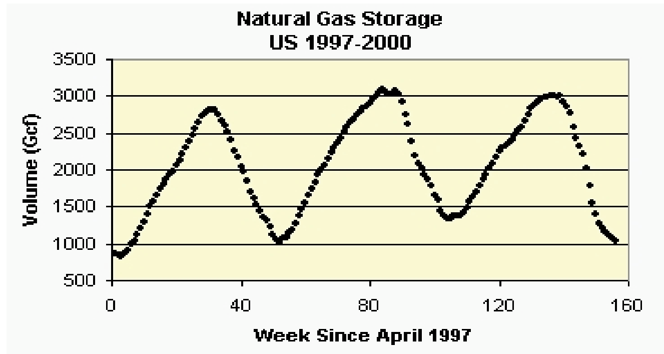
Trigonometric Models of Periodic Phenomena

In this tutorial we will demonstrate how to use trigonometric graphs to model periodic occurrences. A phenomenon is "periodic" if seems to go through a repetitive, predictable cycle, such as the graph of natural gas storage in the United States given below. (The table was captured using Captura from www.hernansoft.com.)



http://www.seattlecentral.org/qelp/sets/ 065/065.html

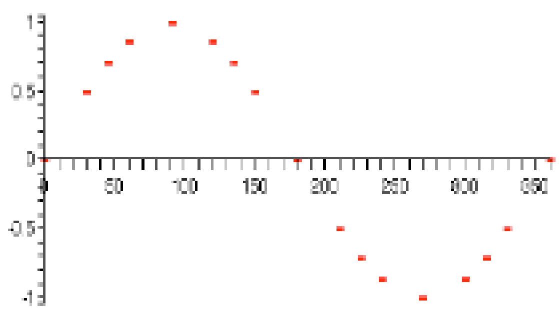
Here we can see that the storage increases at certain times of the year - during the warmer months, when the gas is not much used for heating and reserves can be increased - and then decreases during the colder months because the supply is being used. It is possible to model this with a relatively simple trigonometric graph, but first we examine the simplest trig curves.

The Basics

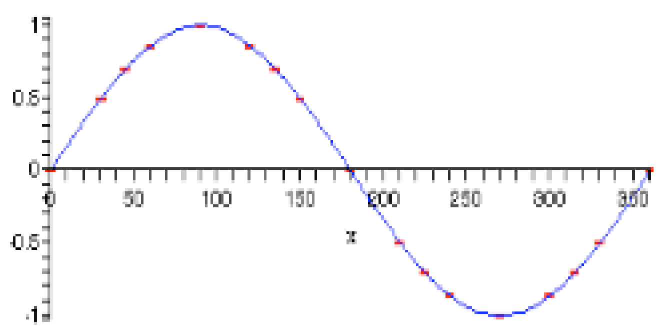
The simplest sine and cosine curves can be generated by plotting the trig-function values against the angles used in each function. The values can be found in the chart below:

ø (˚)	0	30	45	60	90	120	135	150	180	210	225	240	270	300	315	330	360
sin ø	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0
cos ø	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

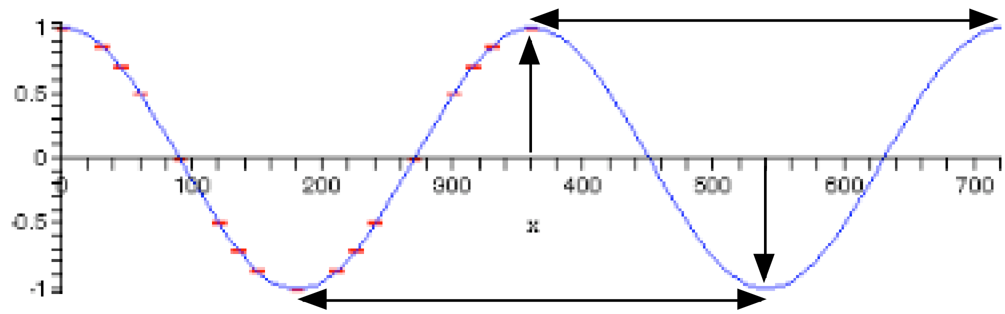
First we plot the sine data, letting the x-axis represent the angles and the y-axis the sine values. The result is the scatter plot below.



Based on just these 17 data points, we can see that the function is following a straightforward pattern - the sine values are increasing to 1, decreasing through the same values to -1, then returning to zero. When we connect the dots, we get the **periodic curve** $y = \sin(x)$ below.



This "sine curve," with equation $y = \sin(x)$, is the simplest of trig graphs. The curve rises and falls by one unit from its "axis of oscillation" - in this case, the x-axis - and after we cycle through 360˚ it appears we are ready to go through the same y-values again. We can see this same pattern in the cosine function on the following page.



Each sine and cosine curve has two basic characteristics:

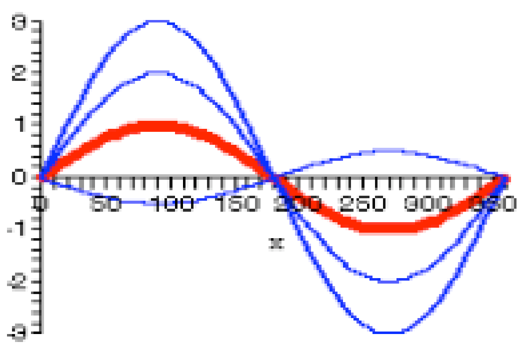
Amplitude - distance the curve rises above, and falls below, its axis of oscillation (marked by the vertical arrows on the cosine graph). The formula is $amp = \dfrac{\max - \min}{2}$.

