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Homework #4 Solutions

11.10 (i)

a) $X(z) = \dfrac{0.4z^2}{(z-1)(z-0.6)}$ Find $x[n]$

$$\frac{X(z)}{z} = \frac{0.4z}{(z-1)(z-0.6)} = \frac{1}{z-1} - \frac{0.6}{z-0.6}$$ via partial fraction expansion

$$x[n] = \mathcal{Z}^{-1}[X(z)] = \mathcal{Z}^{-1}\left[\frac{z}{z-1}\right] - \mathcal{Z}^{-1}\left[\frac{0.6z}{z-0.6}\right]$$

$$x[n] = 1 - 0.6(0.6)^n = 1-(0.6)^{n+1} \quad \text{for} \quad n \geq 0$$

$$= [1-(0.6)^{n+1}]u[n]$$

c) Find 1st three non-zero terms

$$x[0] = 1-(0.6)^{0+1} = 1-0.6 = 0.4$$

$$x[1] = 1-(0.6)^{1+1} = 0.64$$

$$x[2] = 1-(0.6)^{2+1} = 0.784$$

d) long division : denominator $\overline{)\text{numerator}}$ where $(z-1)(z-0.6) = z^2 - 1.6z + 0.6$

$$\begin{array}{r l}
 & 0.4 + 0.64z^{-1} + 0.784z^{-2} \\
\hline
z^2 - 1.6z + 0.6 \,) & 0.4z^2 \\
 & -(0.4z^2 - 0.64z + 0.24) \\
\hline
 & 0 \quad 0.64z - 0.24 \\
 & -(0.64z - 1.024 + \dots) \\
\hline
 & 0 \quad 0.784 + \dots \\
 & -(0.784 + \dots) \\
\hline
 & 0 + \dots
\end{array}$$

$$\sum_{n=0}^{n} x[n]z^{-n} = X(z) = x_0 + x_1 z^{-1} + x_2 z^{-2} + \dots x_n z^{-n}$$

$\therefore$ $x[0] = 0.4 = x_0$

$x[1] = 0.64 = x_1$

$x[2] = 0.784 = x_2$

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11.10 (i)

e) final value theorem

$$x[\infty] = \lim_{z\to 1} (1-z^{-1})X(z) = \lim_{z\to 1} \frac{1}{z}(z-1)X(z)$$

$$= \lim_{z\to 1} \frac{1}{z}\cancel{(z-1)} \frac{0.4z^2}{\cancel{(z-1)}(z-0.6)} = \frac{1}{z}\frac{0.4z^2}{z-0.6}\Bigg|_{z=1} = \frac{0.4}{0.4} = 1$$

f) verify part e

$$x[\infty] = \lim_{n\to\infty} x[n] = \lim_{n\to\infty} 1-(0.6)^{n+1} = 1-(0.6)^{\infty} = 1-0 = 1$$

g) initial value property

$$x[0] = \lim_{z\to\infty} X(z) = \lim_{z\to\infty} \frac{0.4z^2}{(z-1)(z-0.6)} = \lim_{z\to\infty} \frac{0.4z^2}{z^2-1.6z+0.6}$$

Using L'Hospitals rule

$$= \lim_{z\to\infty} \frac{\frac{d}{dz}(0.4z^2)}{\frac{d}{dz}(z^2-1.6z+0.6)} = \lim_{z\to\infty} \frac{0.4(2z)}{2z-1.6} = \lim_{z\to\infty} \frac{\frac{d}{dz}0.4(2z)}{\frac{d}{dz}(2z-1.6)}$$

$$= \lim_{z\to\infty} \frac{0.4\cdot\cancel{2}}{\cancel{2}} = 0.4$$

h) verify part g

$$x[0] = 1-(0.6)^{0+1} = 1-0.6 = 0.4$$

11.10 (ii)

