ECE 120b

6-13-04
RdF

Solutions to Hmk #8

Preliminary Notes on FFT (helpful in the solution of Prob SP-1)

Let $x[n]$ be a N-point finite sequences with $N = 2^P$, i.e. (1)

$$x[n] = \{x[0], x[1], x[2], \ldots, x[N-2], x[N-1]\} \quad (2)$$

Then $x[n]$ can be broken up into two sequences, called <u>even</u> subsequence $x_1[n]$ and <u>odd</u> subsequence $x_2[n]$

where

$$\left\{\begin{aligned} x_1[0] &= x[0] \\ x_1[1] &= x[2] \\ &\vdots \\ x_1[m] &= x[2m] \\ &\vdots \\ x_1\left[\tfrac{N}{2}-1\right] &= x[N-2] \end{aligned}\right\} \quad (3)$$

$$\left\{\begin{aligned} x_2[0] &= x[1] \\ x_2[1] &= x[3] \\ &\vdots \\ x_2[m] &= x[2m+1] \\ &\vdots \\ x_2\left[\tfrac{N}{2}-1\right] &= x[N-1] \end{aligned}\right\} \quad (4)$$

Solutions
to Hmk #8 (CONT)

Now let

$X[k]$ denote the N-pt. DFT of $x[n]$ (5)

$X_1[k]$ " " $\frac{N}{2}$-pt DFT of $x_1[n]$ (6)

$X_2[k]$ " " " " " " $x_2[n]$ (7)

Then the fundamental computational benefit of the FFT algorithm is that it allows one to obtain the components of $X[k]$ from those of $X_1[k]$ and $X_2[k]$ by the simple recursion

$$X[k] = \begin{cases} X_1[k] + W_N^k X_2[k], & k = 0, 1, \ldots, \frac{N}{2}-1 \quad (8a) \\ X_1[k-\frac{N}{2}] - W_N^{k-\frac{N}{2}} X_2(k-\frac{N}{2}), & k = \frac{N}{2}, \ldots, N-1 \quad 8(b) \end{cases}$$

Note that in 8(b) I used the fact that

$$W_N^k = -W_N^{k-\frac{N}{2}} \quad (9)$$

which allows us to use only $\frac{N}{2}$ weights in this iteration

Solns. to Hwk 8 (CONT)

This procedure is exhibited in the below block diagram/signal flow graph. For simplicity we abbreviate

