What is a Field?

Suppose we take the term "field" literally, and consider a soccer field filled with players. We can imagine each player being given a coordinate, and having a way of going from one player to the next. This illustrates the two types of fields possible: scalar and vector fields. The set of coordinates giving the location of the players form a ***scalar field***, and the directions pointing from one player to the next form a ***vector field***. More practical examples of scalar fields include the heat distribution over a surface and the stretching of a drumhead as it vibrates. A vector field may be describing the current flow in a lake, or the wind direction on a weather map.

What can we do with fields?

Essentially, everything we have done in mathematics until now can be done with fields. However, slight adjustments must be made due to the following conditions:

1) A field can generally be n-dimensional, not just the normal 2-dimensional geometry we're used to
2) We will need a way to work with both scalar and vector fields

These two conditions open the doors to a new type of calculus: ***vector calculus***. Basically, you can think of vector calculus as the study of how fields change. Everything you could do in "regular" calculus (differentiation, integration, etc) can be done in field theory, just with some alterations. Vector calculus can tell you how fast wind patterns are changing with time on a weather grid, or how the temperature on a plate varies with position as you investigate different places along the surface.

Divergence and Curl: Measures of Distribution and Flow

In general, there are two quantities needed to fully describe how a field is changing. The ***divergence*** $\nabla \bullet \vec{v}$ measures how the outward flow or magnitude of the field varies, while the ***curl*** $\nabla \times \vec{v}$ measures the amount of curvature or rotation in the flow of the field. Essentially, the total picture of a changing field is simply a combination of these two quantities:

$$\vec{A} = \nabla \bullet \vec{v} + \nabla \times \vec{v}$$

So, if a field has no curl, then its flow is straight, and doesn't curve or rotate. If a field has no divergence, it has no change in intensity.

The Divergence Theorem, AKA Gauss' Theorem

Suppose we wanted to measure the heat inside an extremely hot object. We could try to probe inside the object and determine the amount of heat directly, or we could see how the heat flow through the surface changed all the way around. From this observation, we could infer how the heat was being distributed inside the object. This is the basis for the Divergence Theorem, which is expressed mathematically as

$$\int_{Volume} \left(\nabla \bullet \vec{v}\right) dV = \int \left(\vec{v} \bullet \hat{n}\right) dA$$

In words, this says ***the total change in the outward flow of a field inside a closed space can be inferred by measuring the total flux of field lines through the surface***. The total flux will be zero if there are no field lines originating from within the space. This will mean there is no change in the field intensity within the object.

Another way of stating this is ***the change in the outward flow of a field is caused by a source producing the change***. Without a source present, the field will remain uniform. In differential form, this is expressed as

$$\nabla \bullet \vec{v}\left(\vec{r}\right) = k\rho\left(\vec{r}\right)$$

The Curl Theorem, AKA Stokes' Theorem

Suppose we were headed towards a whirlpool, and wanted to see how strongly the water currents were swirling (curling). We have two options:

1) Allow ourselves to be pulled into the whirlpool and measure the curl directly
2) Find some way of inferring the curl from the currents acting against us as we try to paddle along a safe path

I think everyone will agree the second option is much more practical, and hence is the reason for the Curl Theorem:

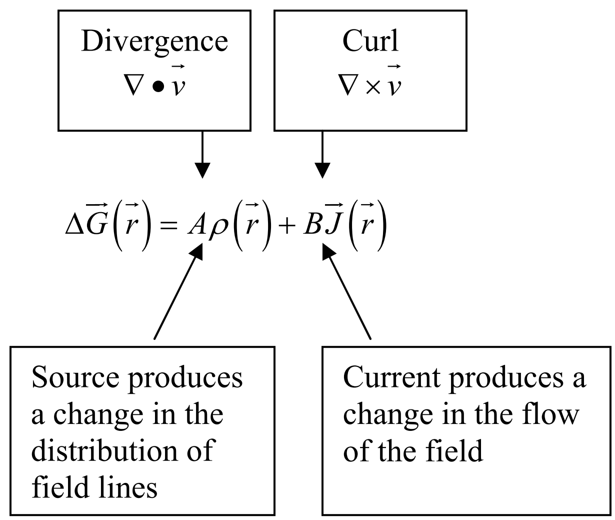
$$\int_{Surface} \left[\left(\nabla \times \vec{v}\right) \bullet \hat{n}\right] dA = \int_{path} \vec{v} \bullet d\vec{L}$$

In words, it states ***the total change in the curl of a field through a surface can be inferred by measuring the field lines while traveling on a path along the surface.***

In other words, if there is a curl in a field, it can be found by the flow of the field:

$$\nabla \times \vec{v}\left(\vec{r}\right) = k\vec{J}\left(\vec{r}\right)$$

Now, we have another way of seeing exactly what is happening in our field. If we allow a vector $\Delta\vec{G}$ to represent the overall change in a field, it will be expressed as



Here, we see something rather interesting. The total change in a field can be written in terms of a static field (due to sources), and a dynamic field (due to currents). Thus, we can express our entire field $\vec{G}\left(\vec{r}\right)$ as a combination of a static field $\vec{E}\left(\vec{r}\right)$ and a dynamic field $\vec{B}\left(\vec{r}\right)$ such that

$$\Delta\vec{G}\left(\vec{r}\right) = \Delta\vec{E}\left(\vec{r}\right) + \Delta\vec{B}\left(\vec{r}\right)$$