Vidyalankar Institute of Technology

Model Solution : S.E. [INFT/CMPN] - Maths IV

1. (a) (i) The **absolute error** of a number, measurement, or calculation is the numerical difference between the true value of the quantity and its approximate value as given, or obtained by measurement or calculations.

Error = True value − Approximate value
Absolute Error = | Error |

(ii) The **relative error** in the absolute error divided by true value of the quantity.

$$\text{Relative error} = \frac{\text{Absolute error}}{\text{True value}}$$

(iii)The **percentage error** is 100 times relative error.
Percentage error = 100 × Relative error

If a number is correct to n significant figures, it is evident that its absolute error can not be more than half a unit in its n'th place. For example, if the number 4.629 is correct to four figures, its absolute error is not greater than 0.001 × 1/2 = 0.0005.

1. (b)

$$\Delta\nabla = \delta^2 = \nabla\Delta$$

$$\begin{aligned}
\Delta\nabla\, y_x &= \Delta\left[y_x - y_{x-h}\right] \\
&= \left(y_{x+h} - y_x\right) - \left(y_x - y_{x-h}\right) \\
&= y_{x+h} - 2y_x + y_{x-h} \\
&= \left[E - 2 + E^{-1}\right] y_x \\
\therefore\ \Delta\nabla &= E^1 - 2 + E^{-1} \\
&= \delta^2 \qquad \left(\because\ \delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right)
\end{aligned}$$

1. (c)

x + y + z = 9
2x − 3y + 4z = 13
3x + 4y + 5z = 40

Interchanging 1^{st} and 3^{rd} equations

3x + 4y + 5z = 40
2x − 3y + 4z = 13
x + y + z = 9

Dividing 1^{st} equation by 3.

x + 1.3333y + 1.6667z = 13.3333
2x − 3y + 4z = 13
x + y + z = 9

Elimination of x from 2^{nd} and 3^{rd} equation

x + 1.3333y + 1.6667z = 13.3333
− 5.6666y + 0.6666z = −13.6666
− 0.3333y − 0.6667z = −4.3333

Dividing 2^{nd} equation by −5.6666

x + 1.3333y + 1.6667z = 13.3333
y + (−0.1176)z = 2.4118
−0.3333y − 0.6667 z = −4.3333

Elimination of y from 1^{st} and 3^{rd} equation

x + 1.3333y + 1.6667z = 13.3333
y + (−0.1176)z = 2.4118
−0.7059z = −3.5294

$$\therefore\ z = \frac{-3.5294}{-0.7059} = 4.9999$$

$\therefore$ y = 2.9998

$\therefore$ x = 1.0003

$$\boxed{\begin{array}{c} x = 1.0003 \\ y = 2.9998 \\ z = 4.9999 \end{array}}$$

1. (d)

Characteristic equation is $\begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$

$(5 - \lambda)(2 - \lambda) - 4 = 0$

$10 - 2\lambda - 5\lambda + \lambda^2 - 4 = 0$

$\therefore$ $\lambda^2 - 6\lambda - \lambda + 6 = 0$

$\therefore$ $\lambda(\lambda - 6) - 1(\lambda - 6) = 0$

$\therefore$ $(\lambda - 6)(\lambda - 1) = 0$

$\lambda = 6, 1$

When $\lambda = 1$, homogenous equation is $(A - \lambda I)X = 0$

$$\begin{bmatrix} 5-1 & 4 \\ 1 & 2-1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$4x + 4y = 0$

$x + y = 0$

$\therefore$ When $x = 1$, $y = -1$, i.e. $X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

When $\lambda = 6$, homogenous equation is $(A - \lambda I) = 0$

$$\begin{bmatrix} 5-6 & 4 \\ 1 & 2-6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$-x + 4y = 0$

$x - 4y = 0$

Put $x = 4$, $y = 1$ $\quad X_2 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

1. (e) Let λ be the characteristic root of the matrix A with corresponding characteristic vector X then

$AX = \lambda X \qquad \ldots (1)$

Taking transpose conjugate of both the sides of (1), we get

$\therefore$ $(AX)^\theta = (\lambda X)^\theta \Rightarrow X^\theta A^\theta = \bar{\lambda} X^\theta$

$\therefore$ $X^\theta A = \bar{\lambda} X^\theta \qquad (\because A^\theta = A)$

Post multiplying by X, we get

$X^\theta AX = \bar{\lambda} X^\theta X$

$\Rightarrow$ $X^\theta \lambda X = \bar{\lambda} X^\theta X \qquad$ [from (1)]

i.e. $\lambda X^\theta X = \bar{\lambda} X^\theta X$

$\Rightarrow$ $(\lambda - \bar{\lambda}) X^\theta X = 0$

Now, since X is the non–zero vector, $X^\theta X \neq 0 \Rightarrow \lambda = \bar{\lambda}$ which shows that λ is real.

1. (f) Equation of the circle with centre at the origin and radius = 2 is $z = 2e^{i\theta}$,

$\therefore\ dz = 2ie^{i\theta}d\theta$

Here, the path C_1 is the upper half of the circle from z = –2 to z = 2

i.e. θ varies from π to 0.

$$\therefore I = \int_{C_1} \frac{2z-3}{z}dz = \int_{\theta=\pi}^{0} \frac{4e^{i\theta}}{2e^{i\theta}} 2ie^{i\theta}d\theta$$

$$= i\int_{\theta=\pi}^{0} (4e^{i\theta} - 3)d\theta = i\left[\frac{4e^{i\theta}}{i} - 3 0\right]_{\pi}^{0}$$

$$= i\left[\frac{4}{i} - 0 - \frac{4e^{i\pi}}{i} + 3\pi\right] = i\left[\frac{4}{i} - \frac{4}{i}(\cos\pi + i\sin\pi) + 3\pi\right]$$

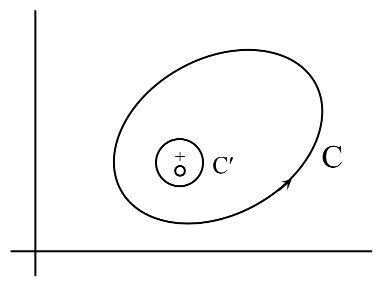
$$= i\left[\frac{4}{i} + \frac{4}{i} + 3\pi\right] = 8\pi + 3\pi\, i$$

2. (a) Cauchy's Integral theorem : If f(z) is single – valued and analytic within and on a closed curve C and if 'a' is any point interior to C, then

$$\int_C \frac{f(z)}{z-a}dz = 2\pi i f(a)$$

Proof : The function,

$$F(z) = \frac{f(z)}{z-a}$$



is analytic everywhere within and on the boundary of C except at z = a. Consider a circle C′ of small radius r around 'a'.

$$\int_C \frac{f(z)}{z-a}dz = \int_{C'} \frac{f(z)}{z-a}dz$$

Let $z - a = re^{i\theta}$ $\therefore dz = rie^{i\theta}.\,d\theta$ (as r is constant)

Also, $f(z) = f(a + re^{i\theta})$.

