PROBLEM 1 : Find two nonnegative numbers whose sum is 9 and so that the product of one number and the square of the other number is a maximum.

PROBLEM 2 : Build a rectangular pen with three parallel partitions using 500 feet of fencing. What dimensions will maximize the total area of the pen ?

PROBLEM 3 : An open rectangular box with square base is to be made from 48 ft.2 of material. What dimensions will result in a box with the largest possible volume ?

PROBLEM 4 : A container in the shape of a right circular cylinder with no top has surface area 3π ft^2. What height h and base radius r will maximize the volume of the cylinder ?

PROBLEM 5 : A sheet of cardboard 3 ft. by 4 ft. will be made into a box by cutting equal-sized squares from each corner and folding up the four edges. What will be the dimensions of the box with largest volume ?

PROBLEM 6 : Consider all triangles formed by lines passing through the point (8/9, 3) and both the x- and y-axes. Find the dimensions of the triangle with the shortest hypotenuse.

PROBLEM 7 : Find the point (x, y) on the graph of y = sqrt(x) nearest the point (4, 0).

PROBLEM 8 : A cylindrical can is to hold 20 π m^3 The material for the top and bottom costs \$10/m^2 and material for the side costs \$8/m^2 Find the radius r and height h of the most economical can.

PROBLEM 9 : You are standing at the edge of a slow-moving river which is one mile wide and wish to return to your campground on the opposite side of the river. You can swim at 2 mph and walk at 3 mph. You must first swim across the river to any point on the opposite bank. From there walk to the campground, which is one mile from the point directly across the river from where you start your swim. What route will take the least amount of time ?

PROBLEM 10 : Construct a window in the shape of a semi-circle over a rectangle. If the distance around the outside of the window is 12 feet, what dimensions will result in the rectangle having largest possible area ?

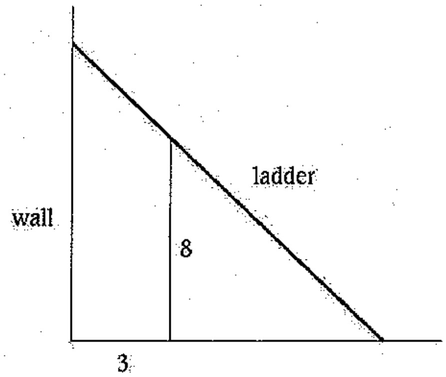
PROBLEM 11 : There are 50 apple trees in an orchard. Each tree produces 800 apples. For each additional tree planted in the orchard, the output per tree drops by 10 apples. How many trees should be added to the existing orchard in order to maximize the total output of trees ?

PROBLEM 12 : Find the dimensions of the rectangle of largest area which can be inscribed in the closed region bounded by the x-axis, y-axis, and graph of $y=8-x^3$. (See diagram.)

PROBLEM 13 : Consider a rectangle of perimeter 12 inches. Form a cylinder by revolving this rectangle about one of its edges. What dimensions of the rectangle will result in a cylinder of maximum volume ?

PROBLEM 14 : Find the dimensions (radius r and height h) of the cone of maximum volume which can be inscribed in a sphere of radius 2.

PROBLEM 15 : Find the length of the shortest ladder that will reach over an 8-ft. high fence to a large wall which is 3 ft. behind the fence. (See diagram.)



PROBLEM 16 : Find the point P = (x, 0) on the x-axis which minimizes the sum of the squares of the distances from P to (0, 0) and from P to (3, 2).

PROBLEM 17 : Car B is 30 miles directly east of Car A and begins moving west at 90 mph. At the same moment car A begins moving north at 60 mph. What will be the minimum distance between the cars and at what time t does the minimum distance occur ?

SOLUTION 1 : Let variables x and y represent two nonnegative numbers. The sum of the two numbers is given to be

$9 = x + y$,

so that

$y = 9 - x$.

We wish to MAXIMIZE the PRODUCT

$P = x y^2$.

However, before we differentiate the right-hand side, we will write it as a function of x only. Substitute for y getting

$P = x y^2$

$P = x (9-x)^2$.

Now differentiate this equation using the product rule and chain rule, getting

$P' = x (2) (9-x)(-1) + (1) (9-x)^2$

$= (9-x) [-2x + (9-x)]$

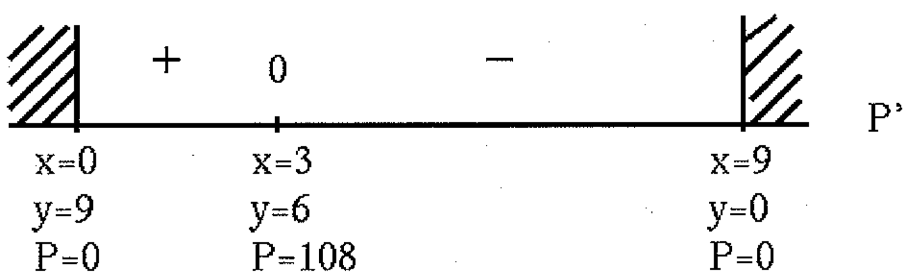
$= (9-x) [9-3x]$

$= (9-x) (3)[3-x]$

$= 0$

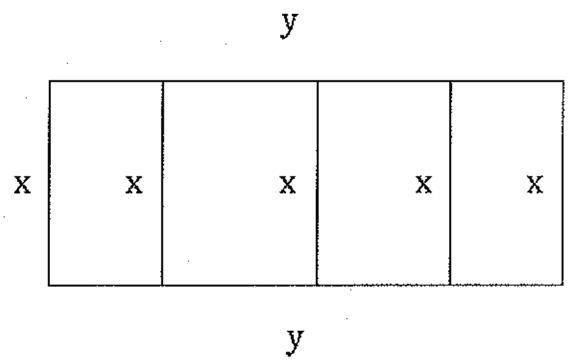
for x=9 or x=3 .

Note that since both x and y are nonnegative numbers and their sum is 9, it follows that $0 \leq x \leq 9$. See the adjoining sign chart for P' .



If x=3 and y=6 , then P= 108 is the largest possible product.

SOLUTION 2 : Let variable x be the width of the pen and variable y the length of the pen. The total amount of fencing is given to be



500 = 5 (width) + 2 (length) = 5x + 2y , so that
2y = 500 - 5x or
y = 250 - (5/2)x .

We wish to MAXIMIZE the total AREA of the pen
A = (width) (length) = x y .

However, before we differentiate the right-hand side, we will write it as a function of x only. Substitute for y getting
A = x y

A = x (250 - (5/2)x)
A = 250x - (5/2)x2 .

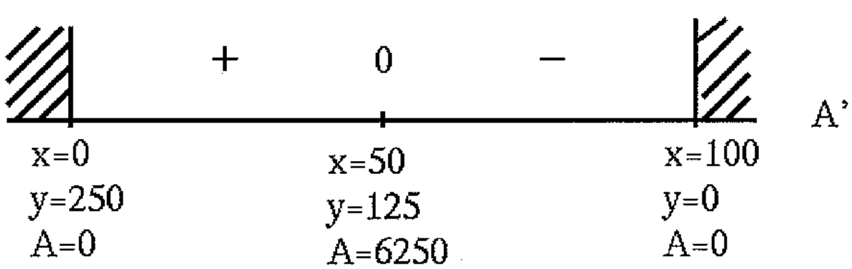
Now differentiate this equation, getting
A' = 250 - (5/2) 2x

A = 250 - 5x

A = 5 (50 - x)

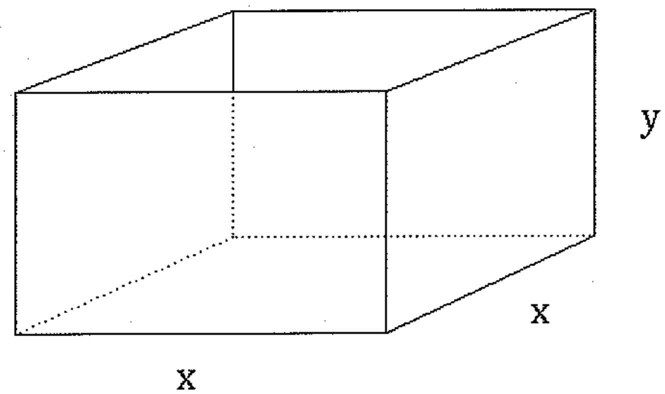
A = 0 for x=50 .

Note that since there are 5 lengths of x in this construction and 500 feet of fencing, it follows that $0 \le x \le 100$. See the adjoining sign chart for A' .



If x=50 ft. and y=125 ft. , then A = 6250 ft^2 is the largest possible area of the pen.

SOLUTION 3 : Let variable x be the length of one edge of the square base and variable y the height of the box.



The total surface area of the box is given to be
48 = (area of base) + 4 (area of one side) = $x^2 + 4(xy)$,

so that $4xy = 48 - x^2$ or

$y = (48 - x^2) / (4x)$ or

$y = 48/(4x) - x^2 / (4x)$ or

$y = 12/x - x/4$

We wish to MAXIMIZE the total VOLUME of the box

V = (length) (width) (height) = (x) (x) (y) = $x^2 y$.

However, before we differentiate the right-hand side, we will write it as a function of x only. Substitute for y getting
$V = x^2 y$

$V = x^2 (12/x - x/4)$

$V = 12x - (1/4)x^3$.

Now differentiate this equation, getting
$V' = 12 - (1/4)3x^2$

$V' = 12 - (3/4)x^2$

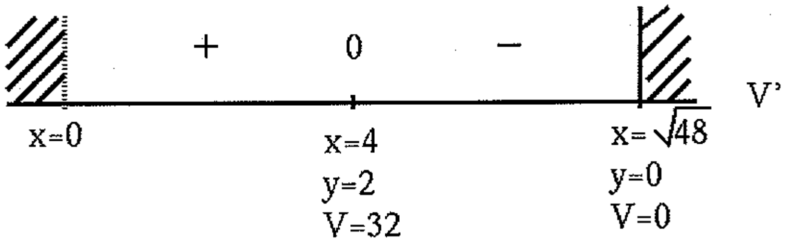
$= (3/4)(16 - x^2)$

$= (3/4)(4 - x)(4 + x)$

$= 0$ for $x = 4$ or $x = -4$.

