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A NUMERICAL SIMULATION MODEL FOR ONE-DIMENSIONAL INFILTRATION

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ABSTRACT

The theory for transient isothermal flow of water into nonswelling unsaturated soil has been developed to a large extent in terms of solutions of the non-linear Richards equation. In the field, the description of infiltration is highly complicated since the initial and boundary conditions are usually not constant while the soil characteristics may vary with time and space. In this study, a model has been formulated for finite difference solution of the non-linear Richards equation applicable to transient, one-dimensional water flow through the unsaturated porous medium. The simulated soil moisture profiles for explicit, Crank-Nicolson and implicit schemes have been compared with the quasi-analytical solution of Philip.

KEY WORDS : Flow through unsaturated porous media, Richards equation, Numerical simulation, Infiltration, One-dimensional model.

INTRODUCTION

Most of the processes involving soil-water interactions in the field, and particularly the flow of water in the rooting zone of most crop plants, occur while the soil is in an unsaturated condition. Unsaturated flow processes are in general complicated and difficult to describe quantitatively, since they often entail changes in the state and content of soil water during flow. Such changes involve complex relations among the variable soil wetness, suction, and conductivity, whose inter-relations may be further complicated by hysteresis. The formulation and solution of unsaturated flow problems very often require the use of indirect methods of analysis, based on approximations or numerical techniques. For this reason, the development of rigorous theoretical and experimental methods for treating these problems was rather late in coming. In recent decades, however, unsaturated flow has become one of the most important and active topics of research and this research has resulted in significant theoretical and practical advances.

Richards (1931) presented the differential equation for soil water flow using an analogy to heat flow in porous media. Up to now this equaiton is used as the basic mathematical expression that underlies unsaturated flow phenomenon. Soil water flow, however, is highly non-linear, as both the hydraulic conductivity and the soil water pressure head depend on the

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soil water content. Exact analytical solutions are only possible for simplified flow cases under a number of restrictive assumptions. Numerical solution of the flow equation on the other had offers a powerful tool in approximating the real nature of the unsaturated zone for a wide variety of soil systems and external conditions.

The objective of the present study is to develop a numerical model (finite difference scheme) for solving the non-linear partial differential equation (Richards equation) describing one-dimensional water flow through the unsaturated porous medium by using explicit, Crank-Nicolson and implicit methods for discretization. The simulated soil moisture profiles at various times in a sandy soil have been compared with the soil moisture profiles obtained through quasi-analytical solution of Philip. Philip's quasi-analytical solution was obtained by solving Richards equation subject to the condition of a constant pressure at the soil surface (Haverkamp et al. 1977).

METHODOLOGY

General Equation of Unsaturated Flow

According to Darcy's law, for one-dimensional vertical flow, the volumetric flux q ($cm^3/cm^2/h$) can be written as

$$q = -K\frac{\partial}{\partial z}(h - z) \quad (cm/h)$$

or

$$q = -K(\frac{\partial h}{\partial z} - 1) \quad (cm/h) \tag{1}$$

where K is the hydraulic conductivity (cm/h), h is the soil water pressure head (relative to the atmosphere) expressed in cm of water and z is the gravitational head (cm) considered positive in downward direction.

In order to get a complete mathematical description for unsaturated flow, we apply the continuity principle (Law of Conservation of Mass)

$$\frac{\partial \theta}{\partial t} = -\frac{\partial q}{\partial z} \quad (1/h) \tag{2}$$

where θ is soil moisture expressed in cm^3/cm^3 and t is time in hours.

Substitution of Eq. (1) into Eq. (2) yields the partial differential equation

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z}[K(\frac{\partial h}{\partial z} - 1)] \tag{3}$$

Equation (3) is a second order, parabolic type of partial differential equation (known as Richards equation) which is non-linear because of the dependency of K and h on θ. To avoid the problem of the two dependent variables θ and h, the derivative of θ with respect to h can be introduced, which is known as the specific water capacity C.

$$C = \frac{d\theta}{dh} \qquad (1/\text{cm}) \tag{4}$$

In Eq. (4) a total instead of a partial derivative notation is used, because h is considered here as a single value function of θ (no hysteresis). Writing

$$\frac{\partial\theta}{\partial t} = \frac{d\theta}{dh}\,\frac{\partial h}{\partial t} \tag{5}$$

and substituting Eq. (4) into Eq. (3) yields

$$C(h)\frac{\partial h}{\partial t} = \frac{\partial}{\partial z}\left[K(h)\left(\frac{\partial h}{\partial z} - 1\right)\right] \tag{6}$$

In Eq. (6) the coefficients C and K are functions of the dependent variable h, but not functions of the derivatives $\partial h/\partial t$ and $\partial h/\partial z$. Written in this form, Eq. (6) provides the basis for predicting soil water movement in layered soils of which each layer may have different physical properties.