If the radius r of C_1 be contracted and allowed to tend to zero then in the limiting position.

$$f(z) = f(a)$$

And hence,

$$\int_{C'} \frac{f(z)}{z-a}dz = \int \frac{f(a)}{re^{i\theta}}.rie^{i\theta}.d\theta = i\,f(a)\int_{C'} d\theta = 2\pi i\; f(a)$$

$$\int_C \frac{f(a)}{z-a}dz = 2\pi i\; f(a)$$

or $$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz$$

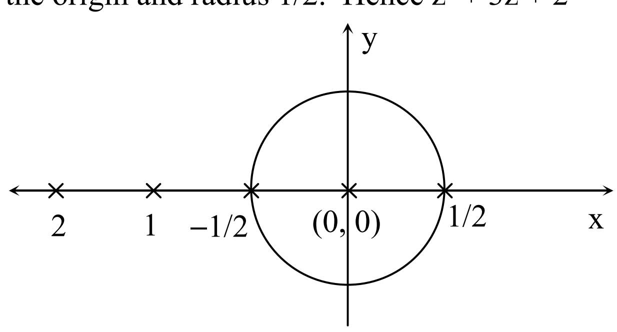
(i) $|z| = 1/2$

$|z| = \frac{1}{2}$ is a circle with centre at the origin and radius 1/2. Hence $z^2 + 3z + 2$

i.e. $(z + 2)(z + 1)$

i.e. z = –2, z = –1, lies outside the circle C and hence f(z) is analytic in C.

By Cauchy's Theorem,



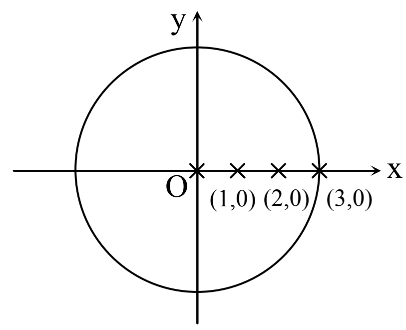
$$\int_C \frac{\sin(\pi z^2)+\cos(\pi z^2)}{z^2+3z+2}\,dz = 0$$

(ii) $|z| = 3$

The singular points are z = 1 and z = 2.

Both the points z = 1 and z = 2 lie within C.

By partial fraction method, we have



$$\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$$

$$\begin{aligned}
\text{Hence } I &= \int_C \frac{e^{2z}}{(z-1)(z-2)}\,dz \\
&= \int_C \frac{e^{2z}}{z-2}\,dz - \int_C \frac{e^{2z}}{z-1}\,dz \\
&= 2\pi i\,(\cos 4\pi + \sin 4\pi) - 2\pi i(\cos \pi + \sin \pi) \\
&= 2\pi i\,(1+0) - 2\pi i\,(-1+0) \\
&= 4\pi i
\end{aligned}$$

2. (b) (i) $f(z) = \dfrac{1}{4z^2+1}$

$\therefore\ f(z) = \dfrac{1}{(2z+i)(2z-i)}$

The poles of f(z) are z = –i/2, i/2

Residue of f(z) at z = –i/2)

$$\begin{aligned}
&= \lim_{z\to -\frac{i}{2}} \left[z+\frac{i}{2}\right] \cdot \frac{1}{(2z+i)(2z-i)} \\
&= \lim_{z\to -\frac{i}{2}} \frac{(2z+i)}{2} \cdot \frac{1}{(2z+i)(2z-i)} \\
&= \frac{1}{2\,.\,(-i-i)} = -\frac{1}{4i}
\end{aligned}$$

Residue of f(z) at z = i/2

$$\begin{aligned}
&= \lim_{z\to \frac{i}{2}} \left[z-\frac{i}{2}\right] \cdot \frac{1}{(2z+i)(2z-i)} \\
&= \lim_{z\to \frac{i}{2}} \frac{(2z-i)}{2} \cdot \frac{1}{(2z+i)(2z-i)} = \frac{1}{2\,.\,(i+i)} = \frac{1}{4i}
\end{aligned}$$

$\oint f(z)\,dz = 2\pi i$ (Sum of the residues)

$$= 2\pi i\left[-\frac{1}{4i}+\frac{1}{4i}\right] = 0$$

2. (b) (ii) The poles of f(z) are given by $z^2(z^2+5z+6) = 0$

i.e. $z^2(z+2)(z+3) = 0$

The poles are z = 0, z = –2, z = –3

The last two poles lie outside the circle | z | = 1 and z = 0 is a pole of order 2.

Residue of f(z) at z = 0

$$\begin{aligned}
&= \lim_{z\to 0} \frac{d}{dz} z^2\,.\,f(z) = \lim_{z\to 0}\frac{d}{dz} z^2\,.\,\frac{(z+4)^2}{z^2(z+2)(z+3)} \\
&= \lim_{z\to 0} \frac{d}{dz} \cdot \frac{(z+4)^2}{z^2+5z+6}
\end{aligned}$$

$$= \lim_{z\to 0}\frac{(z^2+5z+6)\,2\,.\,(z+4)-(z+4)^2(2z+5)}{(z^2+5z+6)^2}$$

$$\therefore \oint f(z)\,dz = 2\pi i\,(\text{Sum of the residues})$$

$$= 2\pi i\left(-\frac{8}{9}\right) = -\frac{16\pi i}{9}$$

2. (c) Since the degree of the numerator is equal to the degree of the denominator we divide the numerator by the denominator.

$$\therefore\ f(z) = \frac{z^2-1}{z^2+5z+6} = 1-\frac{5z+7}{z^2+5z+6}$$

Let $\dfrac{-5z-7}{z^2+5z+6} = \dfrac{a}{z+3}+\dfrac{b}{z+2}$

$-5z - 7 = a(z + 2) + b(z + 3)$

When $z = -2$, $b = 3$

When $z = -3$, $a = -8$

$$\therefore\ f(z) = \frac{z^2-1}{z^2+5z+6} = 1-\frac{8}{z+3}+\frac{3}{z+2}$$

Case (i) : when $|z| < 2$, we write

$$f(z) = 1-\frac{8}{3[1+(z/3)]}+\frac{3}{2[1+(z/2)]}$$

When $|z| < 2$, clearly $|z| < 3$

$$\therefore\ f(z) = 1-\frac{8}{3}\left[1+\frac{z}{3}\right]^{-1}+\frac{3}{2}\left[1+\frac{z}{2}\right]^{-1}$$

$$= 1-\frac{8}{3}\left[1-\left(\frac{z}{3}\right)+\left(\frac{z}{3}\right)^2-\ldots\right]+\frac{3}{2}\left[1-\left(\frac{z}{2}\right)+\left(\frac{z}{2}\right)^2+\ldots\right]$$