$$W_N = W \qquad (9)$$

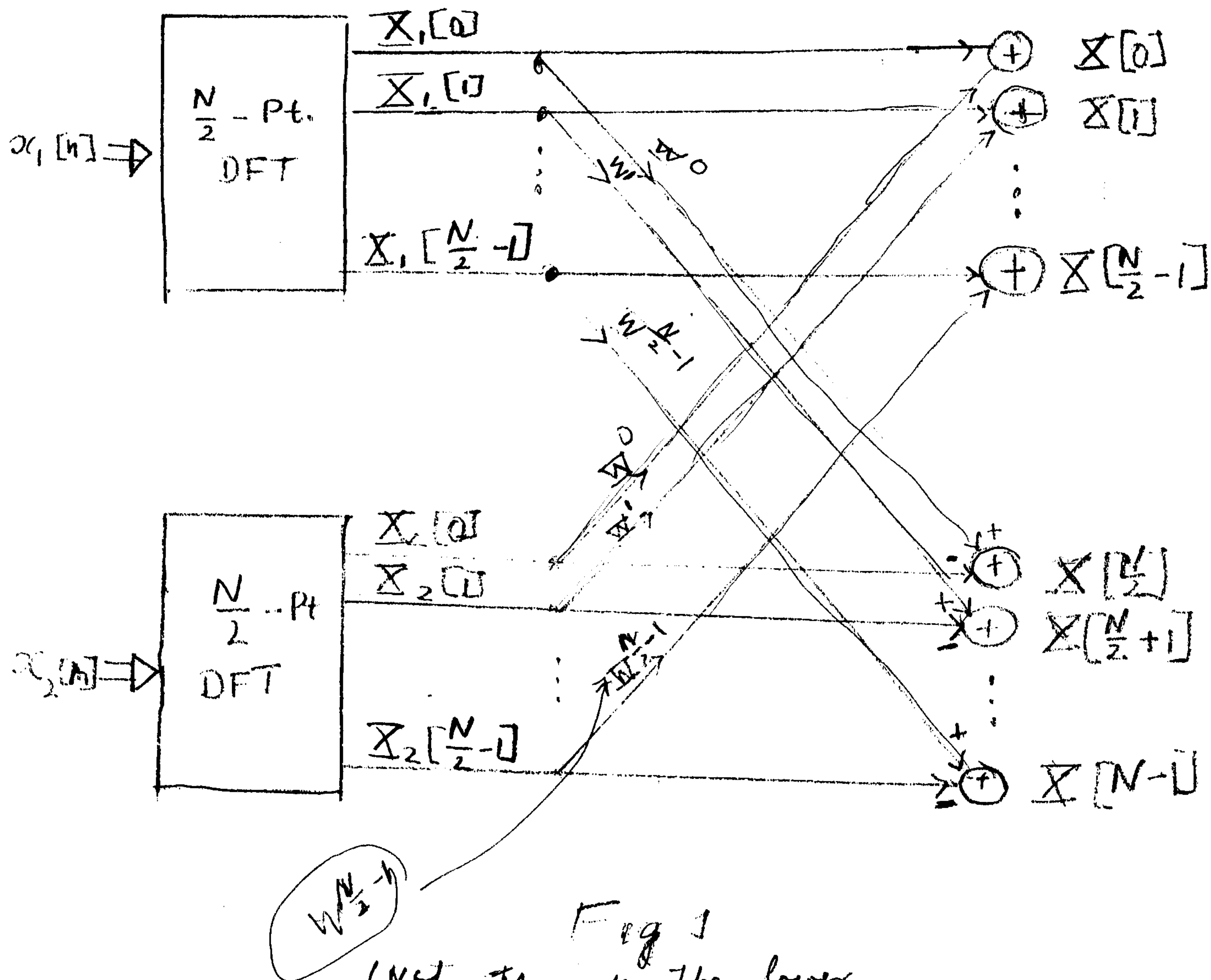


Fig 1

(Note that in the lower components, $k = \frac{N}{2}, \cdots, N-1$, you subtract the weights)

Another way of representing Fig 1 would be through the symbolic representation in Fig 2 done in class and in Notes #6

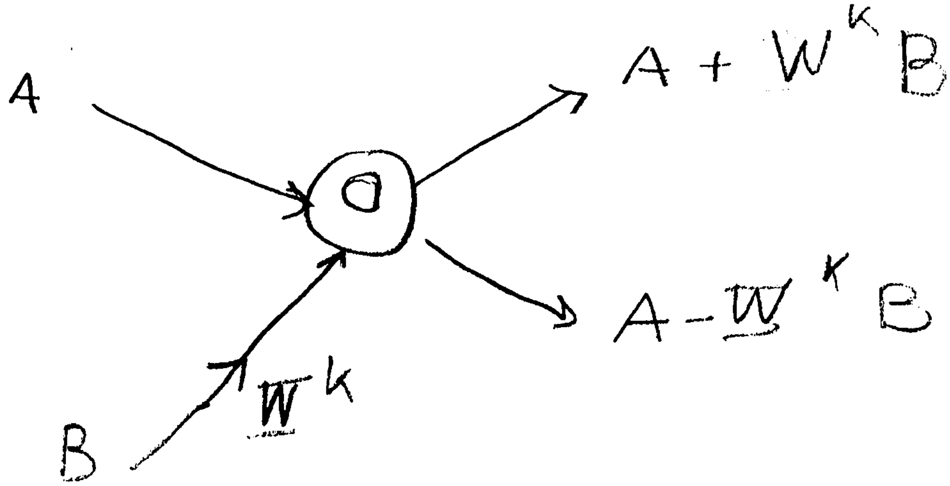


Fig. 2

This representation allows to represent Fig. 1 as Fig 3 on the next page

Solns. to Hwk 8 (con't)