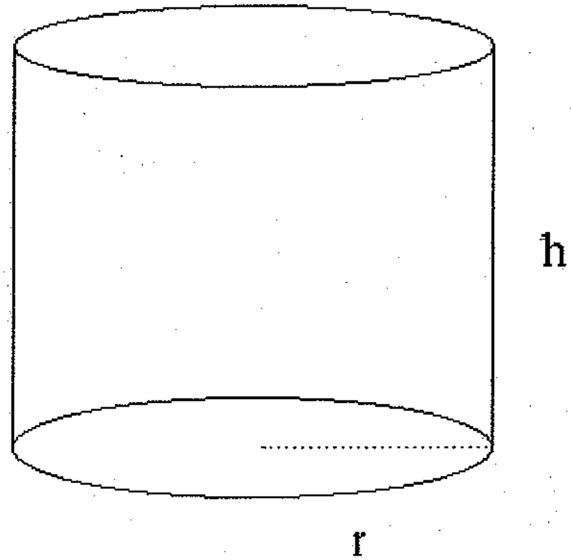
But $x \neq -4$ since variable x measures a distance and $x > 0$. Since the base of the box is square and there are 48 ft^2 of material, it follows that $0 < x \leq$ sqrt(48) .

See the adjoining sign chart for V' .



If $x = 4$ ft. and $y = 2$ ft. , then $V = 32$ ft.^{3} is the largest possible volume of the box.

SOLUTION 4 : Let variable r be the radius of the circular base and variable h the height of the cylinder.



The total surface area of the cylinder is given to be

$$3\pi = \text{(area of base)} + \text{(area of the curved side)}$$

$$= \pi r^2 + (2\pi r)h \ ,$$

so that

$$2\pi r h = 3\pi - \pi r^2$$

or

$$h = \frac{3\pi - \pi r^2}{2\pi r}$$

$$= \frac{3}{2r} - (1/2)r \ .$$

We wish to MAXIMIZE the total VOLUME of the cylinder

$$V = \text{(area of base) (height)} = \pi r^2 h \ .$$

However, before we differentiate the right-hand side, we will write it as a function of r only. Substitute for h getting

$$V = \pi r^2 h$$

$$= \pi r^2 \left(\frac{3}{2r} - (1/2)r\right)$$

$$= (3/2)\pi r - (1/2)\pi r^3 .$$

Now differentiate this equation, getting

$$V' = (3/2)\pi - (1/2)\pi 3r^2$$

$$= (3/2)\pi(1 - r^2)$$

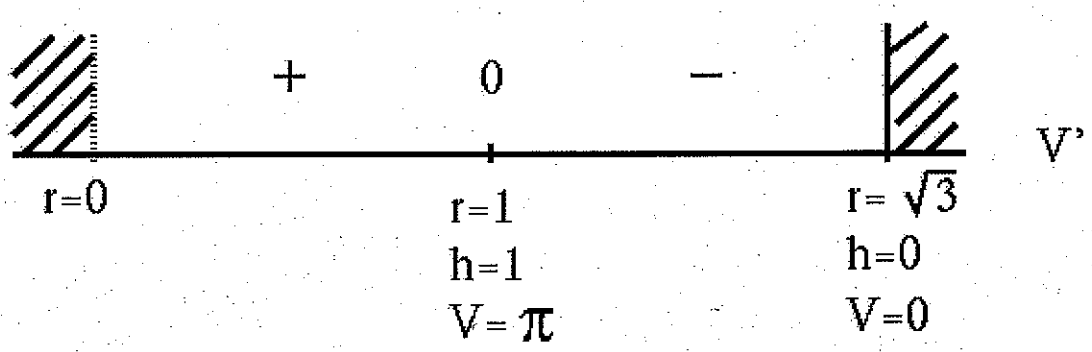
$$= (3/2)\pi(1 - r)(1 + r)$$

$$= 0$$

for

$$r=1 \text{ or } r=-1 .$$

But $r \neq -1$ since variable r measures a distance and $r > 0$. Since the base of the box is a circle and there are 3π ft.^{2} of material, it follows that $0 < r \leq \sqrt{3}$. See the adjoining sign chart for V'.



If

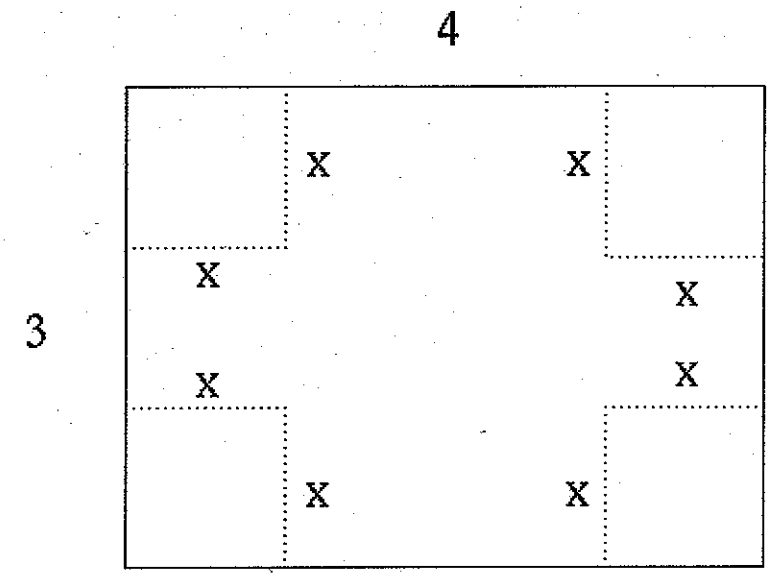
$$r=1 \text{ ft. and } h=1 \text{ ft.},$$

then

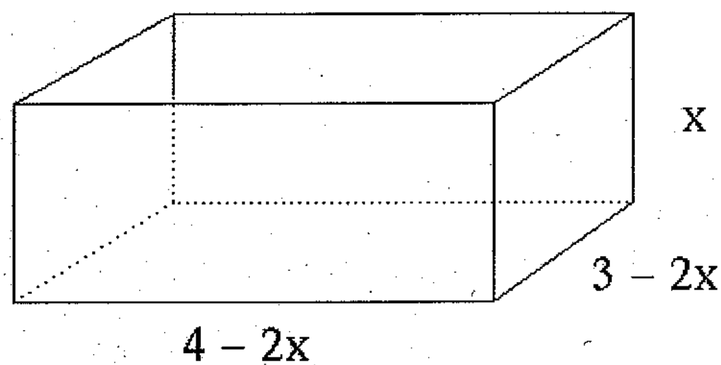
$$V = \pi \text{ ft.}^3$$

is the largest possible volume of the cylinder.

SOLUTION 5 : Let variable x be the length of one edge of the square cut from each corner of the sheet of cardboard.



After removing the corners and folding up the flaps, we have an ordinary rectangular box.



We wish to MAXIMIZE the total VOLUME of the box

$$V = \text{(length) (width) (height)} = (4\text{-}2x)\ (3\text{-}2x)\ (x)\ .$$

Now differentiate this equation using the triple product rule, getting

$$V' = (-2)\ (3-2x)\ (x) + (4-2x)\ (-2)\ (x) + (4-2x)\ (3-2x)\ (1)$$

$$= -6x + 4x^2 - 8x + 4x^2 + 4x^2 - 14x + 12$$

$$= 12x^2 - 28x + 12$$

$$= 4\ (\ 3x^2 - 7x + 3\ )$$

$$= 0$$

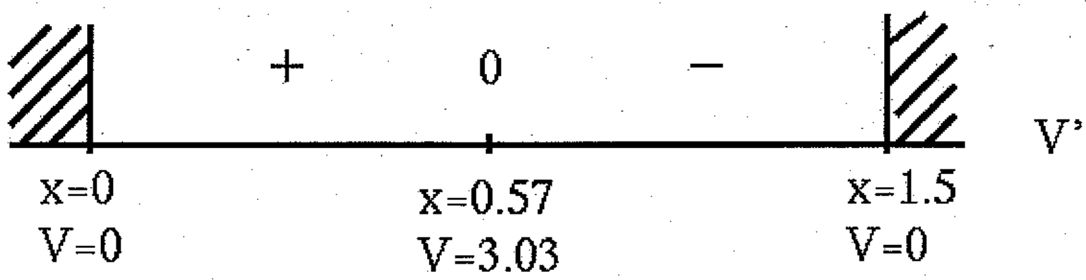
for (Use the quadratic formula.)

$$x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(3)(3)}}{2(3)} = \frac{7 \pm \sqrt{13}}{6}\ ,$$

i.e., for

$$x \approx 0.57 \text{ or } x \approx 1.77 .$$

But $x \neq 1.77$ since variable x measures a distance. In addition, the short edge of the cardboard is 3 ft., so it follows that $0 \le x \le 1.50$. See the adjoining sign chart for V' .