Case (ii) : When $2 < |z| < 3$, we write

$$f(z) = 1-\frac{8}{3[1+(z/3)]}+\frac{3}{z[1+(2/z)]}$$

$$= 1-\frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}+\frac{3}{z}\left(1+\frac{2}{z}\right)^{-1}$$

$$= 1-\frac{8}{3}\left[1-\left(\frac{z}{3}\right)+\left(\frac{z}{3}\right)^2-\ldots\right]+\frac{3}{z}\left[1-\left(\frac{2}{z}\right)+\left(\frac{2}{z}\right)^2-\ldots\right]$$

Case (iii) : when $|z| > 3$, we write

$$f(z) = 1-\frac{8}{z[1+(3/z)]}+\frac{3}{z[1+(2/z)]}$$

When $|z| > 3$, clearly $|z| > 2$

$$\therefore\ f(z) = 1-\frac{8}{z}\left(1+\frac{3}{z}\right)^{-1}+\frac{3}{2}\left(1+\frac{2}{z}\right)^{-1}$$

$$= 1-\frac{8}{z}\left[1-\left(\frac{3}{2}\right)+\left(\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^3+\ldots\right]+\frac{3}{2}\left[1-\left(\frac{2}{z}\right)+\left(\frac{2}{z}\right)^2-\left(\frac{2}{z}\right)^3+\ldots\right]$$

3. (a) (i)

Let $$I = \oint_C \tanh z\,dz$$

$$= \oint_C \frac{e^z-e^{-z}}{e^z+e^{-z}}\,dz$$

The integrand has poles at,

$$e^z+e^{-z} = 0$$

$$e^{2z} = -1$$

$$z = \pm i\,\pi/2, \quad \pm i\,3\pi/2$$

But only $z = \pm i\,\pi/2$ lies within the region $|z| = 3$

$$\therefore \text{Res}(z = i\pi/2) = \lim_{z\to i\pi/2} \frac{(z - i\pi/2)\left(e^{z} - e^{-z}\right)}{e^{z} + e^{-z}} \qquad \frac{0}{0}\text{ form}$$

$\therefore$ Applying L′ Hospital rule, we get

$$= \lim_{z\to i\pi/2} \frac{(z - i\pi/2)\ (e^{z} - e^{-z}) + (e^{z} - e^{-z})\,(1)}{(e^{z} - e^{-z})}$$

$$= \frac{0 + e^{i\pi/2} - e^{i\pi/2}}{e^{i\pi/2} - e^{-i\pi/2}} = 1$$

Similarly,

Res $(z = -i\pi/2) = 1$

$\therefore$ By Cauchy's residue theorem,

$$I = \oint_C \tanh z\,dz = 2\pi i \sum \text{Res}$$

$$I = 4\pi i$$

3. (a) (ii) The residue of the function $f(z) = z^2 e^{1/z}$ at $z = 0$

Note: Finding Residue by method of expansion.

If f (z) has a singular point z = a, f (z) can be expanded in positive and negative powers of (z – a), known as Laurentz series.

i.e. $$f(a) = A_0 + A_1(z-a) + \cdots + A_n(z-a)^n + \cdots + \frac{B_1}{(z-a)} + \frac{B_2}{(z-a)^2} + \cdots + \frac{B_n}{(z-a)^n} + \cdots$$

Here, B_1 is called the residue of f (z) w.r.t. singular point z = a.

Here a = 0. Therefore, to find residue at z = 0, we have of find coefficient of 1/z in the expansion of f (z).

$\therefore$ Consider,

$$f(z) = z^2 e^{1/z}$$

$$= z^2\left(1 + z^{-1} + \frac{z^{-2}}{2!} + \frac{z^{-3}}{3!} + \cdots\right)$$

$$= z^2 + z + \frac{z^{-1}}{2!} + \frac{z^{-2}}{3!} + \cdots$$

Here the coefficient of 1/z is 1/3! = 1/6. $\therefore$ Residue of $z^2 e^{1/z}$ at z = 0 is 1/6.

3. (b) Consider,

$$f(z) = \frac{1}{z\left(z^2 - 3z + 2\right)}$$

$$= \frac{1}{z(z-1)(z-2)}$$

$$= \frac{1}{z}\left[\frac{1}{z-2} - \frac{1}{z-1}\right]$$

$$= \frac{1}{z(z-2)} - \frac{1}{z(z-1)} \qquad \ldots (1)$$

(i) $|z| < 1 \qquad \therefore |z/2| < 1$

Consider, $$\frac{1}{z(z-2)} = \frac{1}{2z\left(z/2 - 1\right)}$$

$$= -\frac{1}{2z}(1-z/2)^{-1}$$
$$= -\frac{1}{2z}\left(1+z/2+z^2/4+...\right)$$
$$= -\frac{1}{2}(1/z+1/2+3/4+...) \quad , \quad |z| < 2 \qquad \therefore \ |z|<1$$

Consider,

$$\frac{1}{z(z-1)} \quad = \quad -\frac{1}{z(1-z)}$$
$$= -\frac{1}{z}(1-z)^{-1}$$
$$= -\frac{1}{z}\left(1+z+z^2+z^3+...\right)$$
$$= -\left(\frac{1}{z}+1+z+z^2+...\right) \quad , \quad |z|<1$$

from (1),

$$f(z) = -\frac{1}{2}\left[\frac{1}{z}+\frac{1}{2}+\frac{z}{4}+...\right]+\left[\frac{1}{z}+1+z+z^4+...\right]$$
$$= \frac{1}{2z}+\frac{3}{4}+\frac{7}{8}z+\frac{15}{16}z^2+...$$

ii) $1<|z|<2$

Now, $$\frac{1}{(z-2)} = \frac{1}{(-2)(1-z/2)}$$
$$= \frac{-1}{2}(1-z/2)^{-1}$$
$$= \frac{-1}{2}\left(1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+...\right)$$

And $$\frac{1}{(z-1)} = \frac{1}{z(1-1/z)}$$
$$= \frac{1}{z}(1-1/z)^{-1}$$
$$= \frac{1}{z}\left(1+1/z+1/z^2+1/z^3+...\right)$$
$$= \left(1/z+1/z^2+1/z^3+1/z^4+...\right)$$

