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LINEAR ALGEBRA

VECTOR SPACES

1. A **vector space** V over a field $\mathcal{F}$ consists of a set on which two operations (addition and scalar multiplication, respectively) are defined so that $\forall\ x, y \in V,\ \exists!\ x + y \in V$, and $\forall\ a \in \mathcal{F}$ and $x \in V,\ \exists!\ ax \in V$, such that the following conditions hold:
   - **a)** $\forall\ x, y \in V,\ x + y = y + x$ (commutativity of addition)
   - **b)** $\forall\ x, y, z \in V,\ (x + y) + z = x + (y + z)$ (associativity of addition)
   - **c)** $\exists!\ 0 \in V$, such that $x + 0 = x,\ \forall\ x \in V$
   - **d)** $\forall\ x \in V,\ \exists\ y \in V$ such that $x + y = 0$
   - **e)** $\forall\ x \in V,\ 1x = x$
   - **f)** $\forall\ a, b \in \mathcal{F}$ and $x \in V,\ (ab)x = a(bx)$
   - **g)** $\forall\ a \in \mathcal{F}$ and $x, y \in V,\ a(x + y) = ax + ay$
   - **h)** $\forall\ a, b \in \mathcal{F}$ and $x \in V,\ (a + b)x = ax + bx$.
2. The set of all $m \times n$ matrices with entries from a field $\mathcal{F}$ is a vector space, which we denote by $\mathcal{M}_{m\times n}(\mathcal{F})$, with the following operations of **matrix addition** and **scalar multiplication**: For $A, B \in \mathcal{M}_{m\times n}(\mathcal{F})$ and $c \in \mathcal{F}$,
   - **a)** $(A + B)_{ij} = A_{ij} + B_{ij}$,
   - **b)** $(cA)_{ij} = cA_{ij}$, for $1 \le i \le m$ and $1 \le j \le n$.
3. A **polynomial of degree** n, or $\mathbf{P}_n$, with coefficients from a field $\mathcal{F}$ is an expression of the form, $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, where n is a nonnegative integer and each a_k, called the **coefficient** of x^k, is in $\mathcal{F}$.
4. A $W \subseteq V$, where V is a vector space over a field $\mathcal{F}$ is called a **subspace** of V if W is a vector space over $\mathcal{F}$ with the operations of addition and scalar multiplication defined on V, and if and only if the follow properties hold:
   - **a)** $x + y \in W$ whenever $x \in W$ and $y \in W$ (closed under addition)
   - **b)** $cx \in W$ whenever $c \in \mathcal{F}$ and $x \in W$ (closed under scalar multiplication)
   - **c)** W has a zero vector
   - **d)** Each vector in W has an additive inverse in W.
5. The **transpose** A^T of an $m \times n$ matrix A is the $n \times m$ matrix obtained by interchanging the rows with the columns, that is, $(A^T)_{ij} = A_{ji}$. A matrix is **symmetric** if and only if $A^T = A$.
6. The **trace** of an $n \times n$ matrix M is the sum of its diagonal entries, that is, $tr(M) = M_{11} + M_{22} + ... + M_{nn}$.
7. A vector space V is called the **direct sum** of W_1 and W_2 if W_1 and W_2 are subspaces of V such that $W_1 \cap W_2 = \{0\}$ and $W_1 + W_2 = V$. We denote the direct sum by $V = W_1 \oplus W_2$.

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LINEAR ALGEBRA

VECTOR SPACES

8. Let S be a nonempty subset of a vector space V. The **span** of S, denoted $\text{span}(S)$, is the set consisting of all linear combinations of the vectors in S. Also, $\text{span}(\varnothing) = \{0\}$.

9. A subset S of a vector space V **generates** (or **spans**) V if $\text{span}(S) = V$.

10. A subset S of a vector space V is called **linearly dependent** if there exist a finite number of distinct vectors $u_1, u_2, \ldots, u_n$ in S and scalars $a_1, a_2, \ldots, a_n$, not all zero, such that $a_1u_1 + a_2u_2 + \cdots + a_nu_n = 0$. A subset S of a vector space V that is not linearly dependent is called **linearly independent**.

11. A **basis** β for a vector space V is a linearly independent subset of V that generates V.

12. A vector space is called **finite-dimensional** if it has a basis consisting of a finite number of vectors. The unique number of vectors in each basis for V is called the **dimension** of V and is denoted $\dim(V)$. A vector space, not finite-dimensional, is called **infinite-dimensional**.