To obtain a solution for the one-dimensional vertical flow equation, Eq. (6) must be supplimented by appropriate initial and boundary conditions.

For the present study, initial condition has been defined as

$$\theta\,(z, t = 0) = 0.10 \tag{7}$$

and upper boundary condition as

$$\theta\,(z{=}0, t) = 0.267 \tag{8}$$

Soil Moisture Characteristics

For the present study, functional relations, as reported by Haverkamp et al. (1977) for characterizing the hydraulic properties of a soil, were used. They compared six models, employing different ways of discretization of non-linear infiltration equation in terms of execution time, accuracy, and programming considerations. The models were tested by comparing water content profiles calculated at given times by each of the model with results obtained from an infiltration experiment carried out in the laboratory. All models yielded excellent agreement with water content profiles measured at various times.

The following analytical expressions, obtained by a least square fit through all data points were chosen for characterizing the soil :

$$K = K_s \frac{A}{A+|h|^{\beta_1}} \tag{9}$$

where, $K_s = 34$ cm/h, $A = 1.175 \times 10^6$, $\beta_1 = 4.74$, and

$$\theta = \frac{\alpha(\theta_s - \theta_r)}{\alpha + |h|^{\beta_2}} + \theta_r \tag{10}$$

where $\theta_s = 0.287$, $\theta_r = 0.075$, $\alpha = 1.611 \times 10^6$, $\beta_2 = 3.96$.

Subscript s refers to saturation, i.e. the value of θ for which h = 0, and the subscript r to residual water content.

Figure 1 presents the relationships between the soil water pressure h, the water content θ and the hydraulic conductivity K for the above soil used in this study.

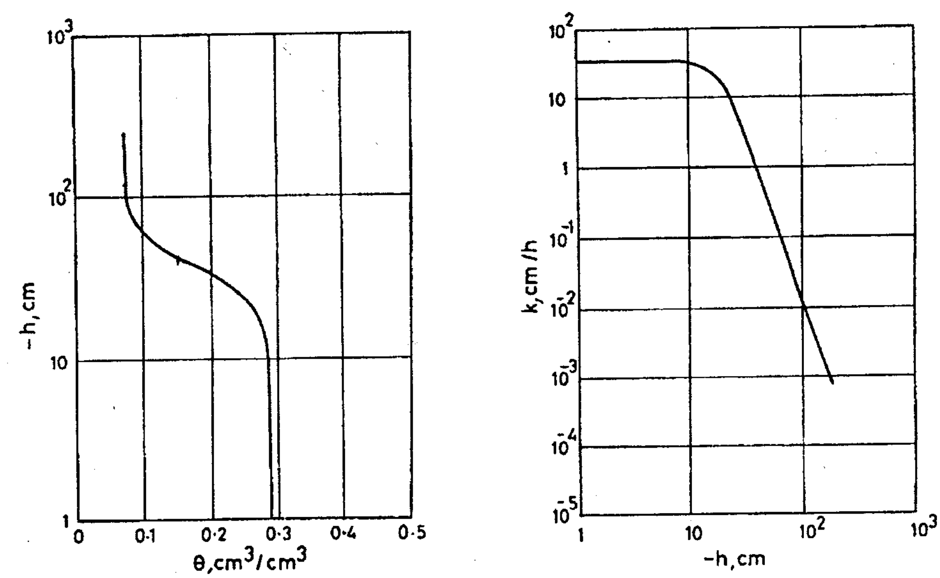

FIG.1 RELATIONSHIPS BETWEEN THE SOIL WATER PRESSURE h, THE WATER CONTENTS θ AND THE HYDRAULIC CONDUCTIVITY K FOR THE SOIL USED IN THE STUDY