$\therefore$ from (1)

$$f(z) = \frac{1}{z}\left[\frac{-1}{2}\left(1+\frac{z}{2}+\frac{z^2}{4}+...\right)-\left(\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+...\right)\right]$$
$$= \frac{-1}{2}\left(\frac{1}{z}+\frac{1}{2}+\frac{z}{4}+...\right)-\left(\frac{1}{z^2}+\frac{1}{z^3}+\frac{1}{z^4}+...\right)$$

iii) $|z|>2$

$\therefore \quad \left|\frac{2}{z}\right|<1$ and $\left|\frac{1}{z}\right|<1$

Hence, $$f(z) = \frac{1}{z}\left[\frac{1}{z-2}-\frac{1}{z-1}\right]$$
$$= \frac{1}{z}\left[\frac{1}{z(1-2/z)}-\frac{1}{z(1-1/z)}\right]$$

$$= \frac{1}{z^2}\left[\left(1-\frac{2}{z}\right)^{-1} - \left(1-\frac{1}{z}\right)^{-1}\right]$$

$$= \frac{1}{z^2}\left[\left(1+\frac{2}{z}+\frac{4}{z^2}+\frac{8}{z^3}+...\right)-\left(1+\frac{1}{z}+\frac{1}{z^2}+...\right)\right]$$

$$= \frac{1}{z^2}\left[1+\frac{2}{z}+\frac{4}{z^2}+\frac{8}{z^3}+...-1-\frac{1}{z}-\frac{1}{z^2}-...\right]$$

$$= \frac{1}{z^2}\left[\frac{1}{z}+\frac{3}{z^2}+\frac{7}{z^3}+...\right]$$

$$= \frac{1}{z^3}+\frac{3}{z^4}+\frac{7}{z^5}+...$$

3. (c) Consider L.H.S. = $I = \int_0^{2\pi} \frac{d\theta}{5-3\cos\theta}$

Let $z = e^{i\theta}$

$\therefore \quad d\theta = \frac{dz}{iz}$ and

$\cos\theta = \frac{z^2+1}{2z}$

$$I = \oint_C \frac{1}{5-3\left(\frac{z^2+1}{2z}\right)}\cdot\frac{dz}{iz}$$

Where C is a unit circle $|z| = 1$

$$I = \frac{2}{i}\oint_C \frac{1}{-3z^2+10z-3}dz$$

$$= -\frac{2}{i}\oint_C \frac{1}{3z^2-10z+3}dz$$

$$= -\frac{2}{i}\oint_C \frac{dz}{3z(z-3)-1(z-3)}$$

$$= -\frac{2}{i}\oint_C \frac{dz}{(3z-1)(z-3)}$$

Let $f(z) = \frac{1}{(3z-1)(z-3)}$

f(z) has poles at z = 1/3 and z = 3

But only pole at z = 1/3 lies within C.

$$\text{Res } (z = 1/3) = \lim_{z\to 1/3} \frac{(z-1/3)}{(3z-1)(z-3)}$$

$$= \lim_{z\to 1/3} \frac{(z-1/3)}{3(z-1/3)(z-3)}$$

$$= \frac{1}{3}\cdot\frac{1}{(1/3-1)} = \frac{1}{3}\cdot\frac{3}{-8} = -\frac{1}{8}$$

By Cauchy's Residue Theorem,

$$I = \int_0^{2\pi} \frac{d\theta}{5-3\cos\theta} = \left(\frac{-2}{i}\right)2\pi i \sum \text{Res.}$$

$$= \frac{-2}{i}2\pi i\left(-\frac{1}{8}\right)$$

$$= \frac{\pi}{2} = \text{R.H.S}$$

4. (a) Step 1:

The characteristic equation of matrix is given by,

$$|A-\lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$\Rightarrow$ $\lambda^3 -$ (sum of principal diagonal elements of A) λ^2 + (sum of minors of the element of the principal diagonal) $\lambda - |A| = 0$

$\Rightarrow$ λ^3 (6 + 3 + 3) λ^2 + [(9–1) + (18–4) + (18–4)]λ – 32 = 0

$\Rightarrow$ $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$ (1)

$\therefore$ $\lambda_1 = 8,\ \lambda_2 = 2,\ \lambda_3 = 2$

Step 2: Matrix equation is,

$$\begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Case i) for $\lambda_1 = 8$, matrix equation becomes

$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By Cramen's rule (using first two rows)

$$\frac{x_1}{\begin{vmatrix} -2 & 2 \\ -5 & -1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -2 & 2 \\ -2 & -1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -2 & -2 \\ -2 & -5 \end{vmatrix}}$$

$$\therefore \quad \frac{x_1}{12} = \frac{-x_2}{6} = \frac{x_3}{6}$$

$$\therefore \quad \frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{1}$$

$$\therefore \quad X_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Case ii) Let $\lambda_2 = 2$, $\therefore$ matrix equation becomes,

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Here, By Cramer's rule, we will get,

$$\frac{x_1}{0} = \frac{x_2}{0} = \frac{x_3}{0} \qquad \therefore \quad X_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

But X_2 must be a non–zero column matrix. This happened because here the roots are repeated. In such cases, use the following method

$\therefore$ By R_1,

$$4x_1 - 2x_2 + 2x_3 = 0$$

Let, $x_1 = 0$, $x_2 = 1$, $x_3 = 1$

$$X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Case iii) Let $\lambda_3 = 2$

$\because$ A is symmetric

$\therefore$ X_1, X_2, X_3 are orthogonal.

Let $X_3 \square \begin{bmatrix} \ell \\ m \\ n \end{bmatrix}$

$\therefore$ X_1, X_2 are orthogonal therefore $X_1' X_3 = 0$

$\therefore$ $2l - m + n = 0$ … (1)

$\because$ X_2, X_3 are orthogonal. Therefore $X_2' X_3 = 0$

$\therefore$ $ol + m + n = 0$ … (2)

Solving (1) and (2) by Cramer's Rule

$$\frac{\ell}{-2} = \frac{-m}{2} = \frac{n}{2}$$

$$\frac{\ell}{1} = \frac{m}{1} = \frac{n}{-1}$$

$$\therefore \quad X_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

4. (b) Proof :

Let A be the given square matrix, λ be the eigen value of A and X be the corresponding eigen vector A.

Then by definition,

$$AX = \lambda X \qquad (1)$$

Premultiply above equation by A

$$A(AX) = A(\lambda X)$$

$$\therefore \quad A^2 X = \lambda(AX)$$

$$= \lambda(\lambda X) \qquad \because AX = \lambda X \text{ from (1)}$$

$$\therefore \quad A^2X = \lambda^2 X$$

$\Rightarrow$ λ^2 is the eigen value of A^2

Again, premultiply above result by A

$$A(A^2X) = A(\lambda^2 X)$$

$$A^3X = \lambda^2(AX)$$

$$A^3X = \lambda^2(\lambda X)$$

$$\therefore \quad A^3X = \lambda^3 X$$

$\Rightarrow$ λ^3 is the eigen value of A^3

In general, λ^n is the eigen value of A^n.

