\mathbf{E}_Q(\mathbf{r}) = -\frac{\partial\varphi(\mathbf{r})}{\partial \mathbf{r}} .$$

1.5. Fundamental laws of electrostatics

1.5.1. Circulation of the electrostatic field vector

From the vector field identity:

(1-16) $$\nabla \times (\nabla \varphi) = \operatorname{rot}(\operatorname{grad} \varphi) = 0 ,$$

it follows that:

(1-17) $$\operatorname{rot} \mathbf{E} = 0 .$$

Integrating this expression over the surface S_L bordered by the line L, we have:

(1-18) $$\int_{S_L} \mathbf{E}_Q \cdot \mathrm{d}\mathbf{S} = \oint_L \operatorname{rot} \mathbf{E}_Q \cdot \mathrm{d}\boldsymbol{l} = 0 ,$$

where we used the Stokes' theorem:

(1-18) $$\int_{S_L} \mathbf{a} \cdot \mathrm{d}\mathbf{S} = \oint_L \operatorname{rot} \mathbf{a} \cdot \mathrm{d}\boldsymbol{l} .$$

Vector fields that have the property (1-17) are said conservative fields.

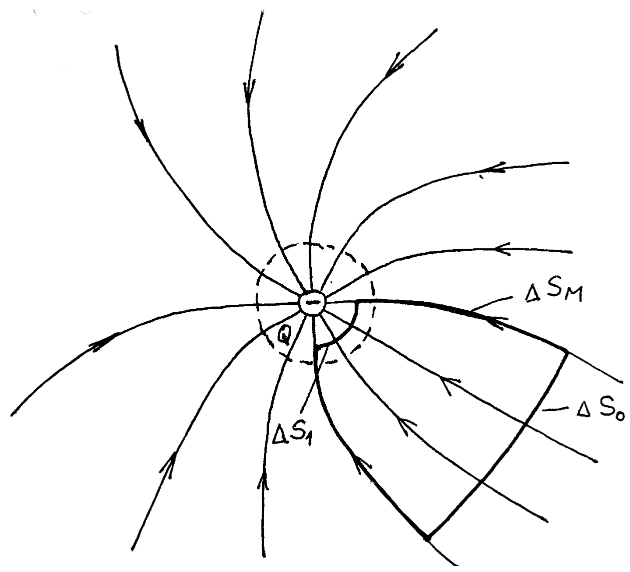

Figure 1-4 Helpful illustration for proving the proportionality between density of field lines with the density of electrostatic flux. By dashed lines is represented sphere inside of which may be considered that lines of electrostatic field $\mathbf{E}_Q$ propagate symmetrically in all directions.

1.5.2. Gauss law

In higher courses of calculus, it is derived so named Dirac-Green identity:

(1-20) $$\Delta \frac{1}{|\mathbf{r}-\mathbf{a}|} = 4\pi\,\delta(\mathbf{r}-\mathbf{a}) ,$$

where Δ denotes:

(1-21) $$\Delta = \nabla\cdot\nabla = \frac{\partial^2}{(\partial x)^2} + \frac{\partial^2}{(\partial y)^2} + \frac{\partial^2}{(\partial z)^2} ,$$

and δ denotes Dirac's delta function. From the previous identity follows that:

(1-22) $$\operatorname{div} \mathbf{E}_Q(\mathbf{r}) = -\Delta\varphi(\mathbf{r}) = \frac{\rho}{\varepsilon_0} ,$$

which is known as differential Gauss' law.

Integrating this equation through a volume V bordered by a closed surface S we get the integral form of the Gauss law:

(1-23) $$\Phi|_S = \oint_S \mathbf{E}_Q\cdot d\,\mathbf{S} = \int_V \nabla\cdot\mathbf{E}_Q\, d\,V = \frac{Q|_S}{\varepsilon_0} ,$$

where $\Phi|_S$ denotes that the flux relates to the whole surface S, while $Q|_S$ denotes the total charge inside the surface S. In this equation is assumed the mathematical Gauss' theorem:

(1-24) $$\oint_S \mathbf{a}\cdot d\,\mathbf{S} = \int_V \nabla\cdot\mathbf{a}\ dV \equiv \int_V \operatorname{div}\mathbf{a}\ dV .$$

1.6. Electrostatic units

SI unit for electric current is Amper (A):

(1-25) $[i] = \text{A}$,

wherefrom it follows the unit for current density:

(1-26) $[j] = \text{A/m}^2$.

The unit for electric charge is Coulomb (C):

(1-27) $[\mathbf{Q}] = \text{C} = \text{A·s}$,

and correspondingly for charge density:

(1-28) $[\rho] = \text{C/m}^3$.

Proton and electron have the values:

(1-29) $p = - e = 1.602176487(40) \times 10^{-19}$ C,

i.e.

(1-30) $1 \text{ C} = 6.241509647 \times 10^{+18}\ p = - 6.241509647 \times 10^{+18}\ e$.

SI unit for electrostatic potential is Volt (V):

(1-31) $[\varphi] = \text{V}$.

From the expression (1-14) it follows that the unit for electrostatic field is V/m:

(1-32) $[\mathbf{E}_Q] = \text{V/m}$.