Arc length problems

1. $y = 3x+5$ between $x=1, x=4$ $\qquad \frac{dy}{dx} = 3$

$$s = \int_1^4 \sqrt{1+3^2}\, dx = \sqrt{10}\int_1^4 dx = \sqrt{10}\,[x]_1^4 = \sqrt{10}(4-1) = 3\sqrt{10}$$

2. $x = 2y-3$ between $y=0, y=2$, $\frac{dx}{dy} = 2$

$$s = \int_0^2 \sqrt{1+2^2}\, dy = \sqrt{5}\int_0^2 dy = \sqrt{5}\,[y]_0^2 = \sqrt{5}(2-0) = 2\sqrt{5}.$$

3. $y = 2x^{3/2}$ between $x=\frac{1}{3}, x=7.$ $\qquad \frac{dy}{dx} = 3x^{\frac{1}{2}}$

$$s = \int_{\frac{1}{3}}^7 \sqrt{1+(3x^{\frac{1}{2}})^2}\, dx = \int_{\frac{1}{3}}^7 (1+9x)^{\frac{1}{2}} dx$$

$$= \left[\frac{(1+9x)^{3/2}}{\frac{3}{2}(9)}\right]_{\frac{1}{3}}^7 = \frac{2}{27}\left(8^3 - 8\right) = \frac{112}{3}.$$

4. $y = \frac{2}{3}(x^2+1)^{3/2}$ between $x=1, x=4.$

$$y' = \frac{2}{3}\cdot\frac{3}{2}(x^2+1)^{\frac{1}{2}}(2x) = 2x\,(x^2+1)^{\frac{1}{2}}$$

$$s = \int_1^4 \sqrt{1+\left(2x(x^2+1)^{\frac{1}{2}}\right)^2}\, dx = \int_1^4 \sqrt{1+4x^2(x^2+1)}\, dx$$

$$= \int_1^4 \sqrt{4x^4+4x^2+1}\, dx = \int_1^4 \sqrt{(2x^2+1)^2}\, dx$$

$$= \int_1^4 (2x^2+1)\, dx = \left[\frac{2x^3}{3}+x\right]_1^4$$

$$= \left(2\cdot\frac{64}{3}+4\right) - \left(\frac{2}{3}+1\right) = 45.$$

5. $y = (4-x^{2/3})^{3/2}$ between $x=1, x=8.$

$y' = \frac{3}{2}(4-x^{\frac{2}{3}})^{\frac{1}{2}}\left(-\frac{2}{3}x^{-\frac{1}{3}}\right) = -x^{-\frac{1}{3}}(4-x^{\frac{2}{3}})^{\frac{1}{2}} \therefore (y')^2 = x^{-\frac{2}{3}}(4-x^{\frac{2}{3}}) = 4x^{-\frac{2}{3}}-1$

$\therefore s = \int_1^8 \sqrt{1+4x^{-2/3}-1}\, dx = 2\int_1^8 x^{-\frac{1}{3}} dx = 2\cdot\frac{3}{2}\left[x^{\frac{2}{3}}\right]_1^8 = 3(4-1) = 9.$

6. $y = (x^4+3)/6x$ between $x=1, x=4.$

$y = \frac{1}{6}x^3+\frac{1}{2}x^{-1} \quad \therefore y' = \frac{1}{2}x^2-\frac{1}{2}x^{-2} \quad \therefore (y')^2 = \frac{1}{4}x^4-\frac{1}{2}+\frac{1}{4}x^{-4}$

$$s = \int_1^4 \sqrt{1+\frac{1}{4}x^4-\frac{1}{2}+\frac{1}{4}x^{-4}}\, dx = \int_1^4 \sqrt{\frac{1}{4}(x^2+x^{-2})^2}\, dx = \frac{1}{2}\int_1^4 (x^2+x^{-2})\,dx$$

$$= \frac{1}{2}\left[\frac{x^3}{3} - x^{-1}\right]_1^4 = \frac{1}{2}\left(\left(\frac{64}{3}-\frac{1}{4}\right)-\left(\frac{1}{3}-1\right)\right) = \frac{87}{8}.$$

$x = \frac{y^4}{16} + \frac{1}{2}y^{-2}$ between $y=-2, y=-1$.