Hence part,

To find characteristic roots i.e. eigen values of A^4 where

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Consider,

$$|A = \lambda I| = 0$$

$$\therefore \quad \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = 0$$

$$\therefore \quad (1-\lambda)^2 - 1 = 0$$

$$\therefore \quad 1 - 2\lambda + \lambda^2 - 1 = 0$$

$$\therefore \quad \lambda^2 - 2\lambda = 0$$
$$\therefore \quad \lambda(\lambda-2) = 0$$
$$\therefore \quad \lambda = 0, 2$$

$\therefore$ $\lambda = 0$ and $\lambda = 2$ are characteristic roots of A.

$\therefore$ By above theorem, λ^4 are characteristic roots of A^4.

$\therefore$ Characteristic roots of A^4 are 0^4, 2^4

i.e. **0, 16**

4. (c) Characteristic matrix of matrix A: For a given non–zero square matrix A of order n, if there exists a scalar λ, then the matrix $(A - \lambda I)$ is called the characteristic matrix of a given matrix A.

Characteristic polynomial of matrix A: The determinant of the characteristic matrix. i.e. $|A - \lambda I|$ which on expansion gives a polynomial in λ of degree n, known as characteristic polynomial of matrix A.

Characteristic equation of matrix A: Characteristic polynomial $|A - \lambda I|$ when equated to zero, the equation obtained is called the characteristic equation of matrix A.

Characteristic roots of matrix A: The roots of the characteristic equation $|A - \lambda I| = 0$ are called characteristic roots or eigen values of matrix A.

Characteristic vector of matrix A: Any non–zero vector X is said to be a characteristic vector of matrix A if there exists a number λ such that, $AX = \lambda X$, then X is a characteristic vector of A corresponding to characteristic root λ. Characteristic vector of A is also called eigen vector of A.

5. (a) It is given that $AX = \lambda X$ … (1)

To prove that $A^{-1}X = \lambda^{-1}X$

Pre–multiplying by A^{-1} on both sides of (1), we get

$$A^{-1}(AX) = \lambda(A^{-1}X)$$
$$\therefore \quad IX = \lambda(A^{-1}X)$$
$$\therefore \quad X = \lambda(A^{-1}X)$$

Since A is non singular, $\lambda \neq 0$, (by the theorem 1), therefore dividing by λ, we get

$$\frac{1}{\lambda}X = A^{-1}X$$

i.e. $A^{-1}X = \lambda^{-1}X$

The characteristic equation is $\begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$

After simplification, we get,

$$\lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$$

$\therefore$ $(\lambda - 1)(\lambda - 1)(\lambda + 5) = 0$ $\quad\therefore$ $\lambda = 1, 1, 5$

Hence 1, 1, 5 are the eigenvalues.

(i) For $\lambda = 1$, $[A - \lambda_1 I]X = 0$ gives

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $\begin{matrix} R_2 - R_1 \\ R_3 - R_1 \end{matrix}$ $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$x_1 + 2x_2 + x_3 = 0$$

We see that the rank of the matrix is 1 and number of variables is 3. Hence, there are 3 − 1 = 2 linearly independent solutions

Putting $x_3 = 0$, $x_2 = -1$, we get $x_1 = 2$

Putting $x_2 = 0$, $x_3 = -1$, we get $x_1 = 1$

Hence, corresponding to the repeated eigenvalue $\lambda = 1$ we get the following two linearly independent eigenvectors.

$$X_1 = [2, -1, 0], \quad X_2 = [1, 0, -1]$$

Further $k_1X_1 + k_2X_2$ is also an eigenvector.

(ii) For $\lambda = 5$, $[A - \lambda_2 I]X = 0$ gives

$$\begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By R_{13}
$$\begin{bmatrix} 1 & 2 & -3 \\ 1 & -2 & 1 \\ -3 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $\begin{matrix} R_2 - R_1 \\ R_3 + 3R_1 \end{matrix}$
$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & -4 & 4 \\ 0 & 0 & -8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_3 + 2R_2$
$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & -4 & 4 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + 2x_2 - 3x_3 = 0 \quad \text{and} \quad -4x_1 + 4x_3 = 0$$

Putting $x_3 = 1$, we get $x_2 = 1$ and then $x_1 = 1$. Hence, corresponding to eigenvalue 5 we get the following eigenvector.

$$X = [1, 1, 1]$$

5. (b) The characteristic equation is $\begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & -1-\lambda & 4 \\ 3 & 1 & -1-\lambda \end{vmatrix} = 0$

After simplification, we get,

$$\lambda^3 + \lambda^2 - 18\lambda - 40 = 0$$

Cayley–Hamilton theorem states that this equation is satisfied by A i.e.

$$A^3 + A^2 - 18A - 40I = 0 \qquad \ldots (1)$$

Now, $A^2 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix}$

$$A^3 = \begin{bmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 44 & 33 & 46 \\ 24 & 13 & 74 \\ 52 & 14 & 8 \end{bmatrix}$$

It can be verified that $A^3 + A^2 - 18A - 40I$

$$= \begin{bmatrix} 44 & 33 & 46 \\ 24 & 13 & 74 \\ 52 & 14 & 8 \end{bmatrix} + \begin{bmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix} - \begin{bmatrix} 18 & 36 & 54 \\ 36 & -18 & 72 \\ 54 & 18 & -18 \end{bmatrix} - \begin{bmatrix} 40 & 0 & 0 \\ 0 & 40 & 0 \\ 0 & 0 & 40 \end{bmatrix}$$

is equal to a zero matrix. Thus, the theorem is verified.