Finite Difference Approximation

Let the entire flow domain be divided into a grid of equal intervals Δz and the time domain be similarly divided into intervals Δt. The partial differential Eq. (6) can be approximated by a finite difference equation replacing ∂t and ∂z by Δt and Δz respectively in the following way :

$$C_i^{j+a} \frac{h_i^{j+1} - h_i^j}{\Delta t} =$$

$$\frac{1}{\Delta z}\left[K_{i+1/2}^{j+a}\left(\frac{h_{i+1}^{j+a} - h_i^{j+a}}{\Delta z} - 1\right) - K_{i-1/2}^{j+a}\left(\frac{h_i^{j+a} - h_{i-1}^{j+a}}{\Delta z} - 1\right)\right] \quad (11)$$

where i and j are the indices of space and time respectively' 'a' is a weighting factor ($0 \leq a \leq 1$) introduced in such a manner that by putting a = 0, it is transformed into explicit scheme, a = 0.5 into Crank-Nicolson scheme, and a = 1 into implicit scheme. Therefore,

$$h_i^{j+a} = (1-a)\ h_i^j + a\ h_i^{j+1} \quad (12a)$$

$$h_{i+1}^{j+a} = (1-a)\ h_{i+1}^j + a\ h_{i+1}^{j+1} \quad (12b)$$

$$h_{i-1}^{j+a} = (1-a)\ h_{i-1}^j + a\ h_{i-1}^{j+1} \quad (12c)$$

The values of C_i^{j+a}, $K_{i+1/2}^{j+a}$ and $K_{i-1/2}^{j+a}$ can be approximated by

$$C_i^{j+a} = F_1 = (1-a)\ C_i^j + a\ C_i^{j+1} \quad (13a)$$

$$K_{i+1/2}^{j+a} = F_2 = (1-a)\ K_{i+1/2}^j + a\ K_{i+1/2}^{j+1}$$

$$= (1-a)\sqrt{(K_i^j\ K_{i+1}^j)} + a\sqrt{(K_i^{j+1}\ K_{i+1}^{j+1})} \quad (13b)$$

$$K_{i-1/2}^{j+a} = F_3 = (1-a)\ K_{i-1/2}^j + a\ K_{i-1/2}^{j+1}$$

$$= (1-a)\sqrt{K_{i-1}^j\ K_i^j}) + a\ K_{i-1/2}^j + a\sqrt{K_{i-1}^{j+1}\ K_i^{j+1}}) \quad (13c)$$

Different methods of weighting interblock hydraulic conductivity values for modelling one-dimensional water transfer in homogeneous unsaturated soil were tested by Haverkamp and Vauclin (1979) for studying their influence upon the accuracy of the finite difference solution. The only method that well simulated their experimental observations was the geometric mean. This approach has therefore been adopted in the Eqs. (13b) and (13c).

Substitution of Eqs. (12) and (13) in Eq. (11) yields the following linear algebraic equation valid for each nodal point :

$$\left[-a\,F_3\,\frac{\Delta t}{(\Delta z)^2}\right] h_{i-1}^{j+1} + \left[F_1 + a\,F_2\,\frac{\Delta t}{(\Delta z)^2} + a\,F_3\,\frac{\Delta t}{(\Delta z)^2}\right] h_i^{j+1}$$

$$-\left[a\,F_2\,\frac{\Delta t}{(\Delta z)^2}\right] h_{i+1}^{j+1} = \left[(1-a)\,F_3\,\frac{\Delta t}{(\Delta z)^2}\right] h_{i-1}^{j}$$

$$+\left[F_1 - (1-a)\,F_2\,\frac{\Delta t}{(\Delta z)^2} - (1-a)\,F_3\,\frac{\Delta t}{(\Delta z)^2}\right] h_i^{j}$$

$$+\left[(1-a)\,F_2\,\frac{\Delta t}{(\Delta z)^2}\right] h_{i+1}^{j} + (F_3 - F_2)\,\frac{\Delta t}{\Delta z} \qquad (14)$$

When Eq. (14) is applied at all nodes, the result is a system of simultaneous linear algebraic equations with a tridiagonal coefficient matrix with zero elements outside the diagonals and unknown values of h. In solving this system of equations, the so-called direct method was used by applying a tridiagonal algorithm of the kind discussed by Remson et al. (1971).