7. $\frac{dx}{dy} = \frac{1}{4}y^3 - y^{-3} \quad \therefore \left(\frac{dx}{dy}\right)^2 = \frac{1}{16}y^6 - \frac{1}{2} + y^{-6}$

$\therefore s = \int_{-2}^{-1} \sqrt{1 + \frac{1}{16}y^6 - \frac{1}{2} + y^{-6}}\, dy = \int_{-2}^{-1} \sqrt{\frac{1}{16}y^6 + \frac{1}{2} + y^{-6}}\, dy$

$= \int_{-2}^{-1} \sqrt{\left(\frac{1}{4}y^3 + y^{-3}\right)^2}\, dy = \int_{-2}^{-1} -\left(\frac{1}{4}y^3 + y^{-3}\right) dy$ (NB $y^3 < 0$ for $-2 < y < -1$)

$= -\left[\frac{1}{16}y^4 - \frac{1}{2}y^{-2}\right]_{-2}^{-1} = \left(\left(\frac{1}{16} - \frac{1}{2}\right) - \left(1 - \frac{1}{8}\right)\right) = \frac{21}{16}.$

$x = \frac{y^5}{30} + \frac{1}{2}y^{-3}$ between $y=1, y=2$.

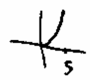
8. $\frac{dx}{dy} = \frac{1}{6}y^4 - \frac{3}{2}y^{-4} \quad \therefore \left(\frac{dx}{dy}\right)^2 = \frac{1}{36}y^8 - \frac{1}{2} + \frac{9}{4}y^{-8}$

$\therefore s = \int_1^2 \sqrt{1 + \frac{1}{36}y^8 - \frac{1}{2} + \frac{9}{4}y^{-8}}\, dy = \int_1^2 \sqrt{\frac{1}{36}y^8 + \frac{1}{2} + \frac{9}{4}y^{-8}}\, dy$

$= \int_1^2 \sqrt{\left(\frac{1}{6}y^4 + \frac{3}{2}y^{-4}\right)^2}\, dy = \int_1^2 \left(\frac{1}{6}y^4 + \frac{3}{2}y^{-4}\right) dy$

$= \left[\frac{1}{30}y^5 - \frac{1}{2}y^{-3}\right]_1^2 = \left(\frac{1}{30}(32) - \frac{1}{2}\cdot\frac{1}{8}\right) - \left(\frac{1}{30} - \frac{1}{2}\right) = \frac{353}{240}$

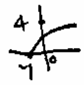
9. Find the length of the arc of the curve $y = x^{3/2}$ from $x = 0$ to $x = 5$.



Since $dy/dx = \frac{3}{2}x^{1/2}$,

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^5 \sqrt{1 + \frac{9}{4}x}\, dx = \left[\frac{8}{27}\left(1 + \frac{9}{4}x\right)^{3/2}\right]_0^5 = \frac{335}{27} \text{ units}$$

10. Find the length of the arc of the curve $x = 3y^{3/2} - 1$ from $y = 0$ to $y = 4$.



Since $dx/dy = \frac{9}{2}y^{1/2}$,

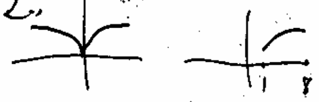
$$s = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_0^4 \sqrt{1 + \frac{81}{4}y}\, dy = \frac{8}{243}(82\sqrt{82} - 1) \text{ units}$$

11. Find the length of the arc of $24xy = x^4 + 48$ from $x = 2$ to $x = 4$.

$\frac{dy}{dx} = \frac{x^4 - 16}{8x^2}$ and $1 + \left(\frac{dy}{dx}\right)^2 = \frac{1}{64}\left(\frac{x^4 + 16}{x^2}\right)^2$. Then $s = \frac{1}{8}\int_2^4 \left(x^2 + \frac{16}{x^2}\right) dx = \frac{17}{6}$ units.