To find A^{-1} multiplying (1) by A^{-1}, we get,

$$A^2 + A - 18I - 40A^{-1} = 0$$

$$\therefore \quad 40A^{-1} = A^2 + A - 18I$$

$$= \begin{bmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix} - 18\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{40}\begin{bmatrix} -3 & 5 & 11 \\ 14 & -10 & 2 \\ 5 & 5 & -5 \end{bmatrix}$$

To find A^{-2} multiply (1) by A^{-2}, we get

$$A + I - 18A^{-1} - 40A^{-2} = 0$$

$$\therefore \quad A^{-2} = \frac{1}{40}\left\{\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix} - \frac{18}{40}\begin{bmatrix} -3 & 5 & 11 \\ 14 & -10 & 2 \\ 5 & 5 & -5 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\}$$

$$= \frac{1}{40}\left\{\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix} + \begin{bmatrix} \frac{54}{40} & -\frac{90}{40} & \frac{198}{40} \\ -\frac{252}{40} & \frac{180}{40} & -\frac{36}{40} \\ -\frac{90}{40} & -\frac{90}{40} & \frac{90}{40} \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\}$$

$$= \frac{1}{40}\begin{bmatrix} \frac{95}{40} & -\frac{50}{40} & \frac{318}{40} \\ -\frac{172}{40} & \frac{141}{40} & \frac{124}{40} \\ \frac{30}{40} & -\frac{50}{40} & \frac{51}{40} \end{bmatrix}$$

$$= \frac{1}{40} \times \frac{1}{40}\begin{bmatrix} 95 & -50 & 318 \\ -172 & 141 & 127 \\ 30 & -50 & 51 \end{bmatrix}$$

To find A^4 multiply (1) by A, we get,

$$A^4 + A^3 - 18A^2 - 40A = 0$$

$$\therefore \quad A^4 = \begin{bmatrix} 40 & 80 & 120 \\ 80 & -40 & 160 \\ 120 & 10 & -40 \end{bmatrix} + \begin{bmatrix} 252 & 54 & 144 \\ 216 & 162 & -36 \\ 36 & 72 & 22 \end{bmatrix} - \begin{bmatrix} 44 & 33 & 46 \\ 24 & 13 & 74 \\ 52 & 14 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 248 & 101 & 218 \\ 272 & 109 & 50 \\ 104 & 98 & 204 \end{bmatrix}$$

5. (c) $$A = \begin{bmatrix} 7 & 4 & -1 \\ 4 & 7 & -1 \\ -4 & -4 & 4 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 7-\lambda & 4 & -1 \\ 4 & 7-\lambda & -1 \\ -4 & -4 & 4-\lambda \end{vmatrix}$$

$$= (-1)^3\lambda^3 + (18)\lambda^2 - (24 + 24 + 33)\lambda + (28 - 4) - 4(16 - 4) - 1(-16 + 28)$$

$$= -\lambda^3 + 18\lambda^2 - 81\lambda + [168 - 48 - 12]$$

$$= -\lambda^3 + 18\lambda^2 - 81\lambda + 108$$

$\therefore \lambda^3 - 18\lambda^2 + 81\lambda - 108 = 0$

$(\lambda - 3)\ (\lambda^2 - 15\lambda + 36) = 0$

$(\lambda - 3)\ (\lambda - 3)\ (\lambda - 12) = 0$

$\lambda = 3, 3, 12$

Let $f(x) = (x - 3)\ (x - 12)$

$= x^2 - 15x + 36$ may be minimal polynomial

Consider $A^2 = \begin{bmatrix} 7 & 4 & -1 \\ 4 & 7 & -1 \\ -4 & -4 & 4 \end{bmatrix}\begin{bmatrix} 7 & 4 & -1 \\ 4 & 7 & -1 \\ -4 & -4 & 4 \end{bmatrix}$

$$= \begin{bmatrix} 69 & 60 & -15 \\ 60 & 69 & -18 \\ -60 & -60 & 24 \end{bmatrix}$$

$\therefore A^2 - 15A + 36I$

$$= \begin{bmatrix} 69 & 60 & -15 \\ 60 & 69 & -15 \\ -60 & -60 & 24 \end{bmatrix} - \begin{bmatrix} 105 & 60 & -15 \\ 60 & 105 & -15 \\ -60 & 60 & 60 \end{bmatrix} + \begin{bmatrix} 36 & 0 & 0 \\ 0 & 36 & 0 \\ 0 & 0 & 36 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore f(x) = x^2 - 15x + 36$ annihilates A.

Thus f(x) is the minimal polynomial of lowest degree that annihilates A. Hence f(x) is the minimal polynomial of A. Since its degree is less than the order of A, A is derogatory.

6. (a) Divide the interval [0, 6] into six parts of width h = 1.

Let $f(x) = \dfrac{1}{1+x^2}$. Then,

x	0	1	2	3	4	5	6
f(x)	1	0.5	0.2	0.1	0.0588	0.0385	0.027
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

i) By Trapezoidal rule,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{2}\left[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)\right]$$

$$= \frac{1}{2}\left[(1 + 0.027) + 2(0.5 + 0.2 + 0.1 + 0.0588 + 0.0385)\right]$$

$$= \frac{1}{2}\left[1.027 + 1.7946\right] = 1.4108.$$

ii) By Simpson's $\frac{1}{3}$ rule,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{3}\left[(y_0 + y_6) + 4(y_2 + y_3 + y_5) + 2(y_2 + y_4)\right]$$

$$= \frac{1}{3}\left[(1 + 0.027) + 4(0.5 + 0.1 + 0.0385) + 2(0.2 + 0.0588)\right]$$

$$= \frac{1}{3}\left[1.027 + 2.554 + 0.5176\right] = \mathbf{1.3662}$$

iii) By Simpson's $\frac{3}{8}$ rule ,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{3h}{8}\left[(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2y_3\right]$$

$$= \frac{3(1)}{8}\left[(1+0.027)+3(0.5+0.2+0.0588+0.0385+2(0.1)\right]$$

$$=\frac{3}{8}[1.027+2.8919+0.2]=1.3571.$$

6. (b) The volume of the solid generated is given by $V = \int_{x_0}^{x_n} \pi y^2 dx$.

Here $x_0 = 0.0$, $x_n = x_4 = 1.0$, $h = 0.25$
Let $f(x) = y^2$. Then

X	0.00	0.25	0.50	0.75	1.00
Y	1	0.9896	0.9589	0.9089	0.8415
y^2	1	0.9793	0.9194	0.8261	0.7081

By Simpson's $\frac{1}{3}$ rule, we have

$$\pi\int_0^1 y^2 dx = \pi\frac{h}{3}\left[(y_0{}^2+y_4{}^2)+4(y_1{}^2+y_3{}^2)+2(y_2)^2\right]$$

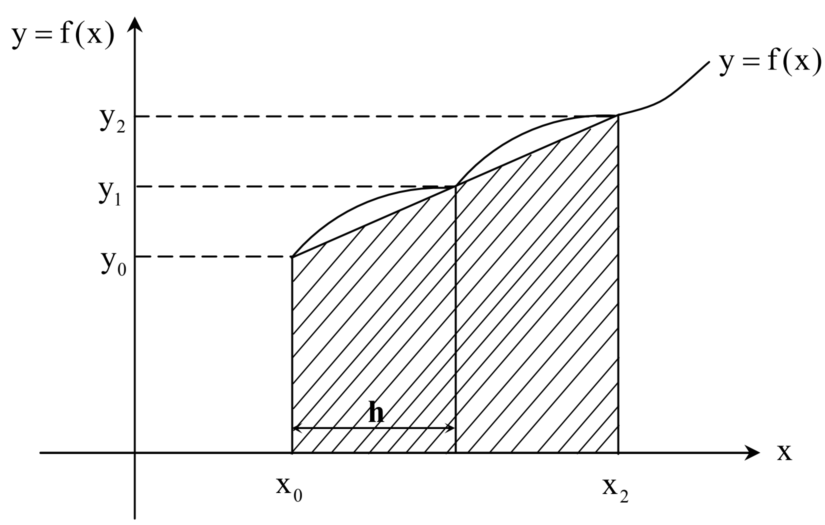
$$= \frac{(3.14)(0.25)}{3}\left[(1+0.7081)+4(0.9793+0.8261)+2(0.9194)\right]$$