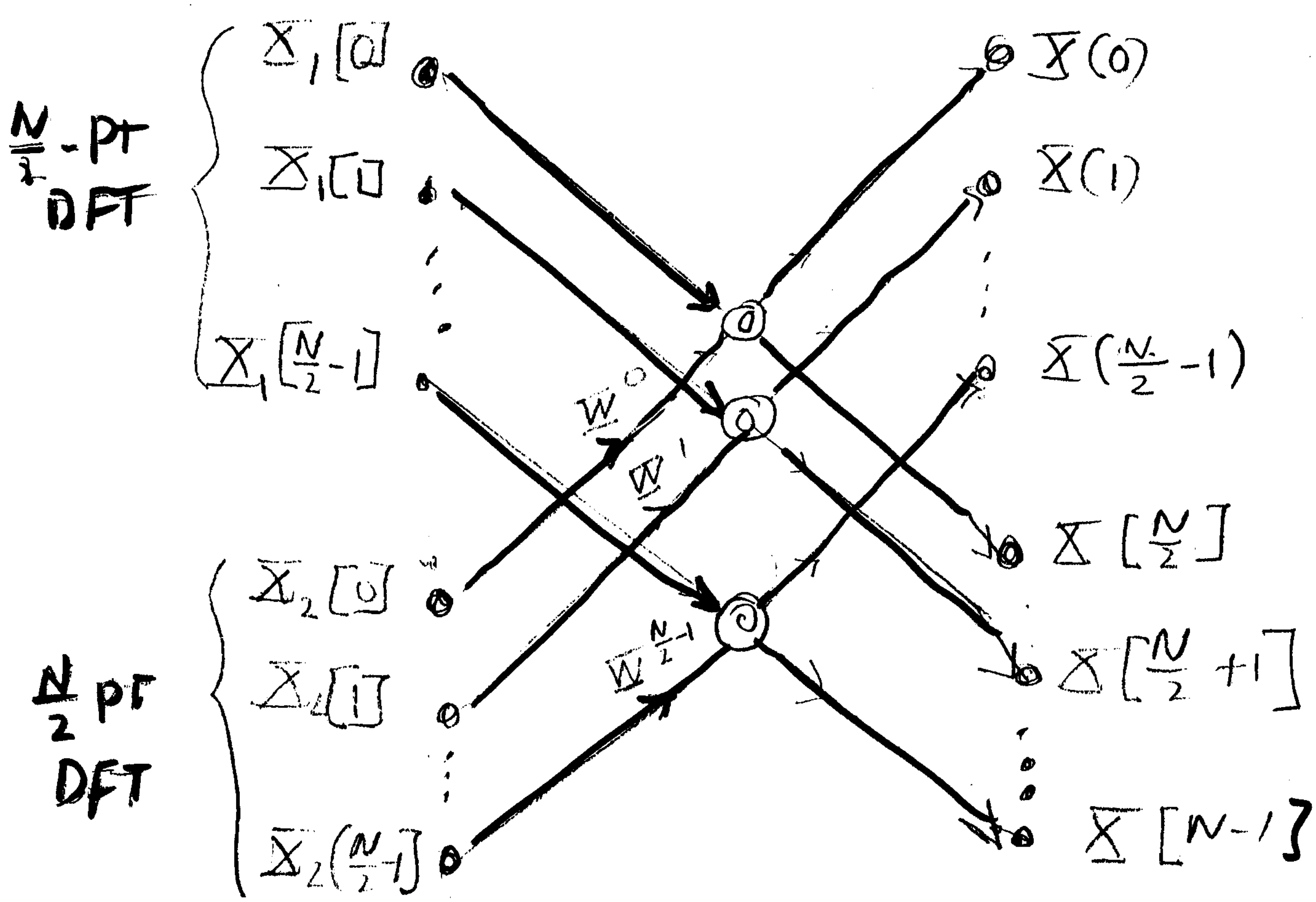


Fig 3

For an eight point FFT see the Figure in p. 5 of Lec. Notes #6.