If

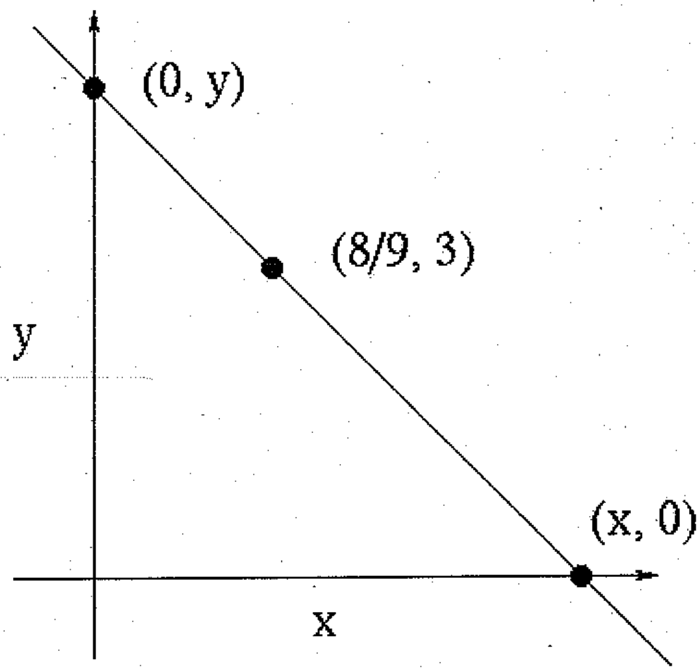
$$x \approx 0.57 \text{ ft.},$$

then

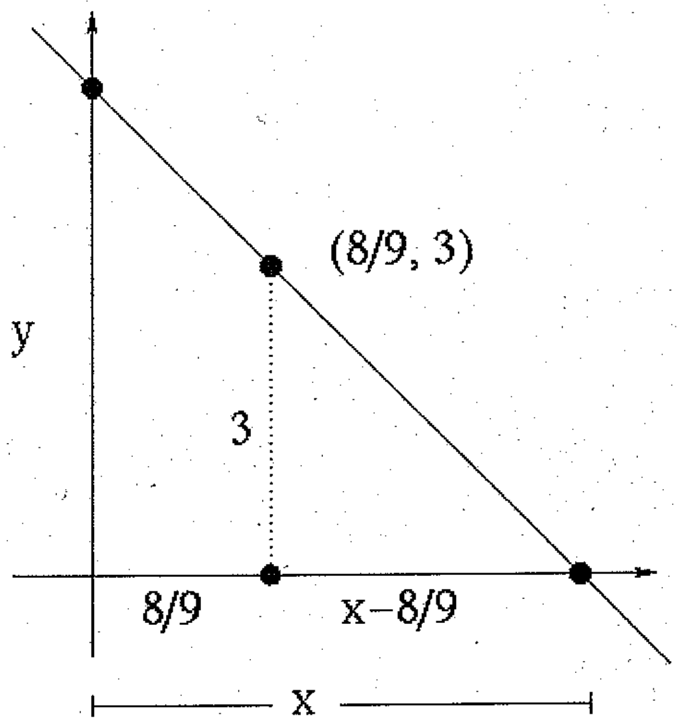
$$V \approx 3.03 \text{ ft.}^3$$

is largest possible volume of the box.

SOLUTION 6 : Let variable x be the x-intercept and variable y the y-intercept of the line passing throught the point (8/9, 3) .



Set up a relationship between x and y using similar triangles.



One relationship is

$$\frac{y}{x} = \frac{3}{x-8/9} ,$$

so that

$$y = \frac{3x}{x-8/9} .$$

We wish to MINIMIZE the length of the HYPOTENUSE of the triangle

$$H = \sqrt{x^2+y^2} .$$

However, before we differentiate the right-hand side, we will write it as a function of *x* only. Substitute for *y* getting

$$H = \sqrt{x^2+y^2}$$

$$= \sqrt{x^2 + \left(\frac{3x}{x-8/9}\right)^2} .$$

Now differentiate this equation using the chain rule and quotient rule, getting

$$H' = (1/2)\left(x^2 + \left(\frac{3x}{x-8/9}\right)^2\right)^{-1/2}\left\{2x + 2\left(\frac{3x}{x-8/9}\right)\frac{(x-8/9)(3)-(3x)(1)}{(x-8/9)^2}\right\}$$

(Factor a 2 out of the big brackets and simplify.)

$$= (1/2)\left(x^2 + \left(\frac{3x}{x-8/9}\right)^2\right)^{-1/2}(2)\left\{x + \frac{3x}{(x-8/9)}\frac{-8/3}{(x-8/9)^2}\right\}$$

$$= \frac{x - \dfrac{8x}{(x-8/9)^3}}{\sqrt{x^2 + \left(\dfrac{3x}{x-8/9}\right)^2}}$$

$$= 0 ,$$

so that (If $\frac{A}{B} = 0$, then $A=0$.)

$$x - \frac{8x}{(x-8/9)^3} = 0 .$$

By factoring out x , it follows that

$$x\left\{1-\frac{8}{(x-8/9)^3}\right\}=0 ,$$

so that (If $AB=0$, then $A=0$ or $B=0$.)

$$x=0$$

(Impossible, since $x> 8/9$. Why ?) or

$$1-\frac{8}{(x-8/9)^3}=0 .$$

Then

$$1=\frac{8}{(x-8/9)^3} ,$$

so that

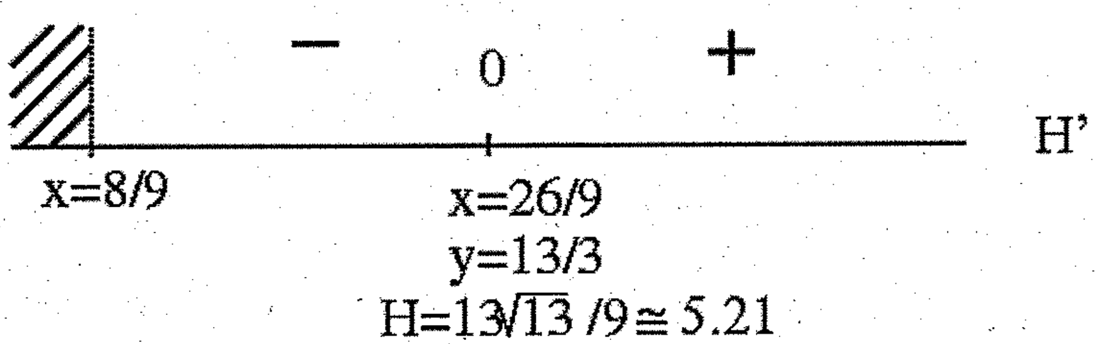
$$(x-8/9)^3 = 8 ,$$

$$x-8/9 = 2 ,$$

and

$$x = 26/9 .$$

See the adjoining sign chart for H' .



If

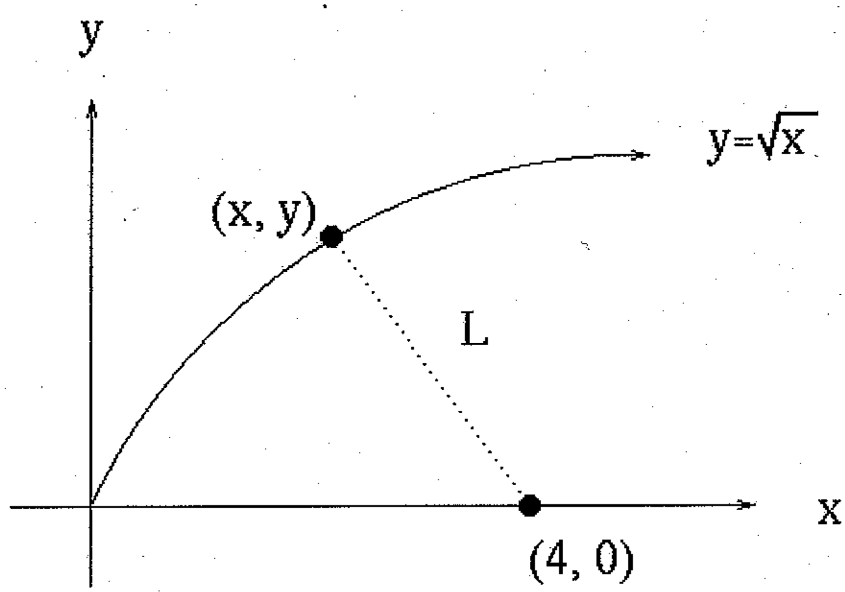
$$x = 26/9 \text{ and } y=13/3 ,$$

then

$$H=\frac{13\sqrt{13}}{9}\approx 5.21$$

is the shortest possible hypotenuse.

SOLUTION 7 : Let (x, y) represent a randomly chosen point on the graph of $y = \sqrt{x}$.



We wish to MINIMIZE the DISTANCE between points (x, y) and $(4, 0)$,

$$L = \sqrt{(x-4)^2 + (y-0)^2}$$

$$= \sqrt{(x-4)^2 + y^2} \ .$$

However, before we differentiate the right-hand side, we will write it as a function of x only. Substitute for y getting

$$L = \sqrt{(x-4)^2 + y^2}$$

$$= \sqrt{(x-4)^2 + (\sqrt{x})^2}$$

$$= \sqrt{(x-4)^2 + x} \ .$$

Now differentiate this equation using the chain rule, getting

$$L' = (1/2)\Big((x-4)^2 + x\Big)^{-1/2}\{2(x-4)+1\}$$

$$= \frac{2x-7}{2\sqrt{(x-4)^2+y^2}}$$

$$= 0 \ ,$$

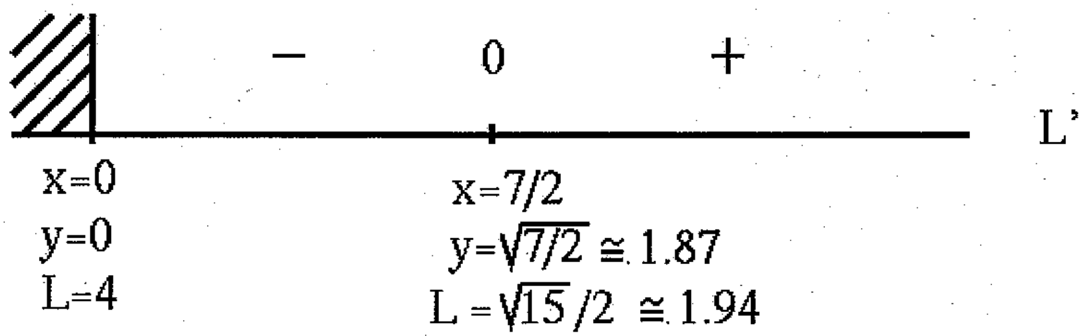
so that (If $\frac{A}{B} = 0$, then A=0 .)

$$2x-7=0\ ,$$

or

$$x=7/2\ .$$

See the adjoining sign chart for L' .