$=$ **2.8178 cubic units.**

6. (c) Trapezoidal rule is given by the equation

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{2}\left[y_0+2(y_1+y_2+.....+y_{n-1})+y_n\right]$$

Geometrical Significance



Geometrically, the function y = f(x) is replaced by n straight lines joining the points (x_0, y_0) and (x_1, y_1), (x_1, y_1) and (x_2, y_2), …… (x_{n-1}, y_{n-1}) and (x_n, y_n). That is, we divide the given area of integration into n trapeziums of height h.

The area bounded by the curve y = f(x) and x − axis (between $x = x_0$ and $x = x_n$) is then approximately equal to the sum of the areas of the n trapeziums obtained.

$\therefore$ Area of curve $= \text{Area}(T_1)+\text{Area}(T_2)+.......+\text{Area}(T_n)$

where $T_1 \Rightarrow$ Trapezium no. 1 and so on.

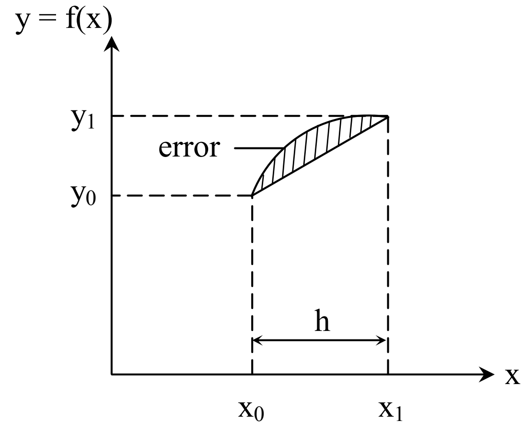
$$\text{i.e.,} \int_{x_0}^{x_n} y\,dx = (y_0 + y_1)\frac{h}{2} + (y_1 + y_2)\frac{h}{2} + ... + (y_{n-1} + y_n)\frac{h}{2}$$

$\because$ Area of trapezium = (sum of parallel sides) $\dfrac{\text{ht.}}{2}$

$$\therefore \int_{x_o}^{x_n} y\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + ... + y_{n-1})\right]$$

Error

Consider the trapezium between the points (x_0, y_0) and (x_1, y_1) as shown.



As the upper extremities are joined by straight lines, there is always some error in the evaluation. It is clear form the above figure that error in the approximation can be substantial when large values of h are used. By taking very small value of h, the error can be minimized.

For linear functions, the error is almost **negligible**.

7. (a) To find f(x) when x = 1.85

The Newton's forward difference table is as follows :

x	f(x)	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
1.7	5.474						
		0.576					
1.8	6.050		0.06				
		0.636		0.007			
1.9	6.686		0.074		0		
		0.703		0.007		0.001	
2.0	7.389		0.074		0.001		−0.002
		0.777		0.008		−0.001	
2.1	8.166		0.082		0		
		0.859		0.008			
2.2	9.025		0.09				
		0.949					
2.3	9.974						

By Newton's forward difference formula

$$f(x) = f(x_0) + p\,\Delta f(x_0) + \frac{p(p-1)}{2!}\Delta^2 f(x_0) + \frac{p(p-1)(p-2)}{3!}\Delta^3 f(x_0)$$
$$+ \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 f(x_0) + \frac{p(p-1)(p-2)(p-3)(p-4)}{5!}\Delta^5 f(x_0)$$
$$+ \frac{p(p-1)(p-2)(p-3)(p-4)(p-5)}{6!}\Delta^6 f(x_0)$$

$$x = x_0 + p(h) = 1.7 + p(0.1)$$

$$p = \frac{x - x_0}{h} = \frac{1.85 - 1.7}{0.1} = 1.5$$

$$\therefore\ f(1.85) = 5.474 + 1.5 \times 0.576 + \frac{1.5\times(1.5-1)}{2!}\times 0.06 + \frac{1.5(1.5-1)(1.5-2)\times 0.007}{3!}$$
$$+\frac{1.5(1.5-1)(1.5-2)\times(1.5-3)\times 0}{4!}$$
$$+\frac{1.5(1.5-1)(1.5-2)(1.5-3)(1.5-4)\times 0.001}{5!}$$
$$+\frac{1.5(1.5-1)(1.5-2)(1.5-3)(1.5-4)(1.5-5)\times -0.002}{6!}$$
$$= 5.474 + 0.576 \times 1.5 + 0.0225 + (-4.375 \times 10^{-4}) + 0 + (-1.1719 \times 10^{-5})$$
$$+ (-1.3672 \times 10^{-5})$$
$$= 6.3600$$

$$\boxed{f(1.85) = 6.3600}$$

To find f(x) when x = 2.35

The Newton's forward difference table is as follows :

x	f(x)	∇	∇^2	∇^3	∇^4	∇^5	∇^6
1.7	5.474						
		0.576					
1.8	6.050		0.06				
		0.636		0.007			
1.9	6.686		0.074		0		
		0.703		0.007		0.001	
2.0	7.389		0.074		0.001		−0.002
		0.777		0.008		−0.001	
2.1	8.166		0.082		0		
		0.859		0.008			
2.2	9.025		0.09				
		0.949					
2.3	9.974						

$$f(x)= f(x_n)+p\nabla f(x_n)+\frac{p(p+1)}{2!}\nabla^2 f(x_n)+\frac{p(p+1)(p+2)}{3!}\nabla^2 f(x_n)$$
$$+\frac{p(p+1)(p+2)}{3!}\nabla^3 f(x_n)+\frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 f(x_n)$$
$$+\frac{p(p+1)(p+2)(p+3)(p+4)}{5!}\nabla\ f(x_n)$$
$$+\frac{p(p+1)(p+2)(p+3)(p+4)(p+5)}{6!}\nabla^6 f(x_n)$$