NOTE To easily obtain the arrangement of data for 2 PT DFT's use bit-reversal as indicated on the next page.

Solns to Hwk 8 (CONT)

Re-Ordering of data by bit reversal

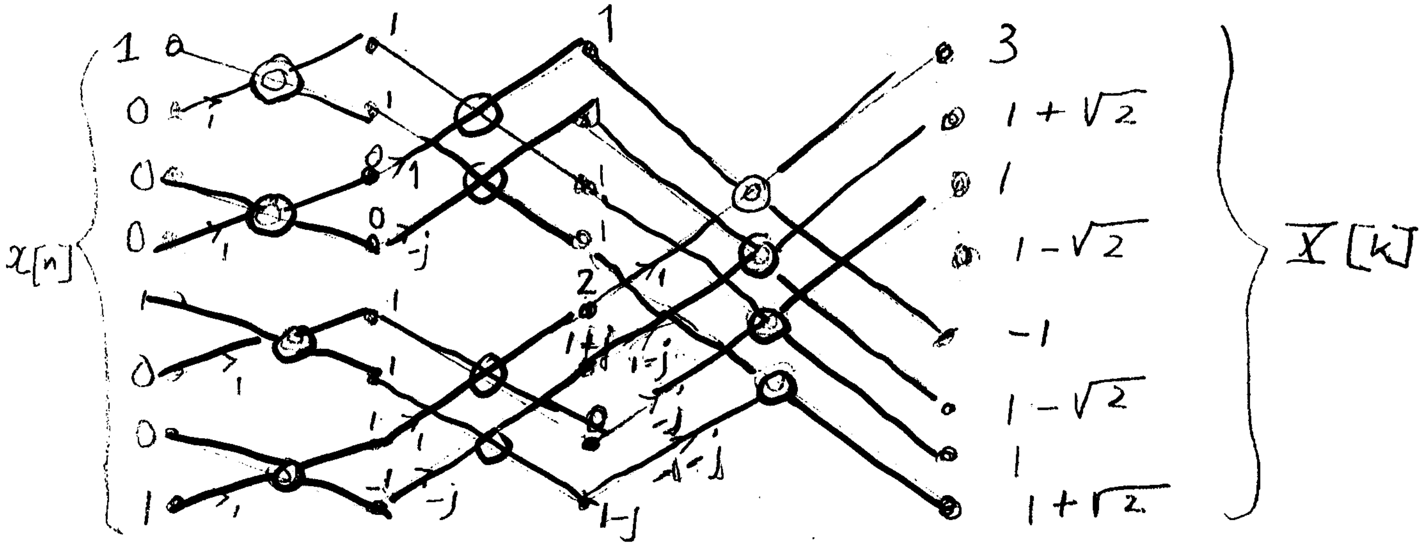
n	Forward order	Reverse Order	n	
0	0 0 0	0 0 0	0	$x[0]$
1	0 0 1	1 0 0	4	$x[4]$
2	0 1 0	0 1 0	2	$x[2]$
3	0 1 1	1 1 0	6	$x[6]$
4	1 0 0	0 0 1	1	$x[1]$
5	1 0 1	1 0 1	5	$x[5]$
6	1 1 0	0 1 1	3	$x[3]$
7	1 1 1	1 1 1	7	$x[7]$

So at the start of the FFT algorithm reorder the data stream by bit reversal and perform the 2-Point DFT on the newly ordered stream