If

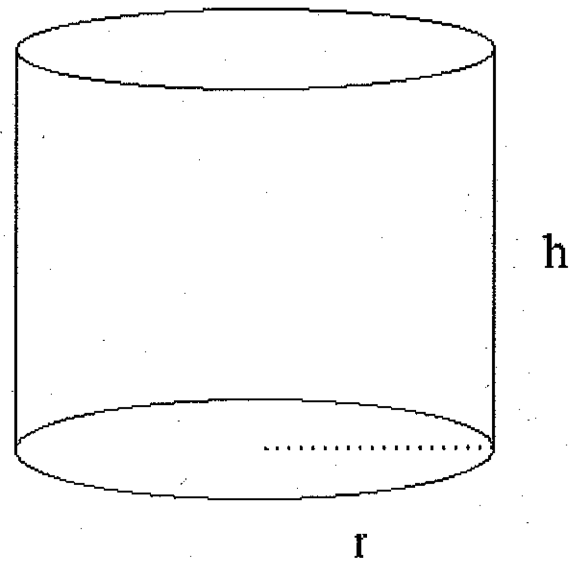
$$x=7/2 \text{ and } y=\sqrt{7/2}\approx 1.87\ ,$$

then

$$L=\frac{\sqrt{15}}{2}\approx 1.94$$

is the shortest possible distance from (4, 0) to the graph of $y=\sqrt{x}$.

SOLUTION 8 : Let variable r be the radius of the circular base and variable h the height of the cylinder.



The total volume of the cylinder is given to be

$$20\pi = \text{(area of base) (height)} = (\pi r^2)h \ ,$$

so that

$$h = \frac{20\pi}{\pi r^2}$$

$$= \frac{20}{r^2} \ .$$

We wish to MINIMIZE the total COST of construction of the cylinder

C = (total cost of bottom) + (total cost of top) + (total cost of side)

= (unit cost of bottom)(area of bottom) + (unit cost of top)(area of top) + (unit cost of side) (area of side)

$$= \$10(\pi r^2) + \$10(\pi r^2) + \$8(2\pi r h)$$

(For convenience drop the **$** signs until the end of the problem.)

$$= 20\pi r^2 + 16\pi r h \ .$$

However, before we differentiate the right-hand side, we will write it as a function of r only. Substitute for h getting

$$C = 20\pi r^2 + 16\pi r h$$

$$= 20\pi r^2 + 16\pi r\left(\frac{20}{r^2}\right)$$

$$= 20\pi r^2 + \frac{320\pi}{r} \ .$$

Now differentiate this equation, getting

$$C' = 40\pi r + 320\pi\left\{\frac{-1}{r^2}\right\}$$

$$= 40\pi r - \frac{320\pi}{r^2}$$

(Get a common denominator and combine fractions.)

$$= 40\pi r\left\{\frac{r^2}{r^2}\right\} - \frac{320\pi}{r^2}$$

$$= \frac{40\pi r^3 - 320\pi}{r^2}$$

$$= \frac{40\pi(r^3 - 8)}{r^2}$$

$$= 0 \ ,$$

so that (If $\frac{A}{B} = 0$, then $A=0$.)

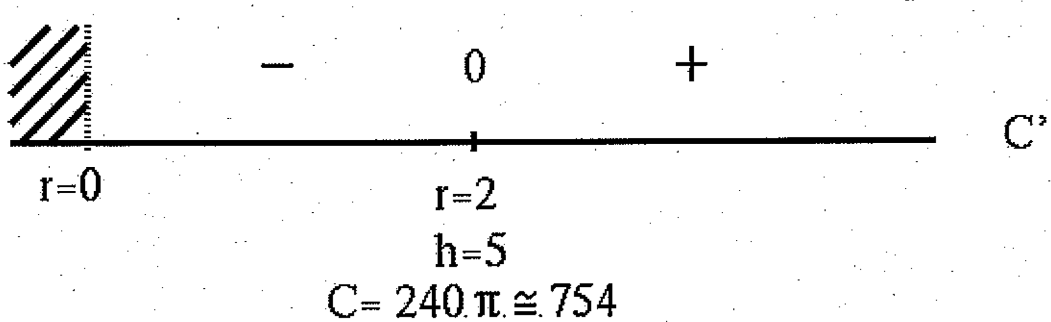
$$40\pi(r^3 - 8) = 0 \ ,$$

$$r^3 = 8 \ ,$$

or

$$r = 2 \ .$$

Since variable r measures a distance, it must satisfy $r > 0$. See the adjoining sign chart for C' .



If

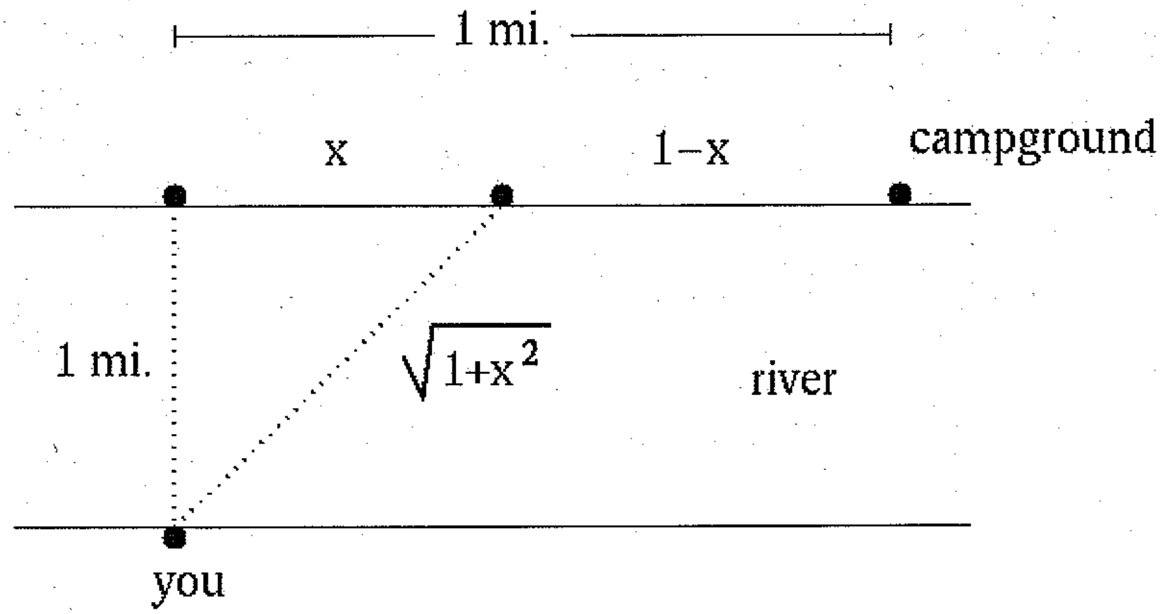
$$r=2 \text{ m. and } h=5 \text{ m.},$$

then

$$C=\$240\pi \approx \$754$$

is the least possible cost of construction.

SOLUTION 9 : Let variable x be the distance denoted in the given diagram.



Assume that you travel at the following rates :

SWIM : 2 mph

WALK : 3 mph .

Recall that if travel is at a CONSTANT rate of speed, then

(distance traveled) = (rate of travel) (time elapsed)

or

$$D = R\,T ,$$

so that time elapsed is

$$T = \frac{D}{R} .$$

We wish to MINIMIZE the total TIME elapsed

$$T = \text{(swim time)} + \text{(walk time)}$$

$$= \text{(swim distance)/(swim rate)} + \text{(walk distance)/(walk rate)}$$

$$= \frac{\sqrt{1+x^2}}{2} + \frac{1-x}{3}$$

$$= (1/2)\sqrt{1+x^2} + (1/3) - (1/3)x .$$

Now differentiate this equation, getting

$$T' = (1/2)(1/2)\{1+x^2\}^{-1/2}(2x) - 1/3$$

$$= \frac{x}{2\sqrt{1+x^2}} - \frac{1}{3}$$

$$= 0 \, ,$$

so that

$$\frac{x}{2\sqrt{1+x^2}} = \frac{1}{3}$$

and

$$3x = 2\sqrt{1+x^2} \, .$$

Square both sides of this equation, getting

$$9x^2 = 4\,(1 + x^2) = 4 + 4x^2 \, ,$$

so that

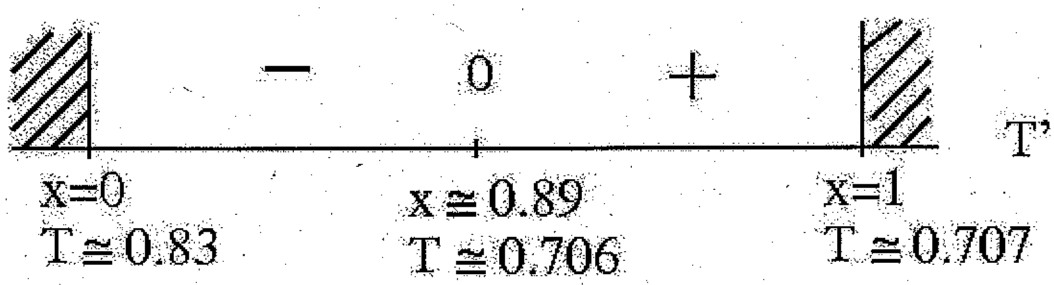
$$5x^2 = 4 \, ,$$

$$x^2 = 4/5 \, ,$$

or

$$x = \frac{\pm 2}{\sqrt{5}} \approx \pm 0.89 \, .$$

But $x \neq \frac{-2}{\sqrt{5}}$ since variable x measures a distance and $0 \le x \le 1$. See the adjoining sign chart for T .