$y^3 = 8x^2 \quad , x=1, x=8 \quad$ arc length

12.



$$\left(\cancel{y = 2x^{2/3}} \quad \cancel{\frac{dy}{dx} = \frac{4}{3}x^{-1/3} \quad \therefore 1+\left(\frac{dy}{dx}\right)^2 = 1 + \frac{16}{9}x^{-2/3}} \quad \leftarrow \text{too hard} \right)$$

For $x \geq 0$ $\quad x = \sqrt{\frac{y^3}{8}} = \frac{y^{3/2}}{2\sqrt{2}} \quad \therefore \frac{dx}{dy} = \frac{3}{2} \cdot \frac{y^{\frac{1}{2}}}{2\sqrt{2}} = \frac{3\sqrt{2}}{8} y^{\frac{1}{2}} \therefore 1 + \left(\frac{dx}{dy}\right)^2 = 1 + \frac{18}{64}y$

$$= 1 + \frac{9}{32}y$$

$x = 1 \Rightarrow y = 2$

$x = 8 \Rightarrow y = 8$

$$s = \int_2^8 \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy = \int_2^8 \sqrt{1+\frac{9}{32}y}\, dy = \frac{\left[\left(1+\frac{9}{32}y\right)^{\frac{3}{2}}\right]_2^8}{\frac{9}{32}\cdot\frac{3}{2}}$$

$$= \frac{\left(1+\frac{9}{32}(8)\right)^{\frac{3}{2}} - \left(1+\frac{9}{32}(2)\right)^{\frac{3}{2}}}{27/64} = \frac{64}{27}\left(\left(\frac{13}{4}\right)^{3/2} - \left(\frac{25}{16}\right)^{3/2}\right) = \frac{64}{27}\left(\frac{13\sqrt{13}}{8} - \frac{125}{64}\right)$$

$$= \frac{64}{27}\left(104\sqrt{13} - 125\right)/64$$

$$= \frac{104\sqrt{13} - 125}{27}$$

13.

$6xy = x^4 + 3 \quad x = 1, \ x = 2.$

$y = \frac{1}{6}x^3 + \frac{1}{2}x^{-1}$

$\frac{dy}{dx} = \frac{1}{2}x^2 - \frac{1}{2}x^{-2}$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{1}{4}x^4 - \frac{1}{2} + \frac{1}{4}x^{-4} = \frac{1}{4}x^4 + \frac{1}{2} + \frac{1}{4}x^{-4}$$

$$= \frac{1}{4}(x^2 + x^{-2})^2$$

$$s = \int_1^2 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = \int_1^2 \frac{1}{2}(x^2 + x^{-2})\, dx$$

$$= \frac{1}{2}\left[\frac{x^3}{3} - x^{-1}\right]_1^2$$

$$= \frac{1}{2}\left(\left(\frac{8}{3} - \frac{1}{2}\right) - \left(\frac{1}{3} - 1\right)\right)$$

$$= \frac{17}{12}$$

Surface area of revolution exercises

$y=6x$, $0\le x\le 1$, about x-axis

1. $SA = \int_0^1 2\pi(6x)\sqrt{1+6^2}\,dx = 12\pi\sqrt{37}\int_0^1 x\,dx$
$= 12\pi\sqrt{37}\left[\frac{x^2}{2}\right]_0^1$
$= 12\pi\sqrt{37}\left(\frac{1}{2}-0\right)$
$= 6\sqrt{37}\pi$

2. $x = 6y+1$, $0\le y\le 2$ about y-axis.
$SA = \int_0^2 2\pi x\sqrt{1+6^2}\,dy = 2\sqrt{37}\pi\int_0^2 (6y+1)\,dy$
$= 2\sqrt{37}\pi\left[3y^2+y\right]_0^2$
$= 2\sqrt{37}\pi\,(14-0)$
$= 28\sqrt{37}\pi$

3. $y=\sqrt{25-x^2}$, $-2\le x\le 3$ about x-axis
$SA = \int_{-2}^{3} 2\pi\cdot\sqrt{25-x^2}\cdot\sqrt{1+\left(\frac{1}{2}(25-x^2)^{-\frac{1}{2}}(-2x)\right)^2}\,dx$
$= \int_{-2}^{3} 2\pi\sqrt{25-x^2}\sqrt{(25-x^2+x^2)/(25-x^2)}\,dx$
$= 10\pi\int_{-2}^{3} dx$
$= 10\pi\left[x\right]_{-2}^{3}$
$= 10\pi(3--2)$
$= 50\pi$.