Period - the length of one cycle (marked by the horizontal arrows on the cosine graph). This can be calculated by measuring the distance between any successive maximum points (or minimum points), or by doubling the horizontal distance between a minimum and the next maximum.

For our first two curves, then, the amplitude is 1 and the period is 360˚ (or 2π if we are in radians).

Adjusting the Amplitude and Frequency: y = A sin(Bx), y = A cos(Bx)

Now that we have our "base" amplitude and frequency . . . let's change them. We can adjust both characteristics by introducing the parameters A and B into the equations. The graph below shows the curves y = A sin(x) for A = 1,2,3, and -0.5, with the original curve y = sin(x) being the thickest line.



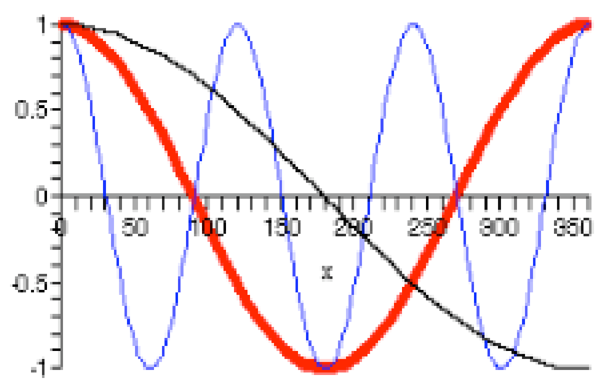
Clearly the amplitude of the curves has changed; when A = 2, the amplitude is 2; when A = 3, the amplitude is 3; and when A = -0.5 the amplitude is 0.5 (since amplitude, a distance, must be positive). However, this last curve has also been flipped upside down (or "reflected in the x-axis"). Since the amplitude must be positive even when A itself is negative, we have the formula

amplitude = | A |.

The relationship between the B-value and the period is not so straightforward. On the right we have y = cos(Bx) for B = 1,3, and 0.5. (The thickest line is again our "base" curve.) When B=3, the period decreases to 120˚; whe B=0.5, the period increases to 720˚ (so only half the curve is shown here). We can describe this reciprocal relationship with the formula

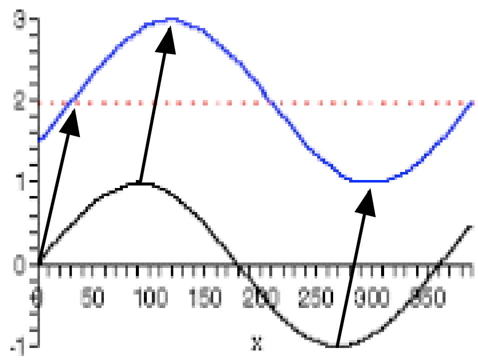
$$B = \frac{360}{period} \quad \text{or} \quad B = \frac{2\pi}{period},$$

using the latter if we are working in radians.



Shifting Horizontally and Vertically: $y = \sin(x\text{-}h) + k$, $y = \cos(x\text{-}h) + k$

To shift (or "translate") a curve along either the x- or y-axis (or both), we introduce the parameters h and k. Below we have the graph of $y = \sin(x - 30) + 2$ and $y = \sin(x)$. (Note that we have returned to an amplitude of 1 and period of 360°). Our original curve $y = \sin(x)$ had its peak at the point (90,1); the shifted curve has its corresponding peak at (120, 3). In the same way, the original curve had a minimum value at (270, -1); now the minimum is at (300,1). And finally, the axis of oscillation has shifted vertically from the line $x = 0$ to the line $x = 2$. In other words, our new curve has been created by shifting $y = \sin(x)$ right 30 and up 2. And, in general, we can create the following shifts:



horizontal shift = h (left if $h < 0$, right if $h > 0$)
vertical shift/new axis of oscillation = k.

Modelling Data: $y = A \sin[B(x\text{-}h)] + k$, $y = A \cos[B(x\text{-}h)] + k$

We have seen how the basic sine and cosine curves can be adjusted: by changing the amplitude and period, and by shifting the curves horizontally and vertically. A summary is below.

amplitude = $|A|$ **vertical shift/axis of oscillation** = k

period = $\dfrac{360}{B}$ or $\dfrac{2\pi}{B}$ **horizontal shift** = h (left if $h < 0$, right if $h > 0$)

Let's use these parameters to model the data from our introductory example of natural gas storage in the U.S. To find our equation, we can use a minimum point and the very next maximum point; in this case, we use the minimum at (52, 1009) and the maximum at (84,3096). (For the full data-set, see http://www.seattlecentral.org/qelp/sets/065/065.html.) Therefore we have that the amplitude must be $amp = \dfrac{3096 - 1009}{2} = 1043.5$. In the same way, the period is double the horizontal distance between the low point and high point, so $period = 2 \cdot (84 - 52) = 64$. For the vertical shift, we know that the axis is in the middle of the graph; therefore we can find axis by using the average of the max and min values: $axis = \dfrac{3096 + 1009}{2} = 2052.5$. And finally, for the horizontal shift, the simplest thing to do is take the x-coordinate of the maximum, which gives us $h = 84$. This gives us the equation

$$y = 1043.5\cos[\frac{2\pi}{64}(x - 84)] + 2052.5$$

which we graph below, along with several of the data-points in the region $44 \le x \le 94$, where the x values are weeks after April 1997 and y-values are billions of cubic feet.