Math 1120, Fall 2000: Solutions to Midterm 3

1. (a) $f'(x) = -3\dfrac{1}{x^4} - \dfrac{1}{2\sqrt{x}} - (\ln 4)4^x + \dfrac{1}{x} - \dfrac{3}{1+x^2}$

   (b) $(\cos x + 2\sec x\tan x)(\tan x - \log_3 x + 2\sin^{-1}(x)) + (\sin x + 2\sec x)(\sec^2 x - \dfrac{1}{(\ln 3)x} + \dfrac{2}{\sqrt{1-x^2}})$

   (c) $\dfrac{(1-(1/x))(2\cos x - 4) - (x - \ln x)(-2\sin x)}{(2\cos x - 4)^2} = \dfrac{(1-(1/x))(2\cos x - 4) + 2(x - \ln x)\sin x}{(2\cos x - 4)^2}$

   (d) $2xe^{x^2+3}$

   (e) $(\ln 3)3^{x/(x-3)}[\dfrac{x-3-x}{(x-3)^2}] = -(\ln 3)3^{x/(x-3)}[\dfrac{3}{(x-3)^2}]$

   (f) $\dfrac{1}{|2x|\sqrt{4x^2-1}}\cos(x^2+2x) - \sec^{-1}(2x)\sin(x^2+2x)(2x+2)$

2. $\displaystyle\lim_{x\to\infty}\frac{5}{3+2f(x)} = \frac{5}{3+\displaystyle\lim_{x\to\infty} f(x)} = \frac{5}{3}$

3. $f'(x) = -\dfrac{2}{\sqrt{1-4x^2}}$, $f'(\dfrac{1}{4}) = -\dfrac{2}{\sqrt{1-4(1/16)}} = -\dfrac{2}{\sqrt{3/4}} = -\dfrac{4}{\sqrt{3}}$. When $x = \dfrac{1}{4}$, $f(\dfrac{1}{4}) = \cos^{-1}(\dfrac{1}{2}) = \dfrac{\pi}{3}$.
   The tangent line is $y - \dfrac{\pi}{3} = -\dfrac{4}{\sqrt{3}}(x - \dfrac{1}{4})$, or $y = -\dfrac{4}{\sqrt{3}}x + \dfrac{1}{\sqrt{3}} + \dfrac{\pi}{3}$.

4. If f is differentiable at a it is continuous there. We need to match function values and derivatives at -1 and 1.
   $$f'(x) = \begin{cases} 2x & \text{if } x < -1 \\ a_1 + 2a^2x + 3a_3x^2 & \text{if } -1 < x < 1 \\ -2 & \text{if } x > 1 \end{cases}$$

At -1, $(-1)^2 = 1 = a_0 - a_1 + a_2 - a_3$ and $2(-1) = -2 = a_1 - 2a_2 + 3a_3$.

At 1, $-2(1) + 1 = -1 = a_0 + a_1 + a_2 + a_3$ and $-2 = a_1 + 2a_2 + 3a_3$. We have:

$$\begin{aligned} a_0 - a_1 + a_2 - a_3 &= 1 \\ a_1 - 2a_2 + 3a_3 &= -2 \\ a_0 + a_1 + a_2 + a_3 &= -1 \\ a_1 + 2a_2 + 3a_3 &= -2 \end{aligned}$$

5. (a) $6x - 4y - 4x\dfrac{dy}{dx} + 2y\dfrac{dy}{dx} = 0$. At $(-1,2)$, $-6 - 8 + 4\dfrac{dy}{dx} + 4\dfrac{dy}{dx} = 0$ and $\dfrac{dy}{dx} = \dfrac{14}{8} = \dfrac{7}{4}$. The tangent line is $y - 2 = \dfrac{7}{4}(x+1)$ or $y = \dfrac{7}{4}x + \dfrac{15}{4}$.

   (b) From (a), $\dfrac{dy}{dx} = \dfrac{4y-6x}{2y-4x} = \dfrac{2y-3x}{y-2x}$. $\dfrac{dy}{dx} = 2$ if $\dfrac{2y-3x}{y-2x} = 2$, $2y - 3x = 2(y-2x)$, $2y - 3x = 2y - 4x$, $3x = 4x$, or $x = 0$. If $x = 0$, $f(0,y) = y^2 = 15$ and $y = \pm\sqrt{15}$.