4. $y = x^3/3$, $1\le x\le\sqrt{7}$ about x-axis
$SA = \int_1^{\sqrt{7}} 2\pi y\sqrt{1+(y')^2}\,dx$
$= \int_1^{\sqrt{7}} 2\pi\cdot\frac{x^3}{3}\cdot\sqrt{1+(x^2)^2}\,dx$
$= \frac{\pi}{6}\int_1^{\sqrt{7}} 4x^3\sqrt{1+x^4}\,dx$ Let $u=1+x^4$ $\therefore du = 4x^3dx$
$x=1 \Rightarrow u=2$
$x=\sqrt{7}\Rightarrow u=50$
$= \frac{\pi}{6}\int_2^{50} u^{\frac{1}{2}}du$
$= \frac{\pi}{6}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_2^{50}$
$= \frac{\pi}{9}\left(50\sqrt{50}-2\sqrt{2}\right)$
$= \frac{\pi}{9}\left(250\sqrt{2}-2\sqrt{2}\right)$
$= \frac{\pi}{9}\left(248\sqrt{2}\right)$

5. $x = y^3$, $0 \le y \le 1$ about y-axis

$$S.A. = \int_0^1 2\pi x \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$
$$= \int_0^1 2\pi y^3 \sqrt{1+(3y^2)^2}\, dy$$
$$= \frac{\pi}{18}\int_0^1 36 y^3 \sqrt{1+9y^4}\, dy$$

Let $u = 1+9y^4$ $\therefore du = 36y^3 dy$
$y = 0 \Rightarrow u = 1$
$y = 1 \Rightarrow u = 10$

$$= \frac{\pi}{18}\int_1^{10} u^{\frac{1}{2}}\, du$$
$$= \frac{\pi}{18}\cdot\frac{2}{3}\left[u^{\frac{3}{2}}\right]_1^{10}$$
$$= \frac{\pi}{27}\left(10\sqrt{10} - 1\right)$$

6. $y = x^2$, $0 \le x \le 2\sqrt{3}$ about y-axis

(i.e. $x = y^{\frac{1}{2}}$)
$(2\sqrt{3})^2 = 12$
$0^2 = 0$.

$$S.A. = \int_0^{12} 2\pi x \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$
$$= \int_0^{12} 2\pi y^{\frac{1}{2}} \sqrt{1+\left(\tfrac{1}{2}y^{-\frac{1}{2}}\right)^2}\, dy$$
$$= \int_0^{12} 2\pi y^{\frac{1}{2}} \sqrt{1+\frac{1}{4y}}\, dy$$
$$= \int_0^{12} \pi \sqrt{4y+1}\, dy$$
$$= \int_0^{12} \pi (4y+1)^{\frac{1}{2}} dy$$
$$= \pi\left[\frac{(4y+1)^{\frac{3}{2}}}{\frac{3}{2}(4)}\right]_0^{12}$$
$$= \frac{\pi}{6}(343 - 1) = 57\pi$$

7. $y = \frac{1}{2}x^2 - 1$, $0 \le x \le 2\sqrt{2}$ about y-axis.
$\therefore x = \sqrt{2y+2}$