Soil Moisture Simulation

Due to the fact that the hydraulic conductivity, K(h) and specific water capacity, C(h) at the end of each time step are unknown, an iterative process was used. In the iteration method, h_i^j is replaced by h_i^j (n), where n is an iteration index. For the first iteration, h_i^j (0) is set equal to h_i^j. The resulting linear equations are then solved for h_i^{j+1}, and h_i^j (1) is obtained from this solution. The parameters C(h) and K(h) are adjusted corresponding to this estimate of h_i^j and the equations are solved again to find h_i^j (2), and the procedure is continued. The iterative procedure is generally terminated when two successive values of h_i^j are close to each other e.g.

$$\sum_{i=1}^{n\,node} \left[h_i^j(n) - h_i^j(n-1)\right]^2 \le 0.0001 \qquad (15)$$

The iteration method is time-consuming, but gives better estiamates.

A specific solution of Richards equation was obtained by Philip (1957) in the case of infiltration in a homogeneous semi-infinite column satisfying the boundary conditions :

$$\left.\begin{array}{lll} t < 0 & z \geq 0 & \theta = \theta_0 \\ t \geq 0 & z = 0 & \theta = \theta_u \end{array}\right\} \quad (16)$$

In a later paper (Philip, 1958), Richards equation was solved for the conditions :

$$\left.\begin{array}{lll} t < 0 & z \geq 0 & h = h_0 \\ t \geq 0 & z = 0 & h = h_u \end{array}\right\} \quad (17)$$

where h_u could take positive values corresponding to an infiltration experiment with submersion. Philip's method led to the solution in the form of a power series in $t^{1/2}$. Since the series converges only for finite t, the solution becomes unreliable as t tends to infinity; the t-range of convergence is dependent upon the characteristics of soil and the initial and boundary conditions.

In the present study, soil moisture profiles were simulated at various times for explicit scheme (a = 0), Crank-Nicolson scheme (a = 0.5) and implicit scheme (a = 1) and compared with the quasi-analytical solution of Philip.

RESULTS

The numerical model described above was tested by comparing the water content profiles calculated at given times with results obtained from quasi-analytical solution of Philip. Using the functional relations given in Eqs. (9) and (10) for characterizing the hydraulic properties of the soil, the water content profiles were determined subject to the following conditions :

$$\left.\begin{array}{lll} t < 0 & z \geq 0 & \theta_0 = 0.10\ cm^3/cm^3 \\ t \geq 0 & z = 0 & \theta_u = 0.267\ cm^3/cm^3 \end{array}\right\} \quad (18)$$

The numerical computations were made with a depth interval $\Delta z = 1$ cm, the total simulation period being 0.8 hour. It was found by trial and error that the numerical scheme is stable for time step $\Delta t = 0.4$ second in case of explicit schem, $\Delta t = 1$ second in case of Crank-Nicolson scheme, and $\Delta t = 5$ seconds in case of implicit scheme. Therefore, these values of time steps were used for condition of stability.

Haverkamp et al. (1977) has reported the infiltration profiles at various times for infiltration in sand (under consideration) obtained by quasi-analytical solution of Philip. Numerical data of Philip's solution are given in Table - 1.

TABLE - 1

WATER CONTENT PROFILES DETERMINED WITH THE SOLUTION OF PHILIP

Water Content (θ)	Depth (z)		
	t = 0.1 hour	t = 0.2 hour	t = 0.8 hour
0.2523	9.4	17.7	65.2
0.2356	12.0	20.7	69.2
0.2189	13.2	22.1	71.1
0.2021	14.1	23.1	72.3
0.1854	14.8	23.8	73.2
0.1686	15.3	24.5	74.0
0.1519	15.9	25.2	74.8
0.1351	16.5	25.9	75.7
0.1184	17.3	26.8	76.8
0.1016	19.5	29.5	78.6