   The two points at which the curve has slope 2 are $(0, \sqrt{15})$ and $(0, -\sqrt{15})$.

6. (a) f increases when f' is positive and f decreases when f' is negative. f' has zeroes at $x = 1$ and is discontinuous at $x = -2$.

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		-2		1	
$(x-1)^2$	$+$		$+$		$+$
$x+2$	$-$		$+$		$+$
$f'(x)$	$-$		$+$		$+$

f is increasing for $x \in (-2,1) \cup (1,\infty)$ and decreasing for $x \in (-\infty,-2)$.

f is concave up when f'' is positive and concave down when f'' is negative. f' is zero for $x=-5$, 1 and discontinuous for $x=-2$.

		-5		-2		1	
$x-1$	$-$		$-$		$-$		$+$
$x+5$	$-$		$+$		$+$		$+$
$(x+2)^2$	$+$		$+$		$+$		$+$
$f''(x)$	$+$		$-$		$-$		$+$

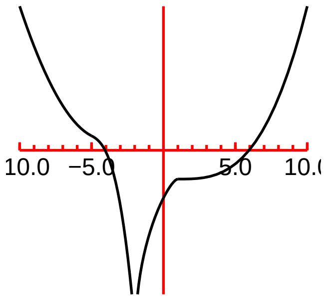
f is concave up for $x \in (-\infty,-5) \cup (1,\infty)$ and concave down for $x \in (-5,-2) \cup (-2,1)$.

(b) $\lim_{x\to\infty} f'(x) = \lim_{x\to\infty} \frac{x^2-2x+1}{x+2} = \lim_{x\to\infty} \frac{x-2+1/x}{1+2/x} = \infty.$

$\lim_{x\to-\infty} f'(x) = \lim_{x\to-\infty} \frac{x-2+1/x}{1+2/x} = -\infty$

$\lim_{x\to-2^-} f'(x) = -\infty, \lim_{x\to-2^+} f'(x) = -\infty$

(c)



7. This was ignored in grading the midterm since the first step was not entirely obvious. We want to compare $f'(x)$ and $g(x) = \frac{f(x)-f(a)}{x-a}$ for $x > a$ using the Race Track Principle. Ideally, we would show that $f'(a) = g(a)$ and obtain some inequality in their derivatives. However, $g(a) = \frac{f(a)-f(a)}{a-a}$ is not defined. Instead, we consider $h(x) = (x-a)f'(x)$ and $g(x) = f(x) - f(a)$.
   - $h(a) = 0$ and $g(a) = 0$ so $h(a) = g(a)$
   - $h'(x) = f'(x) + (x-a)f''(x)$ and $g'(x) = f'(x)$. Since $x > a$, $x - a > 0$, and since $f''(x) < 0$, $(x-a)f''(x) < 0$ as well. $h'(x) < f'(x) = g'(x)$

   By the Race Track Principle, $h(x) < g(x)$ for all $x > a$ so that

$$(x-a)f'(x) < f(x) - f(a)$$

$$f'(x) < \frac{f(x)-f(a)}{x-a}$$

8. Since velocity is the derivative of position, $x(t) = \frac{4^t}{\ln 4} - \frac{t^{e+1}}{e+1} + C$. $x(0) = 3 = \frac{4^0}{\ln 4} + C$ and $C = 3 - \frac{1}{\ln 4}$. $x(t) = \frac{4^t}{\ln 4} - \frac{t^{e+1}}{e+1} + 3 - \frac{1}{\ln 4}$.

9. $f'(x) = 2Ae^{2x}\sin 3x + 3Ae^{2x}\cos 3x$,

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$$\begin{aligned} f''(x) &= 4Ae^{2x}\sin 3x + 6Ae^{2x}\cos 3x + 6Ae^{2x}\cos 3x - 9Ae^{2x}\sin 3x = -5Ae^{2x}\sin 3x + 12Ae^{2x}\cos 3x. \\ y'' - 4y' + 13y &= -5Ae^{2x}\sin 3x + 12Ae^{2x}\cos 3x - 4(2Ae^{2x}\sin 3x + 3Ae^{2x}\cos 3x) + 13Ae^{2x}\sin 3x \\ &= (-5 - 8 + 13)Ae^{2x}\sin 3x + (12 - 12)Ae^{2x}\cos 3x \\ &= 0. \end{aligned}$$

If $f'(0) = 1$, $f'(0) = 3A = 1$ and $A = \dfrac{1}{3}$.