This is clear from the soln of SP-1

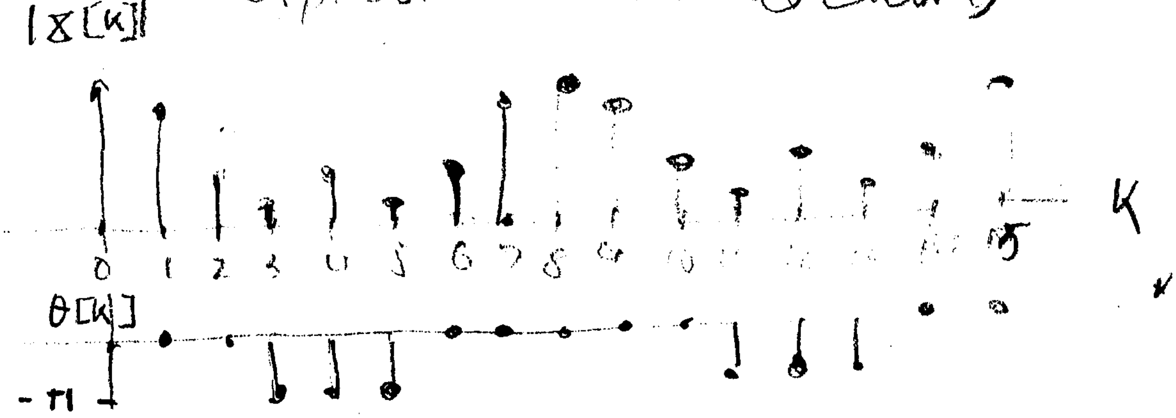
Prob. 5P-4

(a) Detailed Calculation of FFT is depicted in the below signal flow graph



(Note that the input data stream is shuffled automatically by bit reversed or by the procedure explained in lecture)

(b)



Hmk. Prob. Soln.

Prob. SP-2 # 8 (CONT)

(a)

$x[n] = 0 \ 1 \ 1 \ 1$

$y[n] = 1 \ 2 \ 1$

To perform circular convolution extend above to strings by zero padding to length

$$N = 4 + 3 - 1 = 6$$

Extended String

$x[n] = 0 \ 1 \ 1 \ 1 \ 0 \ 0$

$y[n] : 1 \ 2 \ 1 \ 0 \ 0 \ 0$

CONVOLUTION: Using method in Lec. Notes #6, pp. 18-20

Write

$\hat{y} = 1 \ 0 \ 0 \ 0 \ 1 \ 2$

then the circulant matrix

$$Y = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 2 \\ 2 & 1 & 0 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 \end{pmatrix}$$

Soln. to Hwk # 8 (CONT)

The convolution values are given by the vector z below

$$\begin{pmatrix} z[0] \\ z[1] \\ z[2] \\ z[3] \\ z[4] \\ z[5] \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 2 \\ 2 & 1 & 0 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 3 \\ 4 \\ 3 \\ 1 \end{pmatrix}$$

Using DFT Method

- First find DFTs of extended x and y by Matrix Method

$$\underline{X} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & e^{-j\pi/3} & e^{-j2\pi/3} & -1 & e^{-j\frac{4\pi}{3}} & e^{-j5\pi/3} \\ 1 & e^{-j2\pi/3} & e^{-j4\pi/3} & 1 & e^{-j2\pi/3} & e^{-j4\pi/3} \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & e^{j2\pi/3} & e^{-j2\pi/3} & 1 & e^{j2\pi/3} & e^{-j2\pi/3} \\ 1 & e^{j\pi/3} & e^{-j4\pi/3} & -1 & e^{-j2\pi/3} & e^{-j\frac{\pi}{3}} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ e^{-j\pi/3} + e^{-j2\pi/3} - 1 \\ e^{-j2\pi/3} + e^{-j4\pi/3} + 1 \\ -1 \\ e^{j2\pi/3} + e^{-j2\pi/3} + 1 \\ e^{j\pi/3} + e^{-j4\pi/3} - 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -1-j \\ 0 \\ -1 \\ 0 \\ -1 \end{pmatrix}$$

