AP® Calculus BC
2002 Scoring Guidelines

Question 1

Let f and g be the functions given by $f(x) = e^x$ and $g(x) = \ln x$.

(a) Find the area of the region enclosed by the graphs of f and g between $x = \frac{1}{2}$ and $x = 1$.

(b) Find the volume of the solid generated when the region enclosed by the graphs of f and g between $x = \frac{1}{2}$ and $x = 1$ is revolved about the line $y = 4$.

(c) Let h be the function given by $h(x) = f(x) - g(x)$. Find the absolute minimum value of $h(x)$ on the closed interval $\frac{1}{2} \le x \le 1$, and find the absolute maximum value of $h(x)$ on the closed interval $\frac{1}{2} \le x \le 1$. Show the analysis that leads to your answers.

(a) Area $= \int_{1/2}^{1} (e^x - \ln x)\,dx = 1.222$ or 1.223

$2 \begin{cases} 1: \text{ integral} \\ 1: \text{ answer} \end{cases}$

(b) Volume $= \pi \int_{1/2}^{1} \left((4 - \ln x)^2 - (4 - e^x)^2 \right) dx$

$= 7.515\pi$ or 23.609

$4 \begin{cases} 1: \text{ limits and constant} \\ 2: \text{ integrand} \\ \quad < -1 > \text{ each error} \\ \quad \text{Note: } 0/2 \text{ if not of the form} \\ \quad k\int_a^b \left(R(x)^2 - r(x)^2 \right) dx \\ 1: \text{ answer} \end{cases}$

(c) $h'(x) = f'(x) - g'(x) = e^x - \frac{1}{x} = 0$

$x = 0.567143$

Absolute minimum value and absolute maximum value occur at the critical point or at the endpoints.

$h(0.567143) = 2.330$
$h(0.5) = 2.3418$
$h(1) = 2.718$

The absolute minimum is 2.330.
The absolute maximum is 2.718.

$3 \begin{cases} 1: \text{ considers } h'(x) = 0 \\ 1: \text{ identifies critical point} \\ \quad \text{and endpoints as candidates} \\ 1: \text{ answers} \end{cases}$

Note: Errors in computation come off the third point.

Question 2

The rate at which people enter an amusement park on a given day is modeled by the function E defined by

$$E(t) = \frac{15600}{\left(t^2 - 24t + 160\right)}.$$

The rate at which people leave the same amusement park on the same day is modeled by the function L defined by

$$L(t) = \frac{9890}{\left(t^2 - 38t + 370\right)}.$$

Both $E(t)$ and $L(t)$ are measured in people per hour and time t is measured in hours after midnight. These functions are valid for $9 \le t \le 23$, the hours during which the park is open. At time $t = 9$, there are no people in the park.

(a) How many people have entered the park by 5:00 P.M. ($t = 17$)? Round answer to the nearest whole number.

(b) The price of admission to the park is \$15 until 5:00 P.M. ($t = 17$). After 5:00 P.M., the price of admission to the park is \$11. How many dollars are collected from admissions to the park on the given day? Round your answer to the nearest whole number.

(c) Let $H(t) = \int_9^t (E(x) - L(x))\,dx$ for $9 \le t \le 23$. The value of $H(17)$ to the nearest whole number is 3725. Find the value of $H'(17)$ and explain the meaning of $H(17)$ and $H'(17)$ in the context of the park.

(d) At what time t, for $9 \le t \le 23$, does the model predict that the number of people in the park is a maximum?

(a) $\int_9^{17} E(t)\,dt = 6004.270$

6004 people entered the park by 5 pm.

$3 \begin{cases} 1: \text{ limits} \\ 1: \text{ integrand} \\ 1: \text{ answer} \end{cases}$

(b) $15\int_9^{17} E(t)\,dt + 11\int_{17}^{23} E(t)\,dt = 104048.165$

The amount collected was \$104,048.

or

$\int_{17}^{23} E(t)\,dt = 1271.283$

1271 people entered the park between 5 pm and 11 pm, so the amount collected was

$\$15 \cdot (6004) + \$11 \cdot (1271) = \$104,041.$

1 : setup

(c) $H'(17) = E(17) - L(17) = -380.281$

There were 3725 people in the park at $t = 17$.

