Math 1120, Fall 2000: Sample Midterm 3

1. $\lim_{x\to\infty} \frac{x^3-4x+1}{2x^3+2x+1} = \lim_{x\to\infty} \frac{1-4/x^2+1/x^3}{2+2/x^2+1/x^3} = \frac{1}{2}$. $\lim_{x\to-\infty} \frac{x^4+3x-1}{x^2+7x-2} = \lim_{x\to-\infty} \frac{x^2+3/x-1/x^2}{1+7/x-2/x^2} = \infty$. The first function has a horizontal asymptote of $y = \frac{1}{2}$, the second has no horizontal asymptote.

2. (a) $f'(x) = 20x^4 + 6x^{-3} + \frac{5}{2}x^{-1/2} - 2e^x + 6\cos x$

(b) $g'(x) = (\frac{1}{x} + 4\sin x)(\tan x + \sec x) + (\ln x - 4\cos x)(\sec^2 x + \sec x \tan x)$

(c) $h'(x) = \frac{(\frac{1}{3}x^{-2/3})(x-3)e^x - (x^{1/3}+4)[e^x + (x-3)e^x]}{(x-3)^2e^{2x}}$

(d) $f'(x) = \frac{1}{\sqrt{1-(x^2+1)^2}}2x$

(e) $\ln(5)5^{x^3}(3x^2)\log_4(x^5-4x) + 5^{x^3}\frac{1}{(\ln 4)(x^5-4x)}(5x^4-4)$

(g) $\frac{\frac{1}{1+e^{2\sqrt{x}}}e^{\sqrt{x}}\frac{1}{2}x^{-1/2}x\ln x - \tan^{-1}(e^{\sqrt{x}})(\ln x + 1)}{(x\ln x)^2}$

3. $f'(x) = g'(x)h(x) + g(x)h'(x)$, so $h'(x) = \frac{f'(x) - g'(x)h(x)}{g(x)} = \frac{f'(x) - g'(x)f(x)/g(x)}{g(x)} = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2}$.

4. $f'(x) = \frac{(1+3e^{3x})(\sqrt{x}+1) - (x+e^{3x})/(2\sqrt{x})}{(\sqrt{x}+1)^2}$. $f'(4) = \frac{3(1+3e^{12}) - (4+e^{12})/4}{9} = \frac{35e^{12}+12}{36}$. The tangent line is $y - \frac{4+e^{12}}{5} = \frac{35e^{12}+12}{36}(x-4)$, or $y = \frac{35e^{12}+12}{36}(x-4) + \frac{4+e^{12}}{5}$.

5. (a) $f'(x) = \frac{3x^2(x^3+1) - (x^3-1)3x^2}{(x^3+1)^2} = \frac{3x^5+3x^2-3x^5+3x^2}{(x^3+1)^2} = \frac{6x^2}{(x^3+1)^2}$.

$f''(x) = \left(\frac{6x^2}{(x^3+1)(x^3+1)}\right)' = \frac{12x(x^3+1)^2 - 6x^2(3x^2(x^3+1) + (x^3+1)3x^2)}{(x^3+1)^4} = \frac{(x^3+1)(12x(x^3+1) - 6x^2(6x^2))}{(x^3+1)^4} = \frac{-24x^4+12x}{(x^3+1)^3}$.

(b) A local max or min can only occur where $f'(x)$ is discontinuous or zero. The zeroes of $f'(x)$ are $x = 0$ and the discontinuities are $x = -1$. Since $f'(x) = 6\left(\frac{x}{x^3+1}\right)^2$ it is always greater than or equal to 0 so that $f(x)$ is always increasing (though not at -1, of course!). There can be no local maxima or minima and so there can be no global maxima or minima.

$f(x)$ has an inflection point at a if the concavity changes at $x = a$ and $f(x)$ is defined at $x = a$, which can only happen at discontinuities or zeroes of $f''(x)$. The zeroes of $f''(x)$ are $12x(1-2x^3) = 0$ or $x = 0$ and $x = \frac{1}{\sqrt[3]{2}}$ and the discontinuities are $x = -1$.