$$
Y = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & T \\
1 & e^{-j\pi/3} & e^{-j2\pi/3} & -1 & e^{-j4\pi/3} & e^{-j5\pi/3} \\
1 & e^{-j2\pi/3} & e^{-j4\pi/3} & 1 & e^{-j2\pi/3} & e^{-j4\pi/3} \\
1 & -1 & 1 & -1 & 1 & -1 \\
1 & e^{j2\pi/3} & e^{-j2\pi/3} & 1 & e^{j2\pi/3} & e^{-j2\pi/3} \\
1 & e^{j\pi/3} & e^{-j4\pi/3} & -1 & e^{-j2\pi/3} & e^{-j\pi/3}
\end{pmatrix}
= \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}
$$

$$
= \begin{pmatrix}
1 \\
1 + 2e^{-j\pi/3} + e^{-j2\pi/3} \\
1 + 2e^{-j2\pi/3} + e^{-j4\pi/3} \\
0 \\
1 + 2e^{j2\pi/3} + e^{-j2\pi/3} \\
1 + 2e^{j\pi/3} + e^{-j4\pi/3}
\end{pmatrix}
$$

$$
Z = \begin{pmatrix}
(3)(1) \\
-(1+j)\left(1 + 2e^{-j\pi/3} + e^{-j2\pi/3}\right) \\
0 \\
0 \\
0 \\
-\left(1 + 2e^{j\pi/3} + e^{-j4\pi/3}\right)
\end{pmatrix}
= \begin{pmatrix} z[0] \\ z[1] \\ z[2] \\ z[3] \\ z[4] \\ z[5] \end{pmatrix}
$$

Hwk #8 Soln (CONT)

Invert Z by

$$\begin{pmatrix} 0 \\ 1 \\ 3 \\ 4 \\ 3 \\ 1 \end{pmatrix} = z = \begin{pmatrix} z[0] \\ z[1] \\ \vdots \\ z[5] \end{pmatrix} = \underbrace{\frac{1}{6}\underbrace{\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & e^{j\pi/3} & e^{j2\pi/3} & -1 & e^{j4\pi/3} & e^{j5\pi/3} \\ 1 & e^{j2\pi/3} & e^{j4\pi/3} & 1 & e^{j2\pi/3} & e^{j4\pi/3} \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & e^{-j2\pi/3} & e^{j2\pi/3} & 1 & e^{-j2\pi/3} & e^{j2\pi/3} \\ 1 & e^{-j\pi/3} & e^{j4\pi/3} & -1 & e^{j2\pi/3} & e^{j5\pi/3} \end{pmatrix}}_{\underline{W}^*}}_{\underline{W}^{-1}} \begin{bmatrix} Z \\ \vdots \\ Z \end{bmatrix}$$

This is the convolution values which will agree with those obtained by time domain procedure in p. 9 of these Hwk #8 solution

(see eqn. (9) and (10) in Lec Notes #6)

I will skip (b) and move on to SP3.

Solns. to Hwk #8 (CONT)

Prob. SP-3

(a) $x(n)$: 0 1 1 1

$y[n]$: 1 2 1

Extended Strings

$x[n] =$ 0 1 1 1 0 0

$y[n] =$ 1 2 1 0 0 0

CORRELATION

Circulant Matrix $\tilde{Y}$ (See p. 25 of Lee Notes #6)

$$\tilde{Y} = \begin{pmatrix} 1 & 2 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \end{pmatrix}$$

Hmk. Soln. #8 (CONT)

Correlation vector

$$\begin{pmatrix} z[0] \\ z[1] \\ z[2] \\ z[3] \\ z[4] \\ z[5] \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \\ 1 \\ 3 \\ 4 \end{pmatrix}$$