LINEAR TRANSFORMATIONS AND MATRICES

13. Let V and W be vector spaces (over $\mathcal{F}$). We call a function $T : V \to W$ a **linear transformation from** V **to** W, or simply **linear**, if for all $x, y \in V$ and $c \in \mathcal{F}$, we have,

a) $T(x+y) = T(x) + T(y)$,

b) $T(cx) = cT(x)$.

14. For any angle θ, define $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ by the rule: $T_\theta(a_1, a_2)$ is the vector obtained by rotating (a_1, a_2) counterclockwise by θ if $(a_1, a_2) \neq (0,0)$, and $T_\theta(0,0) = (0,0)$. Then $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation that is called the **rotation by** θ. Furthermore, $T_\theta(a_1, a_2) = (a_1\cos\theta - a_2\sin\theta, a_1\sin\theta + a_2\cos\theta)$.

15. Let V and W be vector spaces, and let $T : V \to W$ be linear. We define the **null space** (or **kernal**), $\mathcal{N}(T)$ of T to be the set of all vectors x in V such that $T(x) = 0$, that is, $\mathcal{N}(T) = \{x \in V : T(x) = 0\}$. We define the **range** (or **image**) $\mathcal{R}(T)$ of T to be the subset of W consisting of all images (under T) of vectors in V, that is, $\mathcal{R}(T) = \{T(x) : x \in V\}$.

16. Let V and W be vector spaces, and let $T : V \to W$ be linear. If $\beta = \{v_1, v_2, \ldots, v_n\}$ is a basis for V, then $\mathcal{R}(T) = \text{span}(T(\beta)) = \text{span}(\{T(v_1), T(v_2), \ldots, T(v_n)\})$.

17. Let V and W be vector spaces, and let $T : V \to W$ be linear. If $\mathcal{N}(T)$ and $\mathcal{R}(T)$ are finite-dimensional, then we define the **nullity of** T, denoted $\text{nullity}(T)$, and the **rank of** T, denoted $\text{rank}(T)$, to be the dimensions of $\mathcal{N}(T)$ and $\mathcal{R}(T)$, respectively.

18. Let V and W be vector spaces, and let $T : V \to W$ be linear. If V is finite-dimensional, then $\text{nullity}(T) + \text{rank}(T) = \dim(V)$.

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LINEAR ALGEBRA

LINEAR TRANSFORMATIONS AND MATRICES

19. For the vector space $\mathcal{F}^n$, we call $\{e_1, e_2, ..., e_n\}$ the **standard ordered basis for** $\mathcal{F}^n$. For the vector space $\mathbf{P}_n(\mathcal{F})$, we call $\{1, x, ..., x^n\}$ the **standard ordered basis for** $\mathbf{P}_n(\mathcal{F})$.

20. Let $\beta = \{u_1, u_2, ..., u_n\}$ be an ordered basis for a finite-dimensional vector space V. For $x \in V$ let $a_1, a_2, ..., a_n$ be the unique scalars such that, $x = \sum_{i=1}^{n} a_i u_i$. We define the **coordinate vector of** x **relative to** β, denoted $[x]_\beta$, by $[x]_\beta = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$.

21. Using the previous notation, we call the $m \times n$ matrix A defined by $A_{ij} = a_{ij}$ the **matrix representation of** T **in the ordered bases** β **and** γ and write $A = [T]_\beta^\gamma$. If $V = W$ and $\beta = \gamma$, then we write $A = [T]_\beta$.

22. Let $T, U : V \to W$ be arbitrary functions, where V and W are vector spaces over $\mathcal{F}$, and let $a \in \mathcal{F}$. We define $T + U : V \to W$ by $(T+U)(x) = T(x) + U(x),\ \forall x \in V$, and $aT : V \to W$ by $(aT)(x) = aT(x),\ \forall x \in V$.

23. Let V and W be vector spaces over $\mathcal{F}$. We denote the vector space of all linear transformations from V into W by $\mathcal{L}(V, W)$.