a) $$X(z) = \frac{0.4z}{(z-1)(z-0.6)}$$

$$\frac{X(z)}{z} = \frac{0.4}{(z-1)(z-0.6)} = \frac{1}{z-1} - \frac{1}{z-0.6}$$ via partial fraction expansion

$$X(z) = \frac{z}{z-1} - \frac{z}{z-0.6}$$

$$x[n] = Z^{-1}[X(z)] = Z^{-1}\left[\frac{z}{z-1}\right] - Z^{-1}\left[\frac{z}{z-0.6}\right]$$

$$x[n] = 1-(0.6)^n,\ n\geq 0 = [1-(0.6)^n]u[n]$$

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11.10 (ii)

c) $x[0] = 1-0.6^0 = 0$ $\qquad$ $x[2] = 1-(0.6)^2 = 0.64$

$x[1] = 1-(0.6)^1 = 0.4$ $\qquad$ $x[3] = 1-(0.6)^3 = 0.784$

d)

$$\begin{array}{r|l} & 0.4z^{-1} + 0.64z^{-2} + 0.784z^{-3} + \ldots \\ \hline z^2 - 1.6z + 0.6 & 0.4z \\ & -(0.4z - 0.64 + 0.24z^{-1}) \\ \hline & 0 \quad 0.64 - 0.24z^{-1} \\ & -(0.64 - 1.024z^{-1} + \ldots \\ \hline & 0 \; + \; 0.784z^{-1} + \ldots \end{array}$$

$x[0] = x_0 = 0$ $\qquad$ $x[2] = x_2 = 0.64$

$x[1] = x_1 = 0.4$ $\qquad$ $x[3] = x_3 = 0.784$

e)

$$x[\infty] = \lim_{z\to 1} (\cancel{z-1}) \frac{1}{z} \frac{0.4z}{(\cancel{z-1})(z-0.6)} = \frac{0.4z}{(z-0.6)} \frac{1}{z} \Bigg|_{z=1} = \frac{0.4}{0.4} = 1$$

f) $x[\infty] = \lim_{n\to\infty} 1-(0.6)^n = 1-(0.6)^\infty = 1-0 = 1$

g)

$$x[0] = \lim_{z\to\infty} \frac{0.4z}{z^2-1.6z+0.6} = \text{L'Hospital} \Rightarrow \lim_{z\to\infty} \frac{\frac{d}{dz}(0.4z)}{\frac{d}{dz}(z^2-1.6z+0.6)} =$$

$$= \lim_{z\to\infty} \frac{0.4}{2z-1.6} = \frac{0.4}{\infty} = 0$$

h) $x[0] = 1-(0.6)^0 = 1-1 = 0$

11.10 (iv)

a) $X(z) = \dfrac{z}{z^2-z+1}$ $\qquad$ $\mathcal{Z}[\sin bn] = \dfrac{(\sin b)z}{z^2 - 2\cos b\, z + 1}$

$2\cos b = 1 \Rightarrow \cos b = 1/2 \;\therefore\; b = \pi/3$

$\sin b = \sin \pi/3 = \sqrt{3}/2 = 0.866$

$$x[n] = \mathcal{Z}^{-1}[X(z)] = \frac{2}{\sqrt{3}} \sin\left(\frac{\pi}{3} n\right) = 1.15 \sin\left(\frac{\pi}{3} n\right)$$

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11.10 (iv)

c) $x[0] = 1.15\sin(0) = 0$

$x[1] = 1.15\sin(\pi/3) = \frac{2}{\sqrt{3}}\frac{\sqrt{3}}{2} = 1$

$x[2] = 1.15\sin(\frac{\pi}{3}\cdot 2) = \frac{2}{\sqrt{3}}\frac{\sqrt{3}}{2} = 1$