If

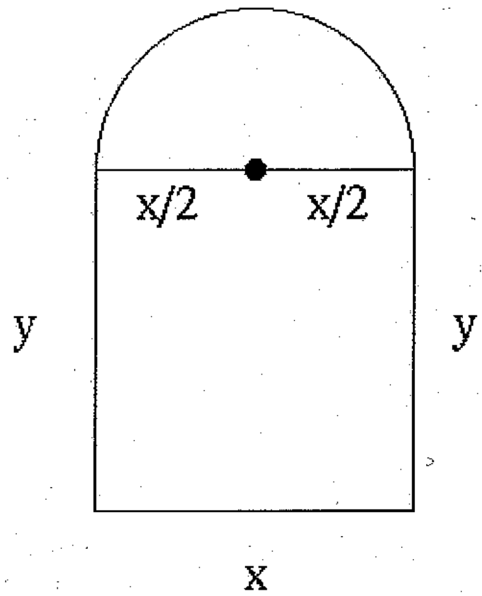
$$x = \frac{2}{\sqrt{5}} \approx 0.89 \text{ mi.}$$

then

$$T \approx 0.71 \text{ hr.}$$

is the shortest possible time of travel.

SOLUTION 10 : Let variable *x* be the width and variable *y* the length of the rectangular portion of the window.



The semi-circular portion of the window has length

$$C = (1/2)(2\pi)(x/2) = \pi x/2$$

The perimeter (distance around outside only) of the window is given to be

$$12$$

so that

$$12 = y + x + y + \pi x/2 = 2y + x + \pi x/2$$

or

$$y = \frac{12 - x - \frac{\pi x}{2}}{2} = 6 - \frac{x}{2} - \frac{\pi x}{4}$$

We wish to MAXIMIZE the total AREA of the RECTANGLE

$$A = \text{(width) (length)} = x\,y\,.$$

However, before we differentiate the right-hand side, we will write it as a function of *x* only. Substitute for *y* getting

$$A = x\,y$$

$$= x(6 - (1/2)x - (\pi/4)x)$$

$$= 6x - (1/2)x^2 - (\pi/4)x^2\,.$$

Now differentiate this equation, getting

$$A' = 6 - (1/2)(2)x - (\pi/4)(2)x$$

$$= 6 - (1 + \pi/2)x$$

$$= 0$$

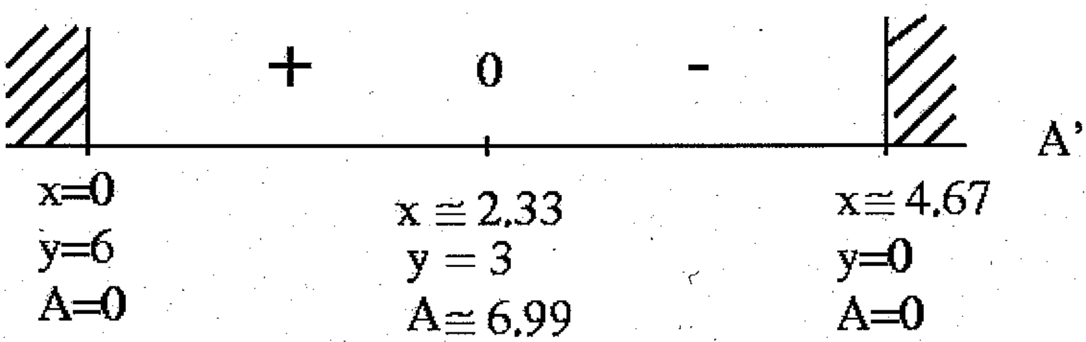
for

$$x = \frac{6}{1+\pi/2} = \frac{12}{2+\pi},$$

i.e.,

$$x \approx 2.33 .$$

Since variable x measures distance, $x \geq 0$. In addition, x is largest when $y = 0$ and the window is in the shape of a semi-circle. Thus, $0 \leq x \leq \frac{24}{2+\pi} \approx 4.67$ (Why ?). See the adjoining sign chart for A' .



If

$x \approx 2.33$ ft. and y=3 ft. ,

then

$A \approx 6.99$ ft.^{2}

is the largest possible area of the rectangle.

SOLUTION 11 : Let variable x be the ADDITIONAL trees planted in the existing orchard. We wish to MAXIMIZE the total PRODUCTION of apples

$$P = \text{(number of trees) (apple output per tree)}$$

$$= (50 + x) (800 - 10x)$$

$$= 40{,}000 + 300\,x - 10\,x^2 .$$

Now differentiate this equation, getting

$$P' = 300 - 20\,x$$

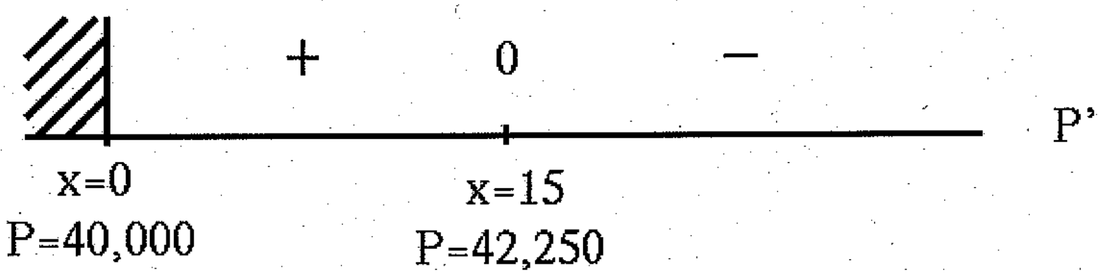
$$= 20 (15 - x)$$

$$= 0$$

for

$$x = 15 .$$

See the adjoining sign chart for P' .



If

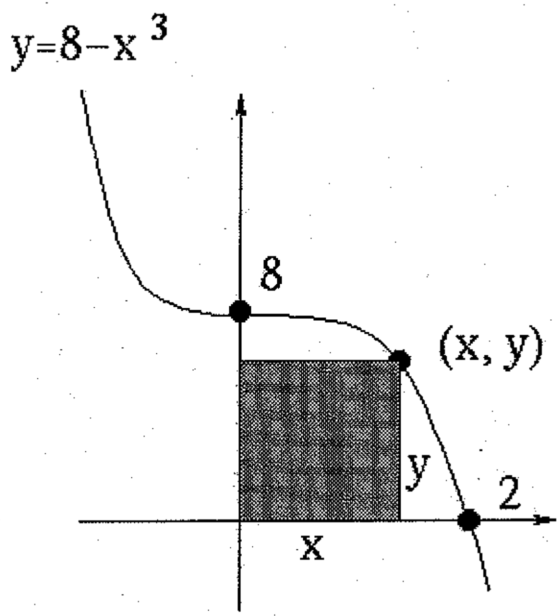
$$x = 15 \text{ additional trees ,}$$

then

$$P = 42{,}250 \text{ apples}$$

is the largest possible production of apples.

SOLUTION 12 : Let variable x be the length of the base and variable y the height of the inscribed rectangle.



We wish to MAXIMIZE the total AREA of the rectangle

$$A = \text{(length of base) (height)} = xy \,.$$

However, before we differentiate the right-hand side, we will write it as a function of x only. Substitute for y getting

$$A = x\,y$$

$$= x\,(\,8 - x^3\,)$$

$$= 8x - x^4 \,.$$

Now differentiate this equation, getting

$$A' = 8 - 4\,x^3$$

$$= 4\,(\,2 - x^3\,)$$

$$= 0 \,,$$

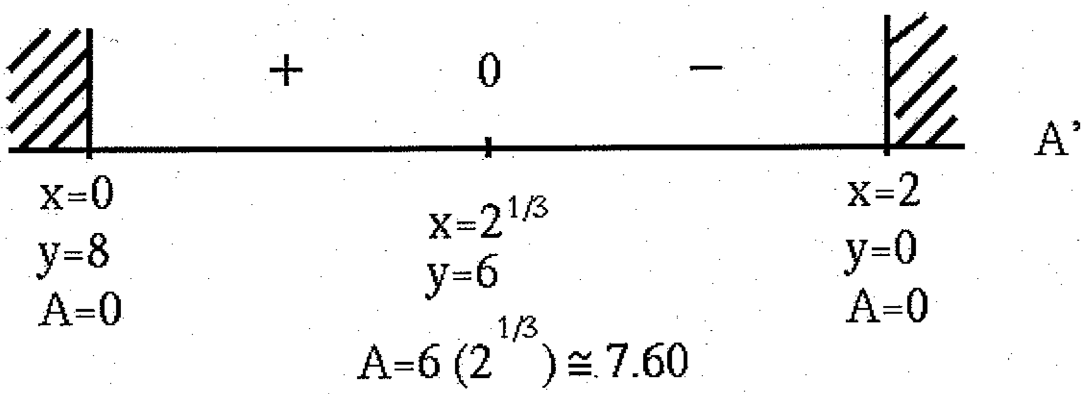
so that

$$x^3 = 2$$

and

$$x = 2^{1/3} \approx 1.26 \ .$$

Note that $0 \leq x \leq 2$. See the adjoining sign chart for A' .



If

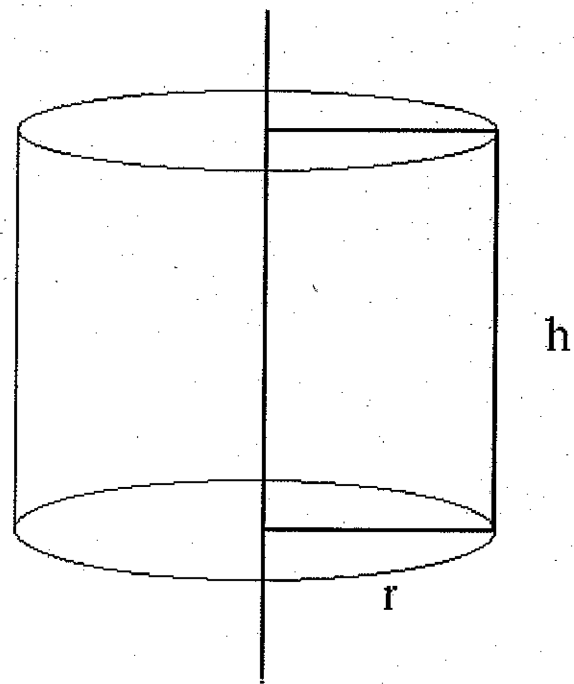
$$x = 2^{1/3} \approx 1.26 \text{ and } y = 6 \ ,$$

then

$$A = 6(2^{1/3}) \approx 7.60$$

is the largest possible area for the inscribed rectangle.