24. Let $A \in \mathcal{M}_{m\times n}(\mathcal{F})$, $B \in \mathcal{M}_{n\times p}(\mathcal{F})$, then the **matrix multiplication** given by $AB \in \mathcal{M}_{m\times p}(\mathcal{F})$ where $(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$, for $1 \le i \le m,\ 1 \le j \le p$. Example:

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} \cdot & \cdot & \cdot & a & \cdot \\ \cdot & \cdot & \cdot & b & \cdot \\ \cdot & \cdot & \cdot & c & \cdot \\ \cdot & \cdot & \cdot & d & \cdot \end{bmatrix} = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & x_{3,4} & \cdot \end{bmatrix}$$

$x_{3,4} = (1, 2, 3, 4) \cdot (a, b, c, d) = 1 \times a + 2 \times b + 3 \times c + 4 \times d.$

25. We define the **Kronecker delta** δ_{ij}, by $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \ne j$. Thus, the $n \times n$ **identity matrix** I_n is defined by $(I_n)_{ij} = \delta_{ij}$.

26. Let $A \in \mathcal{M}_{m\times n}(\mathcal{F})$, we denote by L_A the mapping $L_A : \mathcal{F}^n \to \mathcal{F}^m$ defined by $L_A(x) = Ax$, for each column vector $x \in \mathcal{F}^n$. We call L_A the **left-multiplication transformation**.

27. Let V and W be vector spaces, and let $T : V \to W$ be linear. A function $U : W \to V$ is said to be an **inverse** of T if $TU = I_W$ and $UT = I_V$. If T has an inverse, then T is said to be **invertible**. If T is invertible, then the inverse of T is unique, and is denoted T^{-1}.

28. Let A be an $n \times n$ matrix. A is **invertible** if $\exists$ an $n \times n$ matrix B such that $AB = BA = I$.

LINEAR ALGEBRA

LINEAR TRANSFORMATIONS AND MATRICES

29. Let V and W be vector spaces. We say that V is **isomorphic to** W if there exists a linear transformation $T:V\to W$ that is invertible. Such a linear transformation is called an **isomorphism** from V to W.

30. Let β be an ordered basis for an n-dimensional vector space V over the field $\mathcal{F}$. The **standard representation of** V **with respect to** β is the function $\phi_\beta : V\to\mathcal{F}^n$ defined by $\phi_\beta(x)=[x]_\beta,\ \forall x\in V$.

31. Let V and W be vector spaces of dimension n and m, respectively, and let $T:V\to W$ be a linear transformation. Define $A=[T]_\beta^\gamma$, where β and γ are arbitrary ordered bases of V and W, respectively. We now use ϕ_β and ϕ_γ to form a relationship with the linear transformation T and $L_A:\mathcal{F}^n\to\mathcal{F}^m$. Consider this figure:

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ \phi_\beta\downarrow & & \downarrow\phi_\gamma \\ \mathcal{F}^n & \xrightarrow[L_A]{} & \mathcal{F}^m \end{array}$$

, where we can conclude that $L_A\phi_\beta=\phi_\gamma T$.

32. Let β and β' be two ordered bases for a finite-dimensional vector space V, and let $Q=[I_V]_{\beta'}^{\beta}$, then,

a) Q is invertible,

b) for any $v\in V$, $[v]_\beta=Q[v]_{\beta'}$.

33. The matrix $Q=[I_V]_{\beta'}^{\beta}$ above is called a **change of coordinate matrix**. We say that Q **changes β'-coordinates into β-coordinates**. Observe that if $\beta=\{x_1,x_2,...,x_n\}$ and $\beta'=\{x'_1,x'_2,...,x'_n\}$, then $x'_j=\sum_{i=1}^{n}Q_{ij}x_i,\ j=1,2,...,n$ that is, the jth column of Q is $[x'_j]_\beta$.

34. Let T be a linear operator on a finite-dimensional vector space V, and let β and β' be ordered bases for V. Suppose that Q is the change of coordinate matrix that changes β'-coordinates into β-coordinates, then $[T]_{\beta'}=Q^{-1}[T]_\beta Q$.

35. Let A and B be matrices in $\mathcal{M}_{m\times n}(\mathcal{F})$. We say that B is **similar** to A if there exists an invertible matrix Q such that $B=Q^{-1}AQ$.

36. Let V be the vector space of continuous real-valued functions on the interval $[0,2\pi]$. Fix a function $g\in V$. The function $h:V\to\mathbb{R}$ defined by $h(x)=\frac{1}{2\pi}\int_0^{2\pi}x(t)g(t)dt$ is a linear functional on V. In the cases that $g(t)$ equals $\sin nt$ or $\cos nt$, $h(x)$ is often called the **nth Fourier coefficient of x**.