$\frac{1}{2}(0)^2 - 1 = -1$, $\frac{1}{2}(2\sqrt{2})^2 - 1 = 3$

$$S.A. = \int_{-1}^{3} 2\pi x \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$
$$= \int_{-1}^{3} 2\pi \sqrt{2y+2}\sqrt{1+\left(\tfrac{1}{2}(2y+2)^{-\frac{1}{2}}(2)\right)^2}\, dy$$
$$= \int_{-1}^{3} 2\pi \sqrt{2y+2}\sqrt{1+\frac{1}{2y+2}}\, dy$$
$$= \int_{-1}^{3} 2\pi (2y+3)^{\frac{1}{2}} dy$$
$$= 2\pi\left[\frac{(2y+3)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)(2)}\right]_{-1}^{3}$$
$$= \frac{2\pi}{3}(27 - 1) = 52\pi/3.$$

8. $y = \frac{x^6+2}{8x^2}$, $1 \le x \le 3$ about x-axis $\qquad y = \frac{1}{8}x^4 + \frac{1}{4}x^{-2}$

$\qquad y' = \frac{1}{2}x^3 - \frac{1}{2}x^{-3}$

$$SA = \int_1^3 2\pi y \sqrt{1+(y')^2}\, dx$$

$$= \int_1^3 2\pi \cdot \frac{x^6+2}{8x^2} \sqrt{1+\left(\frac{1}{2}x^3 - \frac{1}{2}x^{-3}\right)^2}\, dx$$

$$= \int_1^3 \pi \frac{x^6+2}{\cancel{8}4x^2} \sqrt{1+\frac{1}{4}x^6 - \frac{1}{2} + \frac{1}{4}x^{-6}}\, dx$$

$$= \int_1^3 \pi \cdot \frac{x^6+2}{4x^2} \sqrt{(x^6+2+x^{-6})/4}\, dx$$

$$= \frac{\pi}{8} \int_1^3 \frac{x^6+2}{x^2} \sqrt{(x^3+x^{-3})^2}\, dx$$

$$= \frac{\pi}{8} \int_1^3 (x^4+2x^{-2})(x^3+x^{-3})\, dx$$

$$= \frac{\pi}{8} \int_1^3 (x^7+2x+x+2x^{-5})\, dx$$

$$= \frac{\pi}{8} \int_1^3 (x^7+3x+2x^{-5})\, dx$$

$$= \frac{\pi}{8} \left[\frac{x^8}{8} + \frac{3x^2}{2} + \frac{2x^{-4}}{-4}\right]_1^3$$

$$= \frac{\pi}{8} \left[\frac{x^8}{8} + \frac{3x^2}{2} - \frac{1}{2}x^{-4}\right]_1^3$$

$$= \frac{\pi}{8}\left(\left(\frac{3^8}{8} + \frac{3(3^2)}{2} - \frac{1}{2}(3^{-4})\right) - \left(\frac{1}{8} + \frac{3}{2} - \frac{1}{2}\right)\right)$$

$$= \frac{\pi}{8}\left(\left(\frac{6561}{8} + \frac{27}{2} - \frac{1}{162} - \frac{1}{8} - \frac{3}{2} + \frac{1}{2}\right)\right.$$

$$= \frac{8429\pi}{81}.$$

9. Find the area of the surface of revolution generated by revolving about the x axis the arc of the parabola $y^2 = 12x$ from $x = 0$ to $x = 3$. (See Fig. 48-2.)

Solution 1: Here $\frac{dy}{dx} = \frac{6}{y}$ and $1 + \left(\frac{dy}{dx}\right)^2 = \frac{y^2 + 36}{y^2}$. Then

$$S_x = 2\pi \int_0^3 y \frac{\sqrt{y^2 + 36}}{y}\, dx = 2\pi \int_0^3 \sqrt{12x + 36}\, dx = 24(2\sqrt{2} - 1)\pi \text{ square units}$$

Solution 2: $\frac{dx}{dy} = \frac{y}{6}$ and $1 + \left(\frac{dx}{dy}\right)^2 = \frac{36 + y^2}{36}$. Hence,

$$S_x = 2\pi \int_0^6 y \frac{\sqrt{36 + y^2}}{6}\, dy = \left[\frac{\pi}{9}(36 + y^2)^{3/2}\right]_0^6 = 24(2\sqrt{2} - 1)\pi \text{ square units}$$

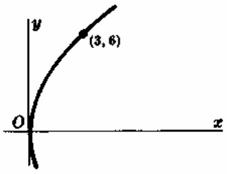


Fig. 48-2

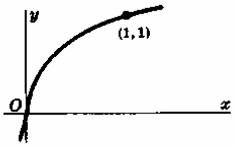


Fig. 48-3

10. Find the area of the surface of revolution generated by revolving about the y axis the arc of $x = y^3$ from $y = 0$ to $y = 1$.