SOLUTION 13 : Let variable r be the length of the base and variable h the height of the rectangle.



It is given that the perimeter of the rectangle is

$$12 = 2r + 2h$$

so that

$$2h = 12 - 2r$$

and

$$h = 6 - r\ .$$

We wish to MAXIMIZE the total VOLUME of the resulting CYLINDER

$$V = \text{(area of base) (height)} = (\pi r^2)h\ .$$

However, before we differentiate the right-hand side, we will write it as a function of r only. Substitute for h getting

$$V = \pi r^2 h$$

$$= \pi r^2 (6 - r)$$

$$= \pi (6r^2 - r^3)\ .$$

Now differentiate this equation, getting

$$V' = \pi (12r - 3r^2)$$

$$= \pi(3r)(4-r)$$

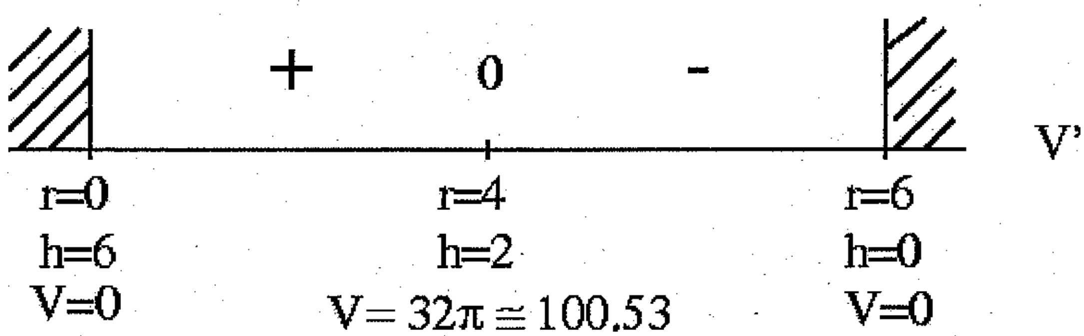
$$= 3\pi r(4-r)$$

$$= 0$$

for

$$r=0 \text{ or } r=4 .$$

Since variable r measures distance and the perimeter of the rectangle is 12, $0 \leq r \leq 6$. See the adjoining sign chart for V' .



If

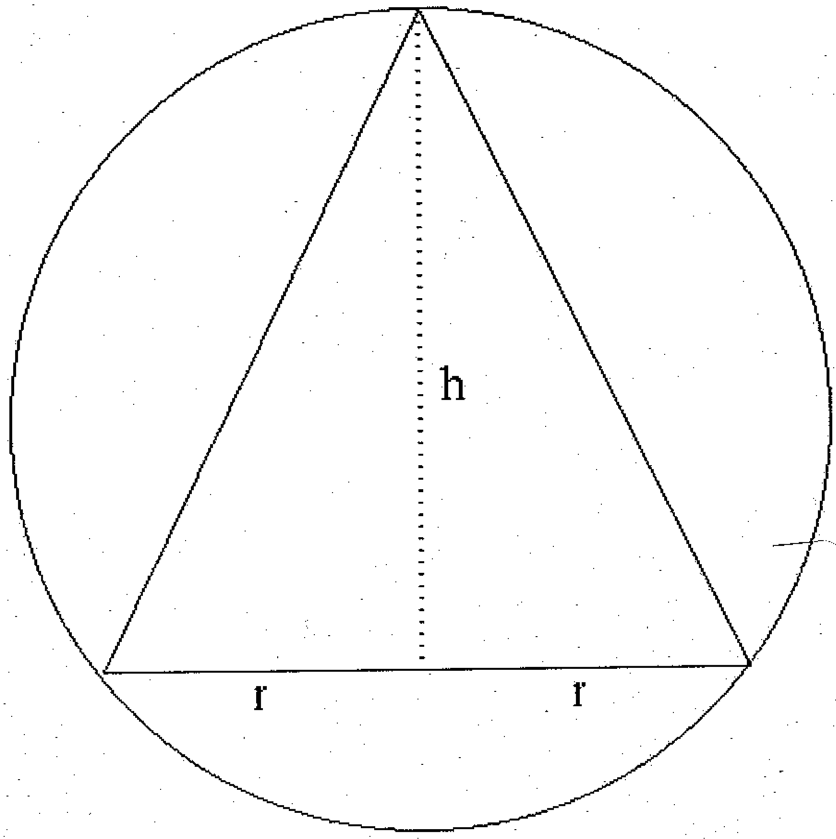
$$r = 4 \text{ ft. and } h = 2 \text{ ft.},$$

then

$$V = 32\pi \text{ ft.}^3 \approx 100.53 \text{ ft.}^3$$

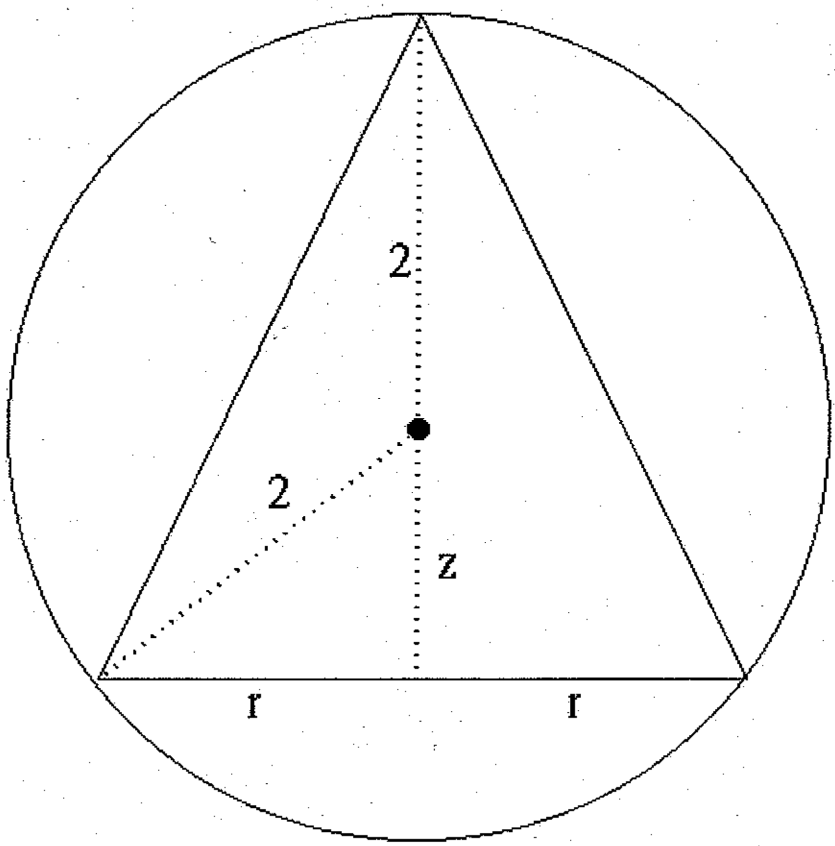
is the largest possible volume for the cylinder.

SOLUTION 14 Let variable r be the radius of the circular base and variable h the height of the inscribed cone as shown in the two-dimensional side view.



two–
dimensional
side view

It is given that the circle's radius is 2. Find a relationship between r and h . Let variable z be the height of the small right triangle.



two–
dimensional
side view

By the Pythagorean Theorem it follows that

$$r^2 + z^2 = 2^2$$

so that

$$z^2 = 4 - r^2$$

or

$$z = \sqrt{4 - r^2} \ .$$

Thus the height of the inscribed cone is

$$h = 2 + z = 2 + \sqrt{4 - r^2} \ .$$

We wish to MINIMIZE the total VOLUME of the CONE

$$V = (1/3)\pi r^2 h \ .$$

However, before we differentiate the right-hand side, we will write it as a function of r only. Substitute for h getting

$$V = (1/3)\pi r^2 h$$

$$= (1/3)\pi r^2(2 + \sqrt{4 - r^2})\ .$$

Now differentiate this equation using the product rule and the chain rule, getting

$$V' = (1/3)\pi r^2\Big\{0 + (1/2)(4 - r^2)^{-1/2}(-2r)\Big\} + (1/3)\pi 2r\big(2 + \sqrt{4 - r^2}\big)$$

(Factor out $(\pi/3)$, get a common denominator, and simplify fractions.)

$$= \frac{\pi}{3}\Big\{\frac{-r^3}{\sqrt{4 - r^2}} + 2r\big(2 + \sqrt{4 - r^2}\big)\Big\}$$

$$= \frac{\pi}{3}\Big\{\frac{-r^3}{\sqrt{4 - r^2}} + 2r\big(2 + \sqrt{4 - r^2}\big)\frac{\sqrt{4 - r^2}}{\sqrt{4 - r^2}}\Big\}$$

$$= \frac{\pi}{3}\Big\{\frac{-r^3 + 2r\big(2 + \sqrt{4 - r^2}\big)\sqrt{4 - r^2}}{\sqrt{4 - r^2}}\Big\}$$

$$= \frac{\pi}{3}\Big\{\frac{-r^3 + 2r\big(2\sqrt{4 - r^2} + (4 - r^2)\big)}{\sqrt{4 - r^2}}\Big\}$$

(Factor out (r) .)

$$= \frac{\pi}{3}(r)\Big\{\frac{-r^2 + 2\big(2\sqrt{4 - r^2} + (4 - r^2)\big)}{\sqrt{4 - r^2}}\Big\}$$

$$= 0\ ,$$

so that (If $AB = 0$, then A=0 or B=0 .)