The number of people in the park was decreasing at the rate of approximately 380 people/hr at time $t = 17$.

$3 \begin{cases} 1: \text{ value of } H'(17) \\ 2: \text{ meanings} \\ \quad 1: \text{ meaning of } H(17) \\ \quad 1: \text{ meaning of } H'(17) \\ \quad < -1 > \text{ if no reference to } t = 17 \end{cases}$

(d) $H'(t) = E(t) - L(t) = 0$

$t = 15.794$ or 15.795

$2 \begin{cases} 1: E(t) - L(t) = 0 \\ 1: \text{ answer} \end{cases}$

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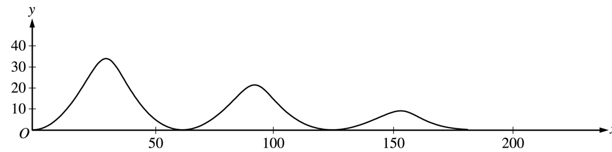
Question 3

The figure above shows the path traveled by a roller coaster car over the time interval $0 \le t \le 18$ seconds. The position of the car at time t seconds can be modeled parametrically by $x(t) = 10t + 4\sin t$, $y(t) = (20 - t)(1 - \cos t)$,



where x and y are measured in meters. The derivatives of these functions are given by

$x'(t) = 10 + 4\cos t$, $y'(t) = (20 - t)\sin t + \cos t - 1$.

(a) Find the slope of the path at time $t = 2$. Show the computations that lead to your answer.

(b) Find the acceleration vector of the car at the time when the car's horizontal position is $x = 140$.

(c) Find the time t at which the car is at its maximum height, and find the speed, in m/sec, of the car at this time.

(d) For $0 < t < 18$, there are two times at which the car is at ground level $(y = 0)$. Find these two times and write an expression that gives the average speed, in m/sec, of the car between these two times. Do not evaluate the expression.

(a) Slope $= \left.\dfrac{dy}{dx}\right|_{t=2} = \dfrac{y'(2)}{x'(2)} = \dfrac{18\sin 2 + \cos 2 - 1}{10 + 4\cos 2}$

$= 1.793$ or 1.794

$$1: \text{ answer using } \frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt}$$

(b) $x(t) = 10t + 4\sin t = 140$; $t_0 = 13.647083$

$x''(t_0) = -3.529$, $y''(t_0) = 1.225$ or 1.226

Acceleration vector is $\langle -3.529, 1.225 \rangle$

or $\langle -3.529, 1.226 \rangle$

$$2\begin{cases} 1: \text{ identifies acceleration vector as derivative of velocity vector} \\ 1: \text{ computes acceleration vector when } x = 140 \end{cases}$$

(c) $y'(t) = (20 - t)\sin t + \cos t - 1 = 0$

$t_1 = 3.023$ or 3.024 at maximum height

Speed $= \sqrt{\left(x'(t_1)\right)^2 + \left(y'(t_1)\right)^2} = |x'(t_1)|$

$= 6.027$ or 6.028

$$3\begin{cases} 1: \text{ sets } y'(t) = 0 \\ 1: \text{ selects first } t > 0 \\ 1: \text{ speed} \end{cases}$$

(d) $y(t) = 0$ when $t = 2\pi$ and $t = 4\pi$

Average speed $= \dfrac{1}{2\pi}\displaystyle\int_{2\pi}^{4\pi} \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}\, dt$

$= \dfrac{1}{2\pi}\displaystyle\int_{2\pi}^{4\pi} \sqrt{(10 + 4\cos t)^2 + ((20 - t)\sin t + \cos t - 1)^2}\, dt$

$$3\begin{cases} 1: \ t = 2\pi, t = 4\pi \\ 1: \text{ limits and constant} \\ 1: \text{ integrand} \end{cases}$$

Question 4

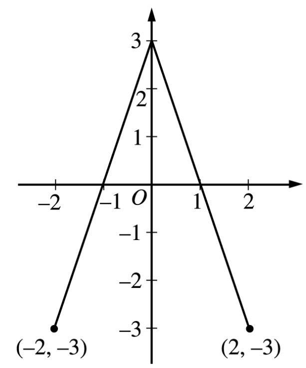
The graph of the function f shown above consists of two line segments. Let g be the function given by $g(x) = \int_0^x f(t)\,dt$.