From Fig. 48-3, we have

$$S_y = 2\pi \int_c^d x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = 2\pi \int_0^1 y^3\sqrt{1 + 9y^4}\, dy = \left[\frac{\pi}{27}(1 + 9y^4)^{3/2}\right]_0^1$$

$$= \frac{\pi}{27}(10\sqrt{10} - 1) \text{ square units}$$

11. Find the area of the surface of revolution generated by revolving a loop of the curve $8a^2y^2 = a^2x^2 - x^4$ about the x axis. (See Fig. 48-4.)

Here $\quad \frac{dy}{dx} = \frac{a^2x - 2x^3}{8a^2y} \quad$ and $\quad 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{(a^2 - 2x^2)^2}{8a^2(a^2 - x^2)} = \frac{(3a^2 - 2x^2)^2}{8a^2(a^2 - x^2)}$

Hence

$$S_x = 2\pi \int_a^b y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = 2\pi \int_0^a \frac{x\sqrt{a^2 - x^2}}{2a\sqrt{2}} \frac{3a^2 - 2x^2}{2a\sqrt{2}\sqrt{a^2 - x^2}}\, dx$$

$$= \frac{\pi}{4a^2} \int_0^a (3a^2 - 2x^2)x\, dx = \frac{1}{4}\pi a^2 \text{ square units}$$

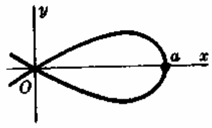
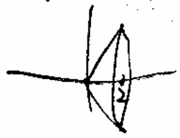


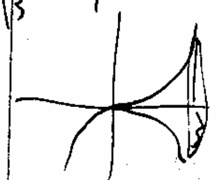
Fig. 48-4

D 12 $y = mx$, $x=0$ to $x=2$ rotated about x-axis.
$(m>0)$



$$SA = 2\pi\int_0^2 y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$
$$= 2\pi\int_0^2 mx\sqrt{1+m^2}\,dx$$
$$= 2\pi m\sqrt{1+m^2}\int_0^2 x\,dx$$
$$= 2\pi m\sqrt{1+m^2}\left[\frac{x^2}{2}\right]_0^2$$
$$= \pi m\sqrt{1+m^2}\left(2^2-0^2\right) = 4\pi\sqrt{1+m^2}.$$

13 $y = \frac{1}{3}x^3$, $x=0$ to $x=3$ about x-axis



$$SA = 2\pi\int_0^3 y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$
$$= 2\pi\int_0^3 \frac{1}{3}x^3\sqrt{1+(x^2)^2}\,dx$$
$$= \frac{2\pi}{3}\int_0^3 x^3\sqrt{1+x^4}\,dx$$

Let $u = 1+x^4$
$\therefore du = 4x^3\,dx$
When $x=0$, $u=1$
When $x=3$, $u=82$

$$= \frac{\pi}{6}\int_0^3 4x^3\sqrt{1+x^4}\,dx$$
$$= \frac{\pi}{6}\int_1^{82} u^{\frac{1}{2}}\,du$$
$$= \frac{\pi}{6}\cdot\frac{2}{3}\left[u^{\frac{3}{2}}\right]_1^{82}$$
$$= \frac{\pi}{9}\left(82\sqrt{82}-1\right).$$

14 $8y^2 = x^2(1-x^2)$, about x-axis

$2\sqrt{2}\,y = \pm x\sqrt{(1-x^2)}$

$y = \pm\frac{x\sqrt{2}}{4}\sqrt{1-x^2} \Rightarrow \frac{dy}{dx} = \frac{\sqrt{2}}{4}\sqrt{1-x^2} + \frac{x\sqrt{2}}{4}\cdot\frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x)$