$x[3] = 1.15\sin(\frac{\pi}{3}\cdot 3) = 1.15\sin(\pi) = 0$

$x[4] = 1.15\sin(\pi/3\cdot 4) = (-\frac{\sqrt{3}}{2})\frac{2}{\sqrt{3}} = -1$

d)

$$\begin{array}{r l}
 & z^{-1} + z^{-2} - z^{-4} \\
z^2 - z + 1 \,\big) & z \\
 & -(z - 1 + z^{-1}) \\ \hline
 & 0 \quad 1 - z^{-1} \\
 & -(1 - z^{-1} + z^{-2}) \\ \hline
 & 0 \quad 0 \quad -z^{-2} \\
 & -(-z^{-2} + \dots) \\ \hline
 & 0 + \dots
\end{array}$$

$x[0] = x_0 = 0$

$x[1] = x_1 = 1$

$x[2] = x_2 = 1$

$x[3] = x_3 = 0$

$x[4] = x_4 = -1$

e) final value $x[\infty]$ does not exist since $x[n]$ is a sinusoidal function

f) see e. $\Rightarrow$ does not exist

g) $x[0] = \lim_{z\to\infty} \frac{z}{z^2 - z + 1} \Rightarrow$ L'Hospital $\lim_{z\to\infty} \frac{\frac{d}{dz}(z)}{\frac{d}{dz}(z^2 - z + 1)} = \lim_{z\to\infty} \frac{1}{2z - 1}$

$= \frac{1}{\infty} = 0$

h) $x[0] = 1.15\sin(0) = 0$

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Homework #4 Solutions

11.12

a) $y[n] - 1.5y[n-1] + 0.5y[n-2] = x[n] = \begin{cases} 1, & n=1 = \delta[n-1] \\ 0, & \text{otherwise} \end{cases}$

$$Y(z) - 1.5z^{-1}Y(z) + 0.5z^{-2}Y(z) = z^{-1}$$

$$Y(z) = \frac{z^{-1}}{1-1.5z^{-1}+0.5z^{-2}} \cdot \frac{z^2}{z^2} = \frac{z}{z^2-1.5z+0.5}$$

$$\frac{Y(z)}{z} = \frac{1}{z^2-1.5z+0.5} = \frac{1}{(z-1)(z-0.5)} = \frac{2}{z-1} + \frac{-2}{z-0.5}$$

$$y[n] = Z^{-1}[Y(z)] = [2 - 2(0.5)^n]u[n]$$

c) Solution from $y[n] = [2-2(0.5^n)]u[n]$ for $n=0,1,2,3,4$

$y[0] = 2-2(0.5)^0 = 0$

$y[1] = 2-2(0.5)^1 = 1$

$y[2] = 2-2(0.5)^2 = 1.5$

$y[3] = 2-2(0.5)^3 = 1.75$

$y[4] = 2-2(0.5)^4 = 1.875$

Using the difference equation

$y[n] = 1.5y[n-1] - 0.5y[n-2] + x[n]$

$y[0] = 1.5y[-1] - 0.5y[-2] + x[0] = 0-0+0 = 0$

$y[1] = 1.5y[0] - 0.5y[-1] + x[1] = 0-0+1 = 1$

$y[2] = 1.5y[1] - 0.5y[0] + x[2] = 1.5-0+0 = 1.5$

$y[3] = 1.5y[2] - 0.5y[1] + x[3] = (1.5)1.5 - 0.5(1) = 1.75$

$y[4] = 1.5y[3] - 0.5y[2] + x[4] = (1.5)(1.75) - 0.5(1.75) = 1.875$

e) initial value property

$$y[0] = \lim_{z\to\infty} Y(z) = \lim_{z\to\infty} \frac{z}{z^2-1.5z+0.5} \Rightarrow \text{L'Hospital } \lim_{z\to\infty}\frac{d/dz\,(z)}{\frac{d}{dz}(z^2-1.5z+0.5)} = \ldots$$

$$\lim_{z\to\infty}\frac{1}{2z-1.5} = \frac{1}{\infty} = 0$$

f) Yes the limit will exist and give the final value.