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	-1	0	$1/\sqrt[3]{2}$	
$12x$	$-$	$-$	$+$	$+$
$1-2x^3$	$+$	$+$	$+$	$-$
$(x^3+1)^3$	$-$	$+$	$+$	$+$
$f''(x)$	$+$	$-$	$+$	$-$

The inflection points are $x=0$ and $x=\dfrac{1}{\sqrt[3]{2}}$.

6. The race track principle states that if $f(a)=g(a)$ and $f'(x)>g'(x)$ for all $x>a$ then $f(x)>g(x)$ for all $x>a$. Using $f(x)=e^x$, $g(x)=1+x+\dfrac{x^2}{2}$ and $a=0$ we see that $f(0)=e^0=1=1+0+\dfrac{0^2}{2}=g(0)$. $f'(x)=e^x$, $g'(x)=1+x$ and we know that $f'(x)=e^x>1+x=g'(x)$. The hypotheses of the race track principle are satisfied so that $e^x>1+x+\dfrac{x^2}{2}$ for all $x>0$.

7. We first see what has to happen at $x=0$. The joining curve $y=f(x)$ must satsify $f(0)=a_0=0$ (it must join the piece along the negative x-axis), $f'(x)=a_1+2a_2x+3a_3x^2+4a_4x^3+5a_5x^4$ and $f'(0)=a_1=0$ (the first derivatives must match), $f''(x)=2a_2+6a_3x+12a_4x^2+20a_5x^3$ and $f''(0)=a_2=0$ (the second derivatives must match). At $x=1$, $f(1)=a_3+a_4+a_5=1$, $f'(1)=3a_3+4a_4+5a_5=1$ and $f''(1)=6a_3+12a_4+20a_5=0$ (since if $g(x)=x$, $g''(x)=0$). The equations are:

$$\begin{aligned} a_0&=0\\ a_1&=0\\ a_2&=0\\ a_3+a_4+a_5&=1\\ 3a_3+4a_4+5a_5&=1\\ 6a_3+12a_4+20a_5&=0 \end{aligned}$$

8. We seem to be differentiating $f(x)=\cos x$ at $x=\dfrac{\pi}{3}$. Checking this, $[\cos x]'\Big|\,\pi/3=\lim\limits_{h\to 0}\dfrac{\cos(\frac{\pi}{3}+h)-\frac{1}{2}}{h}$, which is what we started with. The value of the limit is $-\sin(\dfrac{\pi}{3})=-\sqrt{\dfrac{3}{2}}$.

9. (a) $2x+5y+5xy'+2yy'=-5$, $y'=\dfrac{-2x-5y-5}{5x+2y}$. At $(1,-2)$, $y'=\dfrac{3}{1}=3$ and the tangent line is $y+2=3(x-1)$, or $y=-3x-5$.

(b) The tangent line will be vertical when the slope is either $\pm\infty$. this occurs only when $5x+2y=0$, or $y=-\dfrac{5x}{2}$. Substituting into the original equation, $x^2-\dfrac{25x^2}{2}+\dfrac{25x^2}{4}=-5x$, $-\dfrac{21}{4}x^2+5x=0$ and $x(-\dfrac{21}{4}x+5)=0$. The tangent line will be vertical at $(0,0)$ and $(\dfrac{20}{21},-\dfrac{50}{21})$.

10. We know that $\dfrac{dv}{dt}=9-0.9t$ so that $v=9t-0.45t^2+C$. Since the initial speed is 10m/s, $v(0)=10=C$ and $v=9t-0.45t^2+10$. After 10s the raindrop is falling at a rate of $90-45+10=55$m/s.

11. $y'=4A\cos 4x-4B\sin 4x$, and $y''=-16A\sin 4x-16B\cos 4x$. We have $y''+16y=-16A\sin 4x-16B\cos 4x+16(A\sin 4x+B\cos 4x)=0$. If $y(0)=1$, $B=1$, and if $y'(0)=2$, $4A=2$ and $A=\dfrac{1}{2}$. $y=\dfrac{1}{2}\sin 4x+\cos 4x$.