LINEAR ALGEBRA

ELEMENTARY MATRIX OPERATIONS AND SYSTEMS OF LINEAR EQUATIONS

37. Let $A \in \mathcal{M}_{m\times n}(\mathcal{F})$. Any one of the following three operations on the rows [columns] of A is called an **elementary row [column] operation**:
- **a)** interchanging any two rows [columns] of A,
- **b)** multiplying any row [column] of A by a nonzero scalar,
- **c)** adding any scalar multiple of a row [column] of A to another row [column].

38. An **elementary matrix** is a matrix obtained by performing an elementary operation on I_n.

39. If $A \in \mathcal{M}_{m\times n}(\mathcal{F})$, we define the **rank** of A, denoted $\operatorname{rank}(A)$, to be the rank of the linear transformation $L_A : \mathcal{F}^n \to \mathcal{F}^m$.

40. Elementary operations preserve the rank of a matrix.

41. A system $Ax = b$ of m linear equations in n unknowns is said to be **homogenous** if $b = 0$. Otherwise the system is said to be **nonhomogenous**.

42. Let K be the solution set of a system of linear equations $Ax = b$, and let K_{H} be the solution set of the corresponding homogenous system $Ax = 0$. Then for any solution s to $Ax = b$, $K = \{s\} + K_{\mathrm{H}} = \{s + k : k \in K_{\mathrm{H}}\}$.

43. Two systems of linear equations are called **equivalent** if they have the same solution set.

44. A matrix is said to be in **reduced row echelon form** if the following are satisfied:
- **a)** Any row containing a nonzero entry precedes any row in which all the entries are zero (if any).
- **b)** The first nonzero entry in each row is the only nonzero entry in its column.
- **c)** The first nonzero entry in each row is 1 and it occurs in a column to the right of the first nonzero entry in the preceding row.

45. Gaussian [Gauss-Jordan] Elimination: Elementary row operations are used to reduce a matrix to row [reduced-row] echelon form in order to find a solution set to the system of linear equations.

DETERMINANTS

46. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a 2×2 matrix with entries from a field $\mathcal{F}$, then we define the **determinant** of A, denoted $\det(A)$ or $|A|$, to be the scalar $ad - bc$.

47. Let $A \in \mathcal{M}_{2\times 2}(\mathcal{F})$. Then the determinant of A is nonzero if and only if A is invertible. Moreover, if A is invertible, then $A^{-1} = \dfrac{1}{\det(A)}\begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix}$.

48. Let $A \in \mathcal{M}_{n\times n}(\mathcal{F})$. If $n = 1$, so that $A = (A_{11})$, we define $\det(A) = A_{11}$. For $n \geq 2$, we define $\det(A)$ recursively as, $\det(A) = \sum_{j=1}^{n} (-1)^{1+j} A_{1j} \bullet \det(\tilde{A}_{1j})$. The scalar, $(-1)^{i+j}\det(\tilde{A}_{ij})$ is called the **cofactor** of the entry of A in row i, column j.

LINEAR ALGEBRA

DETERMINANTS

49. If $A \in \mathcal{M}_{n\times n}(\mathcal{F})$ has a row consisting entirely of zeros, then $\det(A) = 0$.

50. The determinant of a square matrix A can be evaluated along any row, such that

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} A_{ij} \bullet \det(\tilde{A}_{ij}), \text{ for any integer } 1 \le i \le n.$$

51. If $A \in \mathcal{M}_{n\times n}(\mathcal{F})$ and B is a matrix obtained from A by interchanging any two rows of A, then $\det(B) = -\det(A)$.

52. If $A \in \mathcal{M}_{n\times n}(\mathcal{F})$ and B is a matrix obtained from A by adding a multiple of one row of A to another row of A, then $\det(B) = \det(A)$.

53. If $A \in \mathcal{M}_{n\times n}(\mathcal{F})$ and B is a matrix obtained from A by multiplying one row of A by some nonzero scalar $k \in \mathcal{F}$, then $\det(B) = k\det(A)$.