$x = x_n + p(h) = 2.35$

x_n = final value

$\therefore\ x = 2.3 + p(0.1)$

$$\therefore\ p = \frac{x-x_n}{h} = \frac{2.35-2.3}{0.1} = 0.5$$

$$\therefore\ f(2.35) = 9.974 + 0.5 \times 0.94 + \frac{0.5\times(0.5+1)}{2!}\times 0.09$$
$$+\frac{0.5\times(0.5+1)(0.5+2)}{3!}\times 0.008 + \frac{0.5\times(0.5+1)(0.5+2)(0.5+3)}{4!}\times 0$$
$$+\frac{0.5\times(0.5+1)(0.5+2)(0.5+3)(0.5+4)}{5!}\times -0.001$$
$$+\frac{0.5\times(0.5+1)(0.5+2)(0.5+3)(0.5+4)(0.5+5)}{6!}\times -0.002$$
$$= 9.974 + 0.4745 + 0.03375 + 2.5 \times 10^{-3} + 0 + (-2.4609 \times 10^{-4})$$
$$+ (-4.5111\times 10^{-4})$$
$$= 10.4841$$

$$\boxed{f(2.35) \quad = 10.4841}$$

7. (b)

	x_0	x_1	x_2	x_3
x	5	6	9	11
y	12	13	14	16
	y_0	y_1	y_2	y_3

$$f(x)= \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}f(x_0)+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}f(x_1)+$$
$$\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}f(x_2)+\frac{(x-x_0)(x-x_0)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}f(x_3)$$

$$\therefore f(10) = \frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)}\times 12+\frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)}\times 13+$$
$$\frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)}\times 16+\frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)}\times 14$$

$$= \frac{4\times 1\times -1\times 12}{-1\times -4\times -6}+\frac{5\times 1\times -1\times 13}{1\times -3\times -5}+\frac{5\times 4\times 1\times 16}{6\times 5\times 2}+\frac{5\times 4\times -1\times 14}{4\times 3\times -2}$$

$$= \frac{-48}{-24}+\frac{-65}{15}+\frac{320}{60}+\frac{-280}{-24}$$

$$= 2+\frac{-13}{3}+\frac{16}{3}+\frac{35}{3}=\frac{6-13+16+35}{3}=\frac{44}{3}$$

$$= 14.6667$$

$$\boxed{f(10) = 14.6667}$$

7. (c)

$$y_{n+1} = y_n + h f(x_n, y_n)$$

$$y_{n+1} = y_n + \frac{1}{2}h\left[f(x_n, y_n)+f(x_{n+1}, y_{n+1})\right]$$

The first approximation for y at x = 1.0

$$y_1 = y_0 + f(x_0, y_0)$$
$$= 2 + h\log(2+0.8)$$
$$= 2 + 0.447 \times 0.2$$
$$= 2.0894$$

1st modified value of y_1 is given by

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0)+f(x_1, y_1)\right]$$
$$= 2+\frac{0.2}{2}\left[0.447+0.4899\right]$$
$$= 2.0937$$

The 2nd modified value of y_1 is given by

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0)+f(x_1, y_1^{(1)})\right]$$
$$= 2+\frac{0.2}{2}\left[0.447+0.4905\right]$$
$$= 2.0937$$

The third modified value of y_1 is given by

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0)+f(x_1, y_1^{(2)})\right]$$
$$= 2+\frac{0.2}{2}\left[0.447+0.4905\right]$$
$$= 2.0937$$

$$
\begin{aligned}
y_1 &= 2.0937\\
y_2 &= y_1 + hf(x_1, y_1)\\
&= 2.0937 + 0.2 \times 0.4905\\
&= 2.1918\\
y_2^{(1)} &= y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2)\right]\\
&= 2.0937 + \frac{0.2}{2}\left[0.4905 + 0.5304\right]\\
&= 2.1958
\end{aligned}
$$

$$
\begin{aligned}
y_2^{(2)} &= y_1 + \frac{0.2}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(1)})\right]\\
&= 2.0937 + 0.1\ [0.4905 + 0.5309]\\
&= 2.1958\\
y_3 &= y_2 + hf(x_2, y_2)\\
&= 2.1958 + 0.2 \times 0.5309\\
&= 2.3020\\
y_3^{(1)} &= y_2 + \frac{h}{2}\left[f(x_2, y_2) + f(x_3, y_3)\right]\\
&= 2.1958 + 0.1\left[0.5309 + 0.5684\right]\\
&= 2.3057\\
y_3^{(2)} &= y_2 + \frac{h}{2}\left[f(x_2, y_2) + f(x_3, y_3^{(1)})\right]\\
&= 2.1958 + 0.1\ [0.5309 + 0.5689]\\
&= 2.3058\\
y_3^{(3)} &= y_2 + \frac{h}{2}\left[f(x_2, y_2) + f(x_3, y_3^{(2)})\right]\\
&= 2.1958 + 0.1\ [0.5309 + 0.5689]\\
&= 2.3058
\end{aligned}
$$

$$\boxed{x_3 = 1.4, \quad y_3 = 2.3058}$$

7. (d)

$$
\begin{aligned}
\Delta^2\left(\frac{5x+12}{x^2+5x+16}\right) &= \Delta\left[\Delta\left(\frac{5x+12}{x^2+5x+16}\right)\right]\\
&= \Delta\left[\frac{5(x-1)+12}{(x-1)^2+5(x-1)+16} - \frac{5x+12}{x^2+5x+16}\right]\\
&= \Delta\left[\frac{5x-5+12}{x^2-2x+1+5x-5+16} - \left(\frac{5x+12}{x^2+5x+16}\right)\right]\\
&= \Delta\left[\frac{5x+7}{x^2+3x+12} - \frac{5x+12}{x^2+5x+16}\right]\\
&= \Delta\left[\frac{5x+7}{x^2+3x+12}\right] - \Delta\left[\frac{5x+12}{x^2+5x+16}\right]\\
&= \left[\frac{5(x-1)+7}{(x-1)^2+3(x-1)+12} - \left(\frac{5x+7}{x^2+3x+12}\right)\right] - \left[\frac{5x+7}{x^2+3x+12} - \left(\frac{5x+12}{x^2+5x+16}\right)\right]\\
&= \frac{5x+2}{(x^2-2x+1)+3x-3+12} - \frac{(5x+7)}{x^2+3x+12} - \left(\frac{5x+7}{x^2+3x+12}\right) + \frac{5x+12}{x^2+5x+6}\\
&= \frac{5x+2}{x^2+x+10} - \frac{5x+7}{x^2+3x+12} - \frac{5x7}{x^2+3x+12} + \frac{5x+12}{x^2+5x+15}\\
&= \frac{5x+2}{x^2+x+10} - \frac{2(5x+7)}{x^2+3x+12} + \frac{5x+12}{x^2+5x+16}
\end{aligned}
$$

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