Using DFT Method

I will outline the procedure

→ obtain the same DFT's as in SP-2

$$\underline{X} = \underline{W}\, x = \begin{pmatrix} X[0] \\ X[1] \\ \vdots \\ X[5] \end{pmatrix}$$

$$Y = \underline{W}\, y = \begin{pmatrix} Y[0] \\ Y[1] \\ \vdots \\ Y[5] \end{pmatrix}$$

→ Obtain the DFT of Correlation vector

$$Z = \begin{pmatrix} z[0] \\ z[1] \\ \vdots \\ z[5] \end{pmatrix} = \begin{pmatrix} X^*[0]\, Y[0] \\ X^*[1]\, Y[1] \\ \vdots \\ X^*[5]\, Y[5] \end{pmatrix} = \begin{bmatrix} Z[0] \\ Z[1] \\ \vdots \\ Z[5] \end{bmatrix}$$

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Hmk Soln #8 (CONT)

Prob SP-4

$h[n] = [1, 2, 1]$

$x[n] = [x_0, x_1, \cdots, x_{95}]$

In order to perform FFTs
You need block length after 0-padding
equal to a power of 2

Since $h[n]$ has length $3 = N_1$
you will need $x[n]$ blocks to
have length 2 to achieve
$N_1 + N_2 - 1 = 3 + 2 - 1 = 4$

So sensible choices are

$N_2 = 2 \longrightarrow$ block length 4

$N_2 = 6 \longrightarrow$ block lenght 8

Total no. of calculations

1st choice $N_2 = 2$ block length $N = 4$ (24 blocks)

Calc. for each block $= \frac{N}{2} \log_2 N = \frac{4}{2} \log_2 4 = (2)(2) = 4$

Total no. of calcs $= (24)(4) = 96$

2nd choice $N_2 = 6$ block length $= 8$ (12 blocks)

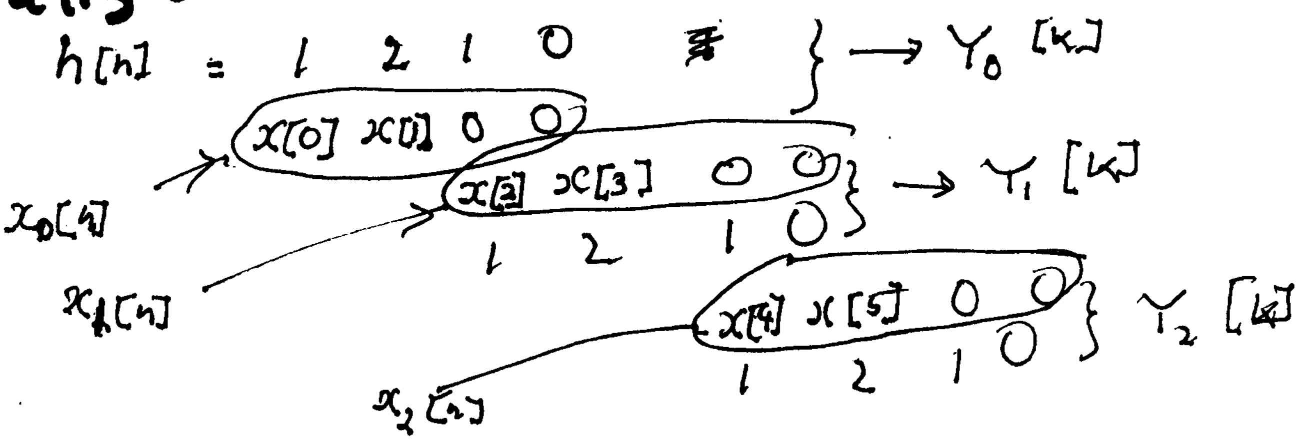
Calc/block $= \frac{8}{2} \log_2 8 = (4)(3) = 12$