54. If $A \in \mathcal{M}_{n\times n}(\mathcal{F})$ has rank less than n, then $\det(A) = 0$.

55. The determinant of an upper triangular matrix is the product of its diagonal entries.

56. A matrix $A \in \mathcal{M}_{n\times n}(\mathcal{F})$ is invertible if and only if $\det(A) \ne 0$. Also, $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

57. For any $A \in \mathcal{M}_{n\times n}(\mathcal{F})$, $\det(A^T) = \det(A)$.

58. Let $Ax = b$ be the matrix form of a system of n linear equations in n unknowns, where $x = (x_1, x_2, \ldots, x_n)^T$. If $\det(A) \ne 0$, then this system has a unique solution, and for each $k = (1, 2, \ldots, n)$, $x_k = \dfrac{\det(M_k)}{\det(A)}$, where M_k is the $n \times n$ matrix obtained from A by replacing column k of A by b.

59. A function $\delta : \mathcal{M}_{n\times n}(\mathcal{F}) \to \mathcal{F}$ is called an n**-linear function** if it a linear function of each row of an $n \times n$ matrix when the remaining $n-1$ rows are held fixed, that is, δ is n-linear if, for every $r = 1, 2, \ldots, n$, we have
$$\delta\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ u + kv \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} = \delta\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ u \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} + k\delta\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ v \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix}$$
whenever k is a scalar and u, v, and each a_i are vectors in $\mathcal{F}^n$.

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LINEAR ALGEBRA

DIAGONALIZATION

60. A linear operator T on a finite-dimensional vector space V is called **diagonalizable** if there is an ordered basis β for V such that $[T]_\beta$ is a diagonal matrix.

61. Let T be a linear operator on a vector space V. A nonzero vector $v \in V$ is called an **eigenvector** of T if there exists a scalar λ such that $T(v) = \lambda v$. The scalar λ is called the **eigenvalue** corresponding to the eigenvector v.

62. Let $A \in \mathcal{M}_{n\times n}(\mathcal{F})$. The polynomial $f(t) = \det(A - tI_n)$ is called the **characteristic polynomial** of A. To find the eigenvalue(s) of a matrix, we compute $\det(A - tI_n)$.

63. Let T be a linear operator on a vector space V, and let λ be an eigenvalue of T. A vector $v \in V$ is an eigenvector of T corresponding to λ if and only if $v \neq 0$ and $v \in \mathcal{N}(T - \lambda I)$.

64. A polynomial $f(t)$ in $\mathbf{P}(\mathcal{F})$ **splits over** $\mathcal{F}$ if there are scalars $c, a_1, \ldots, a_n$ (not necessarily distinct) in $\mathcal{F}$ such that $f(t) = c(t - a_1)(t - a_2)\ldots(t - a_n)$.

65. Let λ be an eigenvalue of a linear operator or matrix with characteristic polynomial $f(t)$. The **(algebraic) multiplicity** of λ is the largest positive integer k for which $(t-\lambda)^k$ is a factor of $f(t)$.

66. Let T be a linear operator on a vector space V, and let λ be an eigenvalue of T. Define $E_\lambda = \{x \in V : T(x) = \lambda x\} = \mathcal{N}(T - \lambda I_V)$. The set E_λ is called the **eigenspace** of T corresponding to the eigenvalue λ.

67. Let T be a linear operator on an n-dimensional vector space V. Then T is diagonalizable if and only if both of the following conditions hold:

a) The characteristic polynomial of T splits.

b) For each eigenvalue λ of T, the multiplicity of λ equals $n - \operatorname{rank}(T - \lambda I)$.

INNER PRODUCT SPACES

68. Let V be a vector space over $\mathcal{F}$. An **inner product** on V is a function that assigns, to every ordered pair of vectors x and y in V, a scalar $\mathcal{F}$, denoted $\langle x, y\rangle$, such that $\forall x, y, z \in V$ and all $c \in \mathcal{F}$, the following hold:

a) $\langle x+z, y\rangle = \langle x, y\rangle + \langle z, y\rangle$

b) $\langle cx, y\rangle = c\langle x, y\rangle$

c) $\overline{\langle x, y\rangle} = \langle y, x\rangle$, where the over-bar denotes complex conjugation

d) $\langle x, x\rangle > 0$, if $x \neq 0$

69. Let $A \in \mathcal{M}_{m\times n}(\mathcal{F})$. We define the **conjugate transpose** or **adjoint** of A to be the $n \times m$ matrix A^* such that $(A^*)_{ij} = \overline{A_{ji}}$, for all i, j.