$$r = 0$$

or

$$\frac{-r^2 + 2(2\sqrt{4 - r^2} + (4 - r^2))}{\sqrt{4 - r^2}} = 0\ ,$$

i.e., (If $\frac{A}{B} = 0$, then A=0 .)

$$-r^2 + 2(2\sqrt{4 - r^2} + (4 - r^2)) = 0\ .$$

Then (Isolate the square root term.)

$$4\sqrt{4 - r^2} + 8 - 2r^2 = r^2\ ,$$

$$4\sqrt{4-r^2} = 3r^2 - 8 \ ,$$

(Square both sides of this equation.)

$$16(\sqrt{4-r^2})^2 = (3r^2-8)^2 \ ,$$

$$16 (4 - r^2) = 9 r^4 - 48 r^2 + 64 \ ,$$

$$64 - 16 r^2 = 9 r^4 - 48 r^2 + 64 \ ,$$

$$32 r^2 - 9 r^4 = 0 \ ,$$

$$r^2 (32 - 9 r^2) = 0 \ ,$$

so that

$$r = 0$$

or

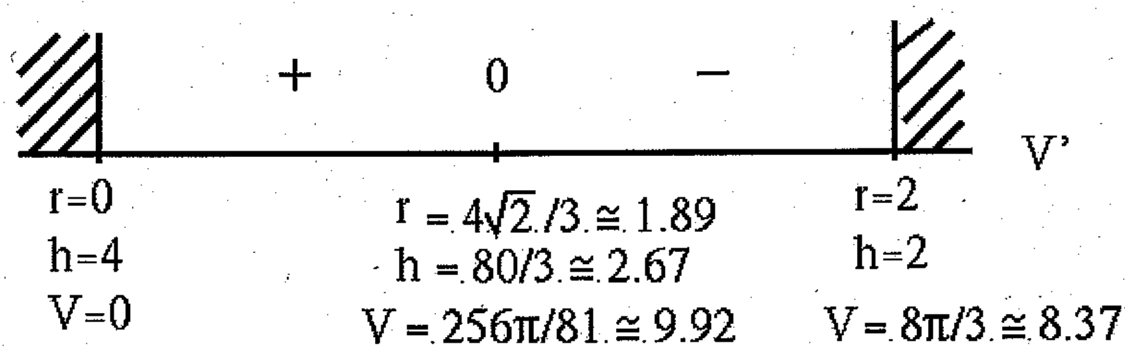
$$32 - 9 r^2 = 0 \ ,$$

$$r^2 = 32/9 \ ,$$

or

$$r = \pm\sqrt{32/9} = \pm 4\sqrt{2}/3 \approx \pm 1.89 \ .$$

But $r \neq -4\sqrt{2}/3$ since variable r measures a distance and $0 \le r \le 2$. See the adjoining sign chart for V' .



If

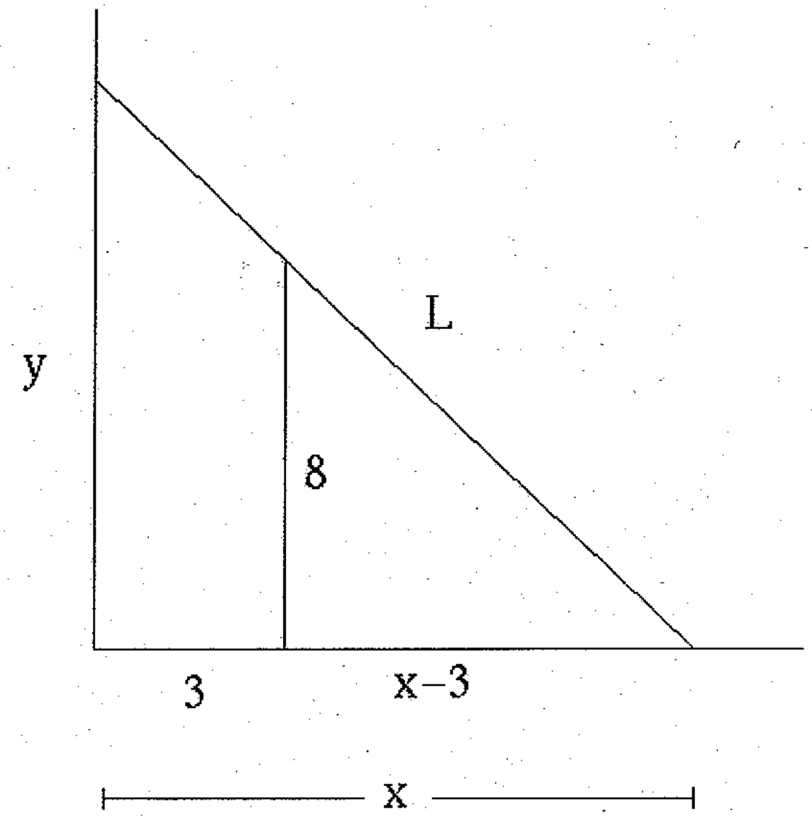
$$r = 4\sqrt{2}/3 \approx 1.89 \text{ and } h = 8/3 \approx 2.67 \ ,$$

then

$$V \approx 9.93$$

is the largest possible volume for the inscribed cone.

SOLUTION 15 Let variable L be the length of the ladder resting on the top of the fence and touching the wall behind it. Let variables x and y be the lengths as shown in the diagram.



Write L as a function of x . First find a relationship between y and x using similar triangles. For example,

$$\frac{y}{x}=\frac{8}{x-3}$$

so that

$$y=\frac{8x}{x-3}.$$

We wish to MINIMIZE the LENGTH of the ladder

$$L=\sqrt{x^2+y^2}.$$

Before we differentiate, rewrite the right-hand side as a function of x only. Then

$$L=\sqrt{x^2+y^2}$$

$$=\sqrt{x^2+\left(\frac{8x}{x-3}\right)^2}$$

$$=\sqrt{x^2+\frac{64x^2}{(x-3)^2}}.$$

Now differentiate this equation using the chain rule and quotient rule, getting

$$D' = (1/2)\left(x^2 + \frac{64x^2}{(x-3)^2}\right)^{-1/2}\left\{2x + \frac{(x-3)^2(128x) - (64x^2)2(x-3)}{(x-3)^4}\right\}$$

(Factor out 64*x* and (*x*-3) from the numerator of the fraction inside the brackets.)

$$= (1/2)\left(x^2 + \frac{64x^2}{(x-3)^2}\right)^{-1/2}\left\{2x + \frac{64x(x-3)[2(x-3) - 2x]}{(x-3)^4}\right\}$$

(Divide out a factor of (*x*-3) and simplify the entire expression.)

$$= \frac{2x + \frac{64x(-6)}{(x-3)^3}}{2\sqrt{x^2 + \frac{64x^2}{(x-3)^2}}}$$

(Factor out 2*x* from the numerator.)

$$= \frac{2x\left\{1 + \frac{-192}{(x-3)^3}\right\}}{2\sqrt{x^2 + \frac{64x^2}{(x-3)^2}}}$$

$$= \frac{x\left\{1 - \frac{192}{(x-3)^3}\right\}}{\sqrt{x^2 + \frac{64x^2}{(x-3)^2}}}$$

$$= 0 ,$$

so that (If $\frac{A}{B} = 0$, then $A = 0$.)

$$x\left\{1 - \frac{192}{(x-3)^3}\right\} = 0 .$$

Then (If $AB = 0$, then $A = 0$ or $B = 0$.)

$$x=0$$

or

$$1 - \frac{192}{(x-3)^3} = 0 ,$$

$$1 = \frac{192}{(x-3)^3} ,$$

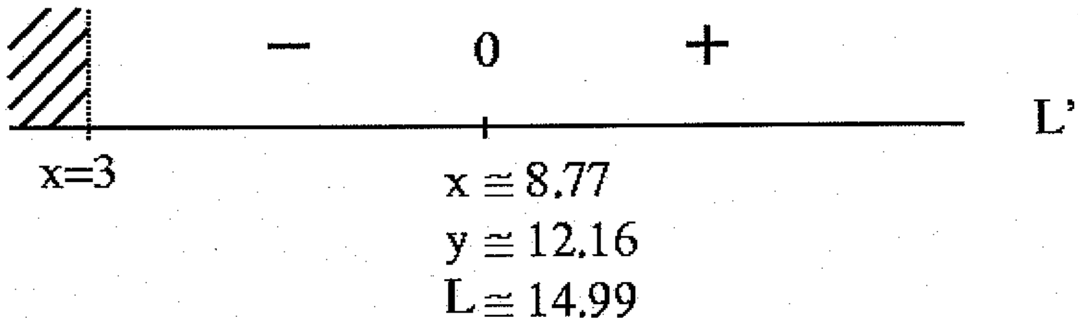
$$(x\text{-}3)^3 = 192 ,$$

$$x - 3 = 192^{1/3} \approx 5.77 ,$$

and

$$x \approx 8.77 .$$

Note that $x > 3$. See the adjoining sign chart for L' .