Total no. of cals $= (12)(8) = 96$

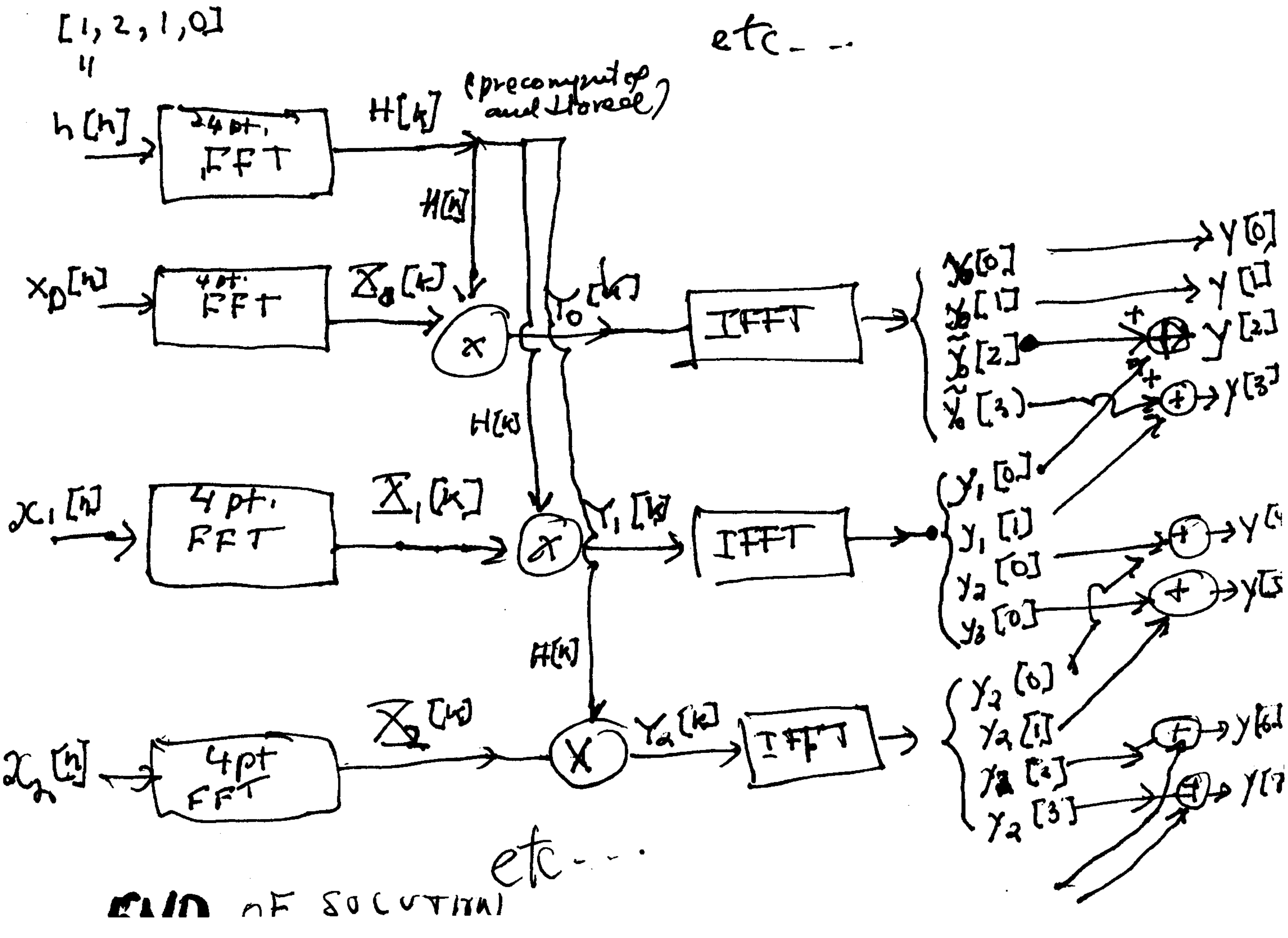
Both block sizes lead to the same amount of calculations

We choose block size = 4

Realization / Architecture



etc...



etc...

END OF SOLUTION