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LINEAR ALGEBRA

INNER PRODUCT SPACES

70. Let V be an inner product space. For $x \in V$, we define the **norm** or **length** of x by $\|x\| = \sqrt{\langle x, x \rangle}$.

71. Let V be an inner product space. Vectors $x, y \in V$ are **orthogonal** (**perpendicular**) if $\langle x, y \rangle = 0$. A subset S of V is **orthogonal** if any two distinct vectors in S are orthogonal. A vector $x \in V$ is a **unit vector** if $\|x\| = 1$. Finally, a subset S of V is **orthonormal** if S is orthogonal and consists entirely of unit vectors.

72. The process of multiplying a nonzero vector by the reciprocal of its length, or norm, is called **normalizing**.

73. Let V be an inner product space. A subset of V is an **orthonormal basis** for V if it an ordered basis that is orthonormal.

74. The Gram-Schmidt Process: Let V be an inner product space and $S = \{w_1, w_2, ..., w_n\}$ be a linearly independent subset of V. Define $S' = \{v_1, v_2, ..., v_n\}$, where $v_1 = w_1$, and $v_k = w_k - \sum_{j=1}^{k-1} \frac{\langle w_k, v_j \rangle}{\|v_j\|^2} v_j$, for $2 \le k \le n$. Then S' is an orthogonal set of nonzero vectors such that $\text{span}(S') = \text{span}(S)$.

75. Let β be an orthonormal subset (possibly infinite) of an inner product space V, and let $x \in V$. The **Fourier coefficients** of x relative to β are the scalars $\langle x, y \rangle$, where $y \in \beta$.

76. Let S be a nonempty subset of an inner product space V. We define $S^\perp$, or "S perp", to be the set of all vectors in V that are orthogonal to every vector in S; that is, $S^\perp = \{x \in V : \langle x, y \rangle = 0 \text{ for all } y \in S\}$. The set $S^\perp$ is called the **orthogonal complement** of S.

77. Let V be an inner product space, and let T be a linear operator on V. We say that T is **normal** if $TT^* = T^*T$.

78. Let T be a linear operator on an inner product space V. We say that T is **self-adjoint** (**Hermitian**) if $T = T^*$.

79. Let T be a linear operator on a finite-dimensional inner product space V over $\mathcal{F}$. If $\|T(x)\| = \|x\|$ for all $x \in V$, we call T a **unitary operator** if $\mathcal{F} = \mathbb{C}$ and an **orthogonal operator** if $\mathcal{F} = \mathbb{R}$. In the infinite-dimensional case, it is generally called an **isometry**.

80. A square matrix A is called an **orthogonal matrix** if $A^T A = AA^T = \mathcal{I}$ and unitary if $A^* A = AA^* = \mathcal{I}$.

81. Let V be a real inner product space. A function $f : V \to V$ is called a **rigid motion** if $\|f(x) - f(y)\| = \|x - y\|$ for all $x, y \in V$.

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INNER PRODUCT SPACES

82. Let V be an inner product space, and let $T: V \to V$ be a projection. We say that T is an **orthogonal projection** if $\mathcal{R}(T)^{\perp} = \mathcal{N}(T)$ and $\mathcal{N}(T)^{\perp} = \mathcal{R}(T)$.

83. The Spectral Theorem: Suppose that T is a linear operator on a finite-dimensional inner product space V over $\mathcal{F}$ with the distinct eigenvalues $\lambda_1, \lambda_2, ..., \lambda_k$. Assume that T is normal if $\mathcal{F} = \mathbb{C}$ and that T is self-adjoint if $\mathcal{F} = \mathbb{R}$. For each i, $1 \le i \le k$, let W_i be the eigenspace of T corresponding to the eigenvalue λ_i, and let T_i be the orthogonal projection of V on W_i. Then the following statements are true:

a) $V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$,

b) If W_i' denotes the direct sum of the subspaces W_j for $j \ne i$, then $W_i^{\perp} = W_i'$,

c) $T_i T_j = \delta_{ij} T_i$, for $1 \le i, j \le k$,

d) $\mathcal{I} = T_1 + T_2 + \cdots + T_k$,

e) $T = \lambda_1 T_1 + \lambda_2 T_2 + \cdots + \lambda_k T_k$.