(a) Find $g(-1)$, $g'(-1)$, and $g''(-1)$.

(b) For what values of x in the open interval $(-2, 2)$ is g increasing? Explain your reasoning.

(c) For what values of x in the open interval $(-2, 2)$ is the graph of g concave down? Explain your reasoning.

(d) On the axes provided, sketch the graph of g on the closed interval $[-2, 2]$.



Graph of f

(a) $g(-1) = \int_0^{-1} f(t)\,dt = -\int_{-1}^{0} f(t)\,dt = -\frac{3}{2}$

$g'(-1) = f(-1) = 0$

$g''(-1) = f'(-1) = 3$

$$3 \begin{cases} 1: & g(-1) \\ 1: & g'(-1) \\ 1: & g''(-1) \end{cases}$$

(b) g is increasing on $-1 < x < 1$ because $g'(x) = f(x) > 0$ on this interval.

$$2 \begin{cases} 1: & \text{interval} \\ 1: & \text{reason} \end{cases}$$

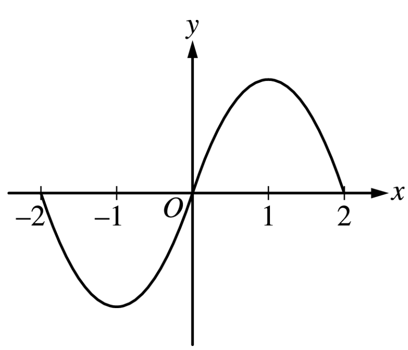
(c) The graph of g is concave down on $0 < x < 2$ because $g''(x) = f'(x) < 0$ on this interval.

or

because $g'(x) = f(x)$ is decreasing on this interval.

$$2 \begin{cases} 1: & \text{interval} \\ 1: & \text{reason} \end{cases}$$

(d)



$$2 \begin{cases} 1: & g(-2) = g(0) = g(2) = 0 \\ 1: & \text{appropriate increasing/decreasing} \\ & \text{and concavity behavior} \\ & < -1 > \text{ vertical asymptote} \end{cases}$$

Question 5

Consider the differential equation $\dfrac{dy}{dx} = 2y - 4x$.

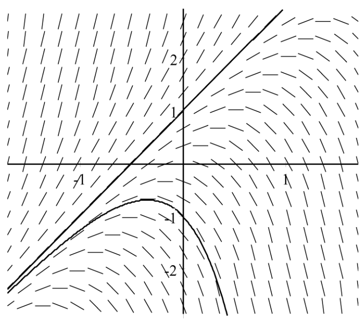
(a) The slope field for the given differential equation is provided. Sketch the solution curve that passes through the point $(0,1)$ and sketch the solution curve that passes through the point $(0,-1)$.

(b) Let f be the function that satisfies the given differential equation with the initial condition $f(0) = 1$. Use Euler's method, starting at $x = 0$ with a step size of 0.1, to approximate $f(0.2)$. Show the work that leads to your answer.

(c) Find the value of b for which $y = 2x + b$ is a solution to the given differential equation. Justify your answer.

(d) Let g be the function that satisfies the given differential equation with the initial condition $g(0) = 0$. Does the graph of g have a local extremum at the point $(0,0)$? If so, is the point a local maximum or a local minimum? Justify your answer.

(a)



$2 \begin{cases} 1: \text{ solution curve through } (0,1) \\ 1: \text{ solution curve through } (0,-1) \end{cases}$

Curves must go through the indicated points, follow the given slope lines, and extend to the boundary of the slope field.