If

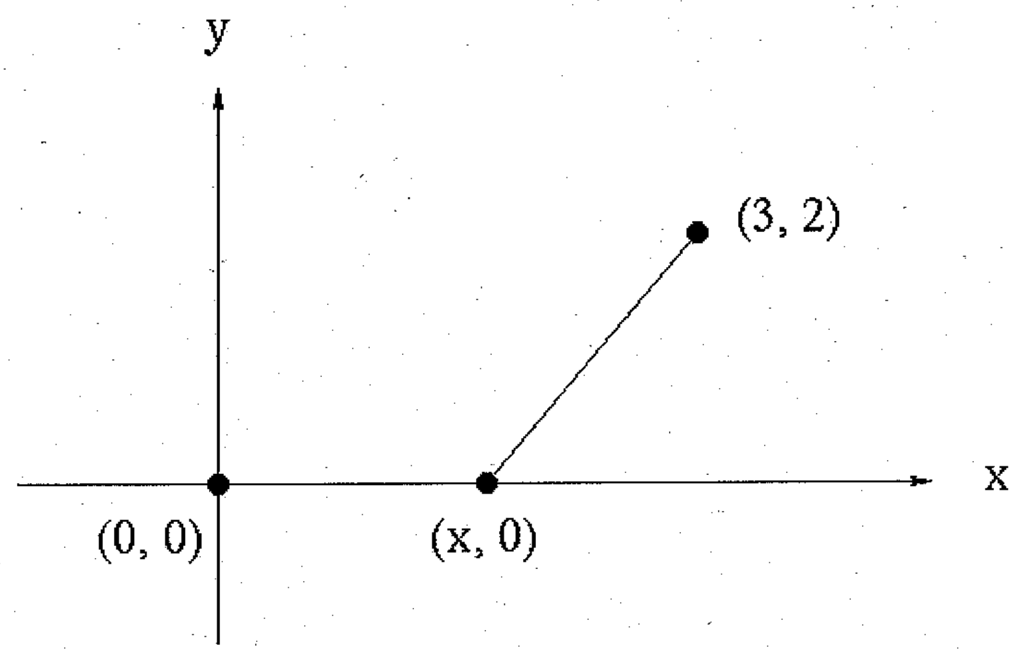
$$x \approx 8.77 \text{ ft. and } y \approx 16.67 \text{ ft.} ,$$

then

$$L \approx 17.64 \text{ ft.}$$

is the length of the shortest possible ladder.

SOLUTION 16



Let variable S be the sum of the squares of the distances between (0, 0) and $(x, 0)$,

$$\left(\sqrt{(x-0)^2+(0-0)^2}\right)^2 = x^2 ,$$

and between (3, 2) and $(x, 0)$,

$$\left(\sqrt{(x-3)^2+(0-2)^2}\right)^2 = (x-3)^2+4 = x^2-6x+13 .$$

We wish to MINIMIZE the SUM of the squares of the distances

$$S = x^2 + (x^2 -6x +13) = 2x^2 -6x +13 .$$

Now differentiate, getting

$$S' = 4x -6$$

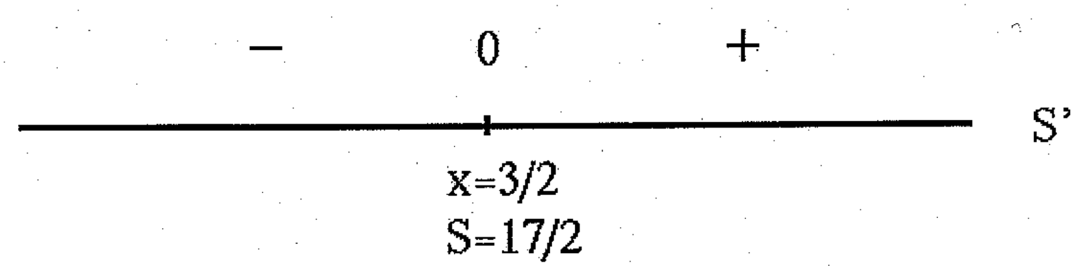
$$= 4(x - 3/2)$$

$$= 0$$

for

$$x= 3/2 .$$

See the adjoining sign chart for S' .



S=17/2

If

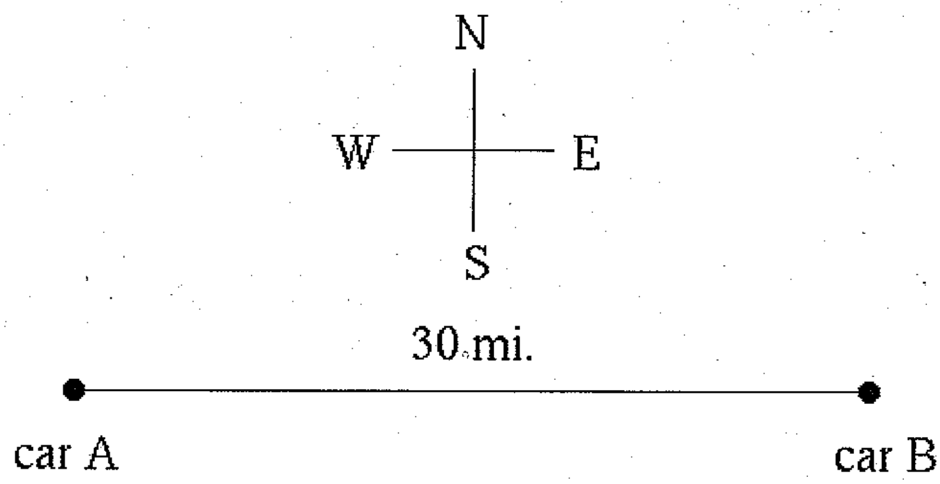
$$x = 3/2\,,$$

then

$$S = 17/2$$

is the smallest sum.

SOLUTION 17

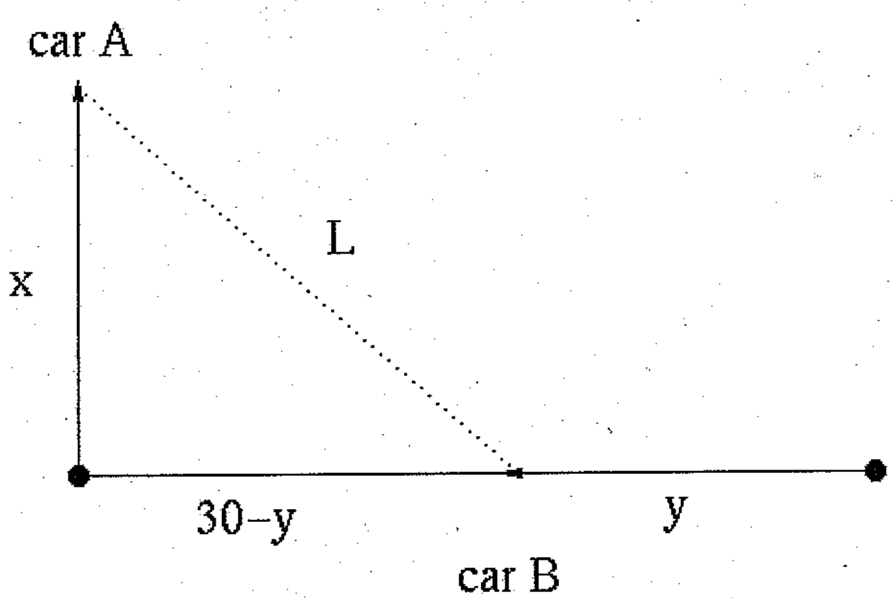


Assume that the two cars travel at the following rates :

CAR A : 60 mph

CAR B : 90 mph

Let variable x be the distance car A travels in t hours, and variable y the distance car B travels in t hours. Let variable L be the distance between cars A and B after t hours.



Thus, by the Pythagorean Theorem distance L is

$$L = \sqrt{x^2 + (30 - y)^2} \ .$$

Before we differentiate, we will rewrite the right-hand side as a function of t only. Recall that if travel is at a CONSTANT rate then

(distance traveled) = (rate of travel) (time elapsed) .

Thus, for car A the distance traveled after t hours is

(Equation 1)

$$x = 60\,t \ ,$$

and for car B the distance traveled after t hours is

(Equation 2)

$$y = 90\,t\,.$$

Use Equations 1 and 2 to rewrite the equation for L as a function of t only. Thus, we wish to MINIMIZE the DISTANCE between the two cars

$$L = \sqrt{x^2 + (30 - y)^2}$$

$$= \sqrt{(60t)^2 + (30 - 90t)^2}$$

$$= \sqrt{3600t^2 + (30 - 90t)^2}\,.$$

Differentiate, using the chain rule, getting

$$L' = (1/2)\left(3600t^2 + (30 - 90t)^2\right)^{-1/2}\{7200t + 2(30 - 90t)(-90)\}$$

$$= \frac{23,400t - 5400}{2\sqrt{3600t^2 + (30 - 90t)^2}}$$

$$= 0$$

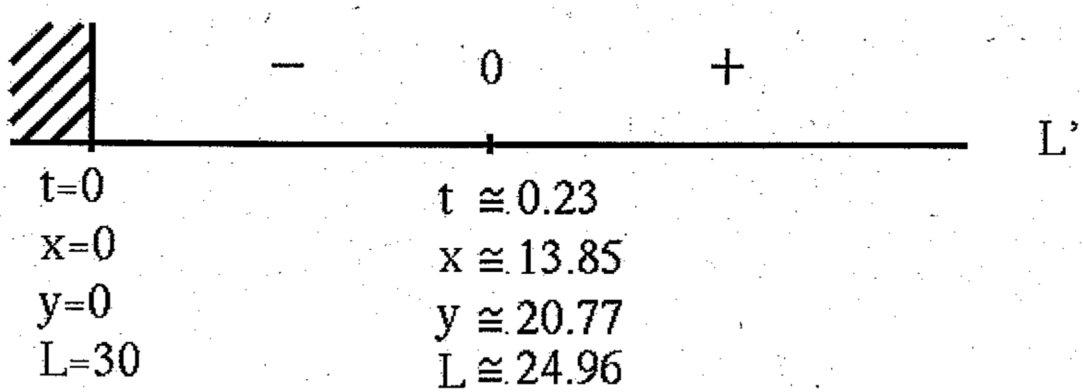
so that (If $\frac{A}{B} = 0$, then $A = 0$.)

$$23,400\,t - 5400 = 0\,,$$

and

$$t \approx 0.23\,.$$

See the adjoining sign chart for L' .



If

$t \approx 0.23$ hrs. = 13.8 min. ,

$$x \approx 13.85 \text{ mi.}, \; y \approx 20.77 \text{ mi.},$$

and

$$L \approx 24.96 \text{ mi.}$$

is the shortest possible distance between the cars.