84. The set $\{\lambda_1, \lambda_2, ..., \lambda_k\}$ of eigenvalues of T is called the **spectrum** of T.

85. The sum $\mathcal{I} = T_1 + T_2 + \cdots + T_k$ from **83. (d)** is called the **resolution of the identity operator** induced by T.

86. The sum $T = \lambda_1 T_1 + \lambda_2 T_2 + \cdots + \lambda_k T_k$ from **83. (e)** is called the **spectral decomposition** of T.

CANONICAL FORMS

87. Let T be a linear operator on a vector space V and let λ be a scalar. A nonzero vector x in V is called a **generalized eigenvector of T corresponding to λ** if $(T - \lambda\mathcal{I})^p(x) = 0$, for some positive integer p.

88. Let T be a linear operator on a vector space V, and let λ be an eigenvalue of T. The **generalized eigenspace of T corresponding to λ**, denoted, K_λ, is the subset of V defined by $K_\lambda = \left\{x \in V : (T - \lambda\mathcal{I})^p(x) = 0, \text{ for some positive integer } p\right\}$.

89. Let T be a linear operator on a vector space V, and let x be a generalized eigenvector of T corresponding to the eigenvalue λ. Suppose that p is the smallest positive integer for which $(T - \lambda\mathcal{I})^p(x) = 0$. Then the ordered set $\left\{(T - \lambda\mathcal{I})^{p-1}(x), (T - \lambda\mathcal{I})^{p-2}(x), ..., (T - \lambda\mathcal{I})(x), x\right\}$ is called a **cycle of generalized eigenvectors** of T corresponding to λ. The vectors $(T - \lambda\mathcal{I})^{p-1}(x)$ and x are called the **initial vector** and the **end vector** of the cycle, respectively. We say that the **length of the cycle** is p.

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90. Let T be a linear operator on a finite-dimensional vector space V, and suppose the characteristic polynomial of T splits, and let β be the union of ordered bases of generalized eigenspaces of V, such that $[T]_\beta = \begin{pmatrix} A_1 & O & \cdots & O \\ O & A_2 & \cdots & O \\ \vdots & \vdots & & \vdots \\ O & O & \cdots & A_k \end{pmatrix}$, where each O is a zero matrix, and each A_i is a square matrix of the form (λ) or $\begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{pmatrix}$ for some eigenvalue λ of T. Such a matrix A_i is called a **Jordan block** corresponding to λ, and the matrix $[T]_\beta$ is called a **Jordan canonical form** of T. We also say that the ordered basis β is a **Jordan canonical basis** for T.

91. Let $A \in \mathcal{M}_{n\times n}(\mathcal{F})$ be such that the characteristic polynomial of A (and hence L_A) splits. Then the **Jordan canonical form** of A is defined to be the Jordan canonical form of the linear operator L_A on $\mathcal{F}^n$. Example:

Let $A = \begin{pmatrix} 3 & 1 & -2 \\ -1 & 0 & 5 \\ -1 & -1 & 4 \end{pmatrix} \in \mathcal{M}_{3\times 3}(\mathbb{R})$, to find the Jordan canonical form for A, we need to find a Jordan canonical basis for $T = L_A$. The characteristic polynomial of A is $f(t) = \det(A - t\mathcal{I}) = -(t-3)(t-2)^2$. Hence $\lambda_1 = 3$ and $\lambda_2 = 2$ are eigenvalues of A with multiplicities 1 and 2, respectively. Then $\dim(K_{\lambda_1}) = 1$, and $\dim(K_{\lambda_2}) = 2$. Then $K_{\lambda_1} = \mathcal{N}(T - 3\mathcal{I})$, and $K_{\lambda_2} = \mathcal{N}\left((T - 2\mathcal{I})^2\right)$. Since $E_{\lambda_1} = \mathcal{N}(T - 3\mathcal{I})$, we have $E_{\lambda_1} = K_{\lambda_1}$. Observe that $(-1,2,1)$ is an eigenvector of T corresponding to $\lambda_1 = 3$; therefore $\beta_1 = \left\{ \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \right\}$ is a basis for K_{λ_1}. Since $\dim(K_{\lambda_2}) = 2$ and a generalized eigenspace has a basis consisting of a union of cycles, this basis is either a union of two cycles of length 1, or a single cycle of length 2. **Continued on next page…**