$$y[\infty] = \lim_{z\to1} \tfrac{1}{z}(z-1)Y(z) = \lim_{z\to1} \tfrac{1}{z}\cancel{(z-1)}\frac{z}{\cancel{(z-1)}(z-0.5)} = \lim_{z\to1}\frac{1}{z}\,\frac{z}{z-0.5}$$

$$= \frac{1}{1-0.5} = 2$$

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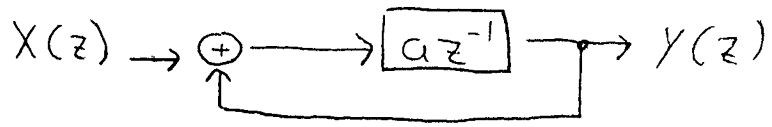
11.15 a)



$$Y(z) = az^{-1}X(z) + az^{-1}Y(z)$$
$$[1-az^{-1}]Y(z) = az^{-1}X(z)$$

$$\therefore y[n] - ay[n-1] = aX[n-1]$$

b) $$H(z) = \frac{Y(z)}{X(z)} = \frac{az^{-1}}{1-az^{-1}} = \frac{a}{z-a}$$

c) BIBO stable

A system pole at $z=1$ must be inside the unit circle $\therefore$
$-1<a<1$ or $|a|<1$ for the system to converge (BIBO stability)

d) Impulse response $x[n] = \delta[n]$

$$X(z) = 1$$
$$H(z) = \frac{a}{z-a} \Rightarrow \frac{H(z)}{z} = \frac{a}{z(z-a)} = \frac{-1}{z} + \frac{1}{z-a}$$

$$\therefore H(z) = -1 + \frac{z}{z-a}$$

$$h[n] = Z^{-1}[H(z)] = Z^{-1}[-1] + Z^{-1}\left[\frac{z}{z-a}\right]$$
$$= (-\delta[n] + a^n)u[n] \begin{cases} -1+a^0 = -1+1 = 0, & n=0 \\ 0 + a^n = a^n, & n \geq 1 \end{cases}$$
$$= a^n u[n-1]$$

Yes, the system is bounded only for $|a|<1$

e) $a = 0.5$

$$y[n] = h[n] * x[n] = h[n] * u[n]$$
$$Y(z) = H(z)X(z) = \left(\frac{0.5}{z-0.5}\right)\left(\frac{z}{z-1}\right)$$
$$\frac{Y(z)}{z} = \frac{0.5}{(z-0.5)(z-1)} = \frac{1}{z-1} + \frac{-1}{z-0.5} \quad \text{(by PFE)}$$
$$Y(z) = \frac{z}{z-1} + \frac{-z}{z-0.5} \qquad \therefore y[n] = Z^{-1}[Y(z)] = 1 - 0.5^n, n \geq 0$$
$$= [1-0.5^n]u[n]$$

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11.15 f) $a = 2.0$

$$Y(z) = H(z)X(z) = \frac{2}{z-2}\,\frac{z}{z-1}$$

$$\frac{Y(z)}{z} = \frac{2}{(z-2)(z-1)} = \frac{2}{z-2} - \frac{2}{z-1}$$

$$Y(z) = \frac{2z}{z-2} - \frac{2z}{z-1}$$

$$y[n] = \mathcal{Z}^{-1}[Y(z)] = [2(2)^n - 2]u[n] = 2[2^n - 1]u[n]$$
$$= 2[2^n - 1],\ n \geq 0$$

11.24 a) $$F_6(z) = \frac{0.6z}{(z-1)(z-0.6)} =$$

$$\frac{F_6(z)}{z} = \frac{0.6}{(z-1)(z-0.6)} = \frac{3/2}{z-1} - \frac{3/2}{z-0.6}$$

$$F_6(z) = \frac{3/2\, z}{z-1} - \frac{3/2\, z}{z-0.6}$$

(i) $|z| < 0.6$ using table 11.5 transform 10 and 9

$f[n] \longleftrightarrow F_6(z)$

$-a^n u[-n-1] \longleftrightarrow \frac{z}{z-a}$ w/ ROC $|z| < |a|$

$$\therefore f[n] = -3/2\,(1)^n u[-n-1] + 3/2\,(0.6)^n u[-n-1]$$
$$= -3/2\, u[-n-1] + 3/2\,(0.6)^n u[-n-1]$$