TABLE - 2

COMPARISON BEWTEEN WATER CONTENT PROFILES AT t = 0.1 HOUR

Depth (z)	Water Content (θ)			
	Philip	Explicit Scheme	Crank-Nicolson Scheme	Implicit Scheme
10	0.2484	0.247574	0.247808	0.248945
11	0.2420	0.241063	0.241411	0.243104
12	0.2356	0.231798	0.232331	0.234931
13	0.2217	0.218237	0.219075	0.223192
14	0.2040	0.198274	0.199584	0.206158
15	0.1787	0.170837	0.172673	0.182437
16	0.1491	0.140986	0.142864	0.154198
17	0.1247	0.118997	0.120258	0.129246
18	0.1130	0.107607	0.108236	0.113437
19	0.1054	0.102811	0.103081	0.105591

TABLE - 3

COMPARISON BEWTEEN WATER CONTENT PROFILES AT t = 0.2 HOUR

Depth (z)	Water Content (θ)			
	Philip	Explicit Scheme	Crank-Nicolson Scheme	Implicit Scheme
18	0.2506	0.247249	0.247470	0.248583
19	0.2451	0.242231	0.242532	0.244049
20	0.2395	0.235506	0.235924	0.238042
21	0.2320	0.226299	0.226895	0.229922
22	0.2201	0.213518	0.214375	0.218780
23	0.2038	0.195913	0.197126	0.203499
24	0.1806	0.173123	0.174694	0.183346
25	0.1567	0.148020	0.149676	0.159615
26	0.1332	0.126794	0.128098	0.136890
27	0.1172	0.113176	0.113972	0.119994
28	0.1109	0.106002	0.106418	0.109876
29	0.1047	0.102619	0.102819	0.104615

TABLE - 4

COMPARISON BEWTEEN WATER CONTENT PROFILES AT t = 0.8 HOUR

Depth (z)	Water Content (θ)			
	Philip	Explicit Scheme	Crank-Nicolson Scheme	Implicit Scheme
66	0.2490	0.246294	0.246633	0.248422
67	0.2448	0.241825	0.242265	0.244585
68	0.2406	0.236003	0.236586	0.239649
69	0.2364	0.228291	0.229075	0.233195
70	0.2286	0.217948	0.219016	0.224639
71	0.2198	0.204080	0.205525	0.213216
72	0.2063	0.186024	0.187897	0.198131
73	0.1891	0.164453	0.166619	0.179147
74	0.1686	0.142607	0.144667	0.157722
75	0.1482	0.124907	0.126460	0.137451
76	0.1305	0.113293	0.114262	0.121852
77	0.1165	0.106719	0.107257	0.111818
78	0.1072	0.103300	0.103580	0.106105

Tables 2, 3 and 4 present the comparison between water content profiles determined with the solution of Philip and the simulated water content profiles for explicit scheme, Crank-Nicolson scheme and implicit scheme, at t = 0.1 hour, 0.2 hour and 0.8 hour respectively. In all cases, the rate of advance of the water front is particularly well described. Some discrepancies are found between numerical water content profiles and quasi-analytical solution in the low water content domain.However, all the numerical schemes yield comparable results which are not significantly different from the quasi-analytical solution.

It can be observed that the implicit scheme gives better agreement with infiltration profiles calculated with Philip's method as compared to the explicit scheme and Crank-Nicolson scheme. Considering Philip's solution as standard, the average relative error in the water content distributions for all the three time simulations was calculated as 1.91 % for implicit scheme, 4.16 % for Crank-Nicolson scheme and 4.71 % for explicit scheme.

Implicit methods are therefore preferable in view of their stability, even for fairly large steps thus keeping computer costs low, and their flexibility for solving flow problems when saturated and unsaturated zones have to be considered simultaneously, since C = 0, one simply has to solve the Laplace's equation.

CONCLUSIONS

A numerical model has been developed for finite difference solution of the non-linear Richards equation describing transient, one-dimensional water flow through the unsaturated porous medium. The solution is applicable to homogeneous and isotropic soils in which functional relationships between hydraulic conductivity, moisture content and soil moisture tension do not show hysteresis properties.

The simulated water content profiles were compared with those computed through quasi-analytical solution of Philip for the condition of a constant pressure at the soil surface. The implicit scheme was found to give better agreement between the two.

The closer agreement between water content distributions obtained with the model and Philip's quasi-analytical solution indicates that numerical model is a reliable tool for predicting infiltration of water into soil. Considering computer time and stability problems, the implicit finite approximation has the widest range of applicability for predicting water movement in soil with both saturated and non-saturated regions.

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