$y = \frac{x\sqrt{2}}{4}\sqrt{1-x^2}$ (above x-axis) $\qquad = \frac{\sqrt{2}}{4}\,\frac{1-x^2-x^2}{\sqrt{1-x^2}}$

$$\therefore 1+\left(\frac{dy}{dx}\right)^2 = \frac{(1-2x^2)^2}{8(1-x^2)}+1 = \frac{1-4x^2+4x^4+8-8x^2}{8(1-x^2)}$$
$$= \frac{9-12x^2+4x^4}{8(1-x^2)} = (3-2x^2)^2/8(1-x^2)$$

$$\therefore S.A. = 2\pi \int_0^1 y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= 2\pi \int_0^1 \frac{x\sqrt{2}}{4}\sqrt{1-x^2} \cdot \frac{3-2x^2}{2\sqrt{2}\sqrt{1-x^2}}\,dx$$

$$= \frac{\pi}{4}\int_0^1 (3x - 2x^3)\,dx$$

$$= \frac{\pi}{4}\left[\frac{3x^2}{2} - \frac{x^4}{2}\right]_0^1$$

$$= \frac{\pi}{8}(3 - 1 - 0)$$

$$= \frac{\pi}{4}$$

15. One loop of $9y^2 = x(3-x)^2$, y-axis

$$y = \frac{1}{3}\sqrt{x}(3-x) = x^{\frac{1}{2}} - \frac{1}{3}x^{\frac{3}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{\frac{1}{2}}$$

$$\therefore 1+\left(\frac{dy}{dx}\right)^2 = \frac{1}{4}x^{-1} - \frac{1}{2} + \frac{1}{4}x + 1$$

$$= \frac{1}{4}x^{-1} + \frac{1}{2} + \frac{1}{4}x$$

$$= \frac{1}{4}(x + 2 + x^{-1}) = \frac{1}{4}(x^{\frac{1}{2}} + x^{-\frac{1}{2}})^2$$

$$\therefore S.A. = 2\pi\int_0^3 x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= 2\pi\int_0^3 x \cdot \frac{1}{2}(x^{\frac{1}{2}} + x^{-\frac{1}{2}})\,dx$$

$$= \pi\int_0^3 (x^{\frac{3}{2}} + x^{\frac{1}{2}})\,dx$$

$$= \pi\left[\frac{2}{5}x^{\frac{5}{2}} + \frac{2}{3}x^{\frac{3}{2}}\right]_0^3$$

$$= \pi\left(\frac{2}{5}(9\sqrt{3}) + 2\sqrt{3} - 0\right)$$

$$= 28\pi\sqrt{3}/5.$$

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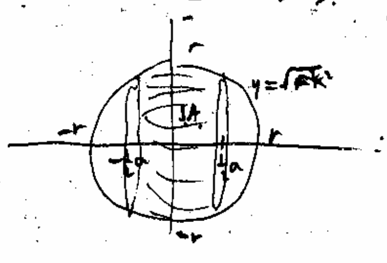

$$\frac{dy}{dx} = \frac{1}{2}(r^2-x^2)^{-\frac{1}{2}}(-2x) = \frac{-x}{\sqrt{r^2-x^2}}$$

$$1+\left(\frac{dy}{dx}\right)^2 = 1+\left(\frac{-x}{\sqrt{r^2-x^2}}\right)^2 = 1+\frac{x^2}{r^2-x^2} = \frac{r^2}{r^2-x^2}$$

$$\therefore S.A. = 2\int_0^{\frac{1}{2}a} 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \quad \text{by symmetry}$$

$$= 4\pi\int_0^{\frac{1}{2}a}\sqrt{r^2-x^2}\cdot\frac{r}{\sqrt{r^2-x^2}}\,dx$$

$$= 4\pi r\int_0^{\frac{1}{2}a} dx = 4\pi r[x]_0^{\frac{1}{2}a} = 4\pi r\left(\frac{1}{2}a - 0\right)$$

$$= 2\pi r a$$