(ii) $|z| > 1$ using table 11.5 transform 3 and 5

$u[n] \longleftrightarrow \frac{z}{z-1}$ w/ ROC $|z| > 1$

$$\therefore f[n] = \frac{3}{2} u[n] - 3/2\,(0.6)^n u[n]$$

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11.24 a)

(iii) $0.6 < |z| < 1$ using table 11.5 transforms 5 and 10/9

$\underset{\text{right side}}{\downarrow}$ $\underset{\text{left side}}{\downarrow}$

Right side $0.6 < |z|$

$$\therefore \frac{-3/2\, z}{z-0.6} \xleftrightarrow[\text{using transform 5}]{} f[n]_{\text{right side}} = -3/2\,(0.6)^n u[n]$$

Left side $|z| < 1$

$$\therefore \frac{3/2\, z}{z-1} \xleftrightarrow[\text{using transform 10 or 9}]{} f[n]_{\text{left side}} = -3/2\, u[-n-1]$$

$$f[n] = f[n]_{rs} + f[n]_{LS} = -3/2\,(0.6)^n u[n] - \tfrac{3}{2} u[-n-1]$$

b) (i) $\lim_{n\to\infty} -3/2\, u[-n-1] + 3/2\,(0.6)^n u[n-1] = 0 + 0 = 0$

(ii) $\lim_{n\to\infty} 3/2\, u[n] - 3/2\,(0.6)^n u[n] = 3/2 - 0 = 3/2$

(iii) $\lim_{n\to\infty} -3/2\,(0.6)^n u[n] - 3/2\, u[-n-1] = 0 + 0 = 0$

c) (i) $f[\infty] = \lim_{z\to 1} \frac{1}{z}(z-1)\frac{0.6z}{(z-1)(z-0.6)}$ but since $|z| < 0.6$

this limit is not valid and hence $f[\infty] = 0$ from part b.

(ii) $f[\infty] = \lim_{z\to 1} \frac{1}{z}\cancel{(z-1)}\frac{0.6z}{\cancel{(z-1)}(z-0.6)} = \lim_{z\to 1} \frac{1}{z}\frac{0.6z}{z-0.6}$

$$= \frac{0.6}{0.4} = \frac{3}{2}$$

(iii) This limit is valid for the right side of the equation $0.6 < |z|$ only and is zero for the left side since $|z| < 1$

$$\therefore f[\infty] = \lim_{z\to 1} \frac{1}{z}(z-1)\frac{-3/2}{z-0.6} + 0 = 0 + 0 = 0$$

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SP-2

$$y[n] - \frac{1}{6}y[n-1] - \frac{1}{6}y[n-2] = x[n] - \frac{1}{2}x[n-1]$$

$$y[-1] = y_{-1} = 0 \;\vdots\; y[-2] = y_{-2} = 1 \;\vdots\; x[n] = \left(\tfrac{1}{3}\right)^n u[n]$$

$$Z[y[n]] = Y(z)$$

$$Y(z) - \frac{1}{6}z^{-1}\left[Y(z) + y_{-1}z\right] - \frac{1}{6}z^{-2}\left[Y(z) + y_{-1}z + y_{-2}z^2\right]$$
$$= X(z) - \frac{1}{2}z^{-1}X(z)$$

$$Y(z)\left[1 - \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}\right] + \left[-\frac{1}{6}y_{-1} - \frac{1}{6}y_{-1}z^{-1} - \frac{1}{6}y_{-2}\right]$$
$$= X(z)\left[1 - \frac{1}{2}z^{-1}\right]$$