(b) $f(0.1) \approx f(0) + f'(0)(0.1)$

$= 1 + (2 - 0)(0.1) = 1.2$

$f(0.2) \approx f(0.1) + f'(0.1)(0.1)$

$\approx 1.2 + (2.4 - 0.4)(0.1) = 1.4$

$2 \begin{cases} 1: \text{ Euler's method equations or equivalent table applied to (at least) two iterations} \\ 1: \text{ Euler approximation to } f(0.2) \text{ (not eligible without first point)} \end{cases}$

(c) Substitute $y = 2x + b$ in the DE:

$2 = 2(2x + b) - 4x = 2b$, so $b = 1$

OR

Guess $b = 1$, $y = 2x + 1$

Verify: $2y - 4x = (4x + 2) - 4x = 2 = \dfrac{dy}{dx}$.

$2 \begin{cases} 1: \text{ uses } \dfrac{d}{dx}(2x + b) = 2 \text{ in DE} \\ 1: \ b = 1 \end{cases}$

(d) g has local maximum at $(0,0)$.

$g'(0) = \left.\dfrac{dy}{dx}\right|_{(0,0)} = 2(0) - 4(0) = 0$, and

$g''(x) = \dfrac{d^2y}{dx^2} = 2\dfrac{dy}{dx} - 4$, so

$g''(0) = 2\,g'(0) - 4 = -4 < 0$.

$3 \begin{cases} 1: \ g'(0) = 0 \\ 1: \text{ shows } g''(0) = -4 \\ 1: \text{ conclusion} \end{cases}$

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Question 6

The Maclaurin series for the function f is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{(2x)^{n+1}}{n+1} = 2x + \frac{4x^2}{2} + \frac{8x^3}{3} + \frac{16x^4}{4} + \cdots + \frac{(2x)^{n+1}}{n+1} + \cdots$$

on its interval of convergence.

(a) Find the interval of convergence of the Maclaurin series for f. Justify your answer.

(b) Find the first four terms and the general term for the Maclaurin series for $f'(x)$.

(c) Use the Maclaurin series you found in part (b) to find the value of $f'\left(-\frac{1}{3}\right)$.

(a) $$\lim_{n\to\infty} \left| \frac{\frac{(2x)^{n+2}}{n+2}}{\frac{(2x)^{n+1}}{n+1}} \right| = \lim_{n\to\infty} \left| \frac{(n+1)}{(n+2)} 2x \right| = \left| 2x \right|$$

$\left| 2x \right| < 1$ for $-\frac{1}{2} < x < \frac{1}{2}$

At $x = \frac{1}{2}$, the series is $\sum_{n=0}^{\infty} \frac{1}{n+1}$ which diverges since this is the harmonic series.

At $x = -\frac{1}{2}$, the series is $\sum_{n=0}^{\infty} (-1)^{n+1} \frac{1}{n+1}$ which converges by the Alternating Series Test.

Hence, the interval of convergence is $-\frac{1}{2} \le x < \frac{1}{2}$.

5 {
- 1 : sets up ratio
- 1 : computes limit of ratio
- 1 : identifies interior of interval of convergence
- 2 : analysis/conclusion at endpoints
  - 1 : right endpoint
  - 1 : left endpoint
- $< -1 >$ if endpoints not $x = \pm\frac{1}{2}$
- $< -1 >$ if multiple intervals

(b) $f'(x) = 2 + 4x + 8x^2 + 16x^3 + \ldots + 2(2x)^n + \ldots$

2 {
- 1 : first 4 terms
- 1 : general term

(c) The series in (b) is a geometric series.

$$f'\left(-\frac{1}{3}\right) = 2 + 4\left(-\frac{1}{3}\right) + 8\left(-\frac{1}{3}\right)^2 + \ldots + 2\left(2 \cdot \left(-\frac{1}{3}\right)\right)^n + \ldots$$

$$= 2 - \frac{4}{3} + \frac{8}{9} - \frac{16}{27} + \ldots + 2\left(-\frac{2}{3}\right)^n + \ldots$$

$$= \frac{2}{1 + \frac{2}{3}} = \frac{6}{5}$$

OR

$f'(x) = \frac{2}{1 - 2x}$ for $-\frac{1}{2} < x < \frac{1}{2}$. Therefore,

$$f'\left(-\frac{1}{3}\right) = \frac{2}{1 + \frac{2}{3}} = \frac{6}{5}$$

2 {
- 1 : substitutes $x = -\frac{1}{3}$ into infinite series from (b) or expresses series from (b) in closed form
- 1 : answer for student's series