using the initial conditions

$$\therefore Y(z) = \frac{z^2 - \frac{1}{2}z}{z^2 - \frac{1}{6}z - \frac{1}{6}}X(z) + \frac{z^2\left[\frac{1}{6}\overset{0}{\cancel{y_{-1}}} + \frac{1}{6}\overset{0}{\cancel{y_{-1}}}z^{-1} + \frac{1}{6}\overset{1}{y_{-2}}\right]}{z^2 - \frac{1}{6}z - \frac{1}{6}}$$

$$Y(z) = \frac{z^2 - \frac{1}{2}z}{z^2 - \frac{1}{6}z - \frac{1}{6}}X(z) + \frac{\frac{1}{6}z^2}{z^2 - \frac{1}{6}z - 1/6}$$

$$= \underbrace{\frac{z^2 - \frac{1}{2}z}{(z - 1/2)(z + 1/3)}X(z)}_{Y_{zs}(z)} + \underbrace{\frac{\frac{1}{6}z^2}{(z - 1/2)(z + 1/3)}}_{Y_{zi}(z)}$$

Zero-State Response

$$Y_{zs}(z) = \frac{z^2 - \frac{1}{2}z}{(z - \frac{1}{2})(z + 1/3)} \cdot X(z) = \frac{z^2 - \frac{1}{2}z}{(z - \frac{1}{2})(z + \frac{1}{3})}\,\frac{z}{(z - \frac{1}{3})}$$

$$\frac{Y_{zs}(z)}{z} = \frac{z^2 - \frac{1}{2}z}{(z - 1/2)(z + 1/3)(z - \frac{1}{3})}$$

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SP-2 (continued)

$$\frac{Y_{ZS}(z)}{z} = \frac{0}{z-\frac{1}{2}} + \frac{1/2}{z+\frac{1}{3}} + \frac{\frac{1}{2}}{z-\frac{1}{3}}$$ via partial fraction expansion

$$\therefore Y_{ZS}(z) = \left(\frac{1}{2}\right)\frac{z}{z+\frac{1}{3}} + \left(\frac{1}{2}\right)\frac{z}{z-\frac{1}{3}}$$

$$y_{ZS}[n] = Z^{-1}\left[Y(z)\right] = \left[\left(\frac{1}{2}\right)\left(-\frac{1}{3}\right)^n + \left(\frac{1}{2}\right)\left(\frac{1}{3}\right)^n\right]u[n]$$

Zero-Input Response

$$Y_{ZI}(z) = \frac{\frac{1}{6}z^2}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{3}\right)}$$

$$\frac{Y_{ZI}(z)}{z} = \frac{\frac{1}{6}z}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{3}\right)} = \frac{\frac{1}{10}}{z-\frac{1}{2}} + \frac{\frac{1}{15}}{z+\frac{1}{3}}$$ via partial fraction expansion

$$\therefore Y_{ZI}(z) = \left(\frac{1}{10}\right)\frac{z}{z-\frac{1}{2}} + \left(\frac{1}{15}\right)\frac{z}{z+\frac{1}{3}}$$

$$y_{ZI}[n] = Z^{-1}\left[Y_{ZI}(z)\right] = \left[\frac{1}{10}\left(\frac{1}{2}\right)^n + \frac{1}{15}\left(-\frac{1}{3}\right)^n\right]u[n]$$

Complete Response

$$y[n] = y_{ZS}[n] + y_{ZI}[n]$$

$$= \left[\left(\frac{1}{2}\right)\left(-1/3\right)^n + \left(\frac{1}{2}\right)\left(\frac{1}{3}\right)^n + \left(\frac{1}{10}\right)\left(\frac{1}{2}\right)^n + \left(\frac{1}{15}\right)\left(-\frac{1}{3}\right)^n\right]u[n]$$

$$= \left[\frac{1}{10}\left(\frac{1}{2}\right)^n + \frac{17}{30}\left(-\frac{1}{3}\right)^n + \frac{1}{2}\left(\frac{1}{3}\right)^n\right]u[n]$$