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91. …Continued. The former case (union of two cycles of length 1) is impossible because the vectors in the basis would be eigenvectors—contradicting the face that $\dim\left(E_{\lambda_2}\right)=1$. Therefore the desired basis is a single cycle of length 2. A vector v is the end vector of such a cycle if and only if $(A-2I)v\neq 0$, but $(A-2I)^2 v=0$. Then $\left\{\begin{pmatrix}1\\-3\\-1\end{pmatrix},\begin{pmatrix}-1\\2\\0\end{pmatrix}\right\}$ is a basis for the solution space of the homogenous system $(A-2I)^2 x=0$. Now choose a vector v in this set so that $(A-2I)v\neq 0$. The vector $v=(-1,2,0)$ is an acceptable candidate. Since $(A-2I)v=(1,-3,-1)$, we obtain the cycle of generalized eigenvectors

$\beta_2=\left\{(A-2I)v,v\right\}=\left\{\begin{pmatrix}1\\-3\\-1\end{pmatrix},\begin{pmatrix}-1\\2\\0\end{pmatrix}\right\}$ as a basis for K_{λ_2}. Finally, we can take the union of these two bases to obtain $\beta=\beta_1\cup\beta_2=\left\{\begin{pmatrix}-1\\2\\1\end{pmatrix},\begin{pmatrix}1\\-3\\-1\end{pmatrix},\begin{pmatrix}-1\\2\\0\end{pmatrix}\right\}$, which is a Jordan canonical basis for A. Therefore, $J=[T]_\beta=\begin{pmatrix}3&0&0\\0&2&1\\0&0&2\end{pmatrix}$ is a Jordan canonical form for A. Notice that A is similar to J; in fact, $J=Q^{-1}AQ$, where Q is the matrix whose columns are the vectors in β.

92. Each generalized eigenspace K_{λ_i} contains an ordered basis β_i consisting of a union of disjoint cycles of generalized eigenvectors corresponding to λ_i. So the union $\beta=\bigcup_{i=1}^{k}\beta_i$ is a Jordan canonical basis for T. For each i, let T_i be the restriction of T to K_{λ_i}, and let $A_i=[T_i]_{\beta_i}$. Then A_i is the Jordan canonical form of T_i, and $J=[T]_\beta=\begin{pmatrix}A_1&O&\cdots&O\\O&A_2&\cdots&O\\\vdots&\vdots&&\vdots\\O&O&\cdots&A_k\end{pmatrix}$ is the Jordan canonical form of T.

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93. To help visualize each of the matrices A_i in a Jordan canonical form, and ordered bases β_i, we use an array of dots called a **dot diagram** of T_i, where T_i is the restriction of T to K_{λ_i}. Suppose that β_i is a disjoint union of cycles of generalized eigenvectors $\gamma_1, \gamma_2, ..., \gamma_{n_i}$ with lengths $p_1 \geq p_2 \geq \cdots \geq p_{n_i}$, respectively. The dot diagram of T_i contains one dot for each vector in β_i, and the dots are configured according to the following rules:

a) The array consists of n_i columns (one column for each cycle).

b) Counting from left to right, the jth column consists of the p_j dots that correspond to the vectors of γ_j starting with the initial vector at the top and continuing down to the end vector.

Denote the end vectors of the cycles by $v_1, v_2, ..., v_{n_i}$. Example:

$$\begin{array}{lll}
\bullet(T-\lambda_i I)^{p_1-1}(v_1) & \bullet(T-\lambda_i I)^{p_1-1}(v_2) \quad \cdots & \bullet(T-\lambda_i I)^{p_{n_i}-1}(v_{n_i}) \\
\bullet(T-\lambda_i I)^{p_1-2}(v_1) & \bullet(T-\lambda_i I)^{p_1-2}(v_2) & \bullet(T-\lambda_i I)^{p_{n_i}-2}(v_{n_i}) \\
\vdots & \vdots & \vdots \\
\vdots & \vdots & \bullet(T-\lambda_i I)(v_{n_i}) \\
\vdots & \bullet(T-\lambda_i I)(v_2) & \bullet v_{n_i} \\
\bullet(T-\lambda_i I)(v_1) & \bullet v_2 & \\
\bullet v_1 & &
\end{array}$$

94. A linear operator T on a vector space V, [$n \times n$ matrix A], is called **nilpotent** if $T^p = T_0$, [$A^p = O$], for come positive integer p.

95. Let T be a linear operator on a finite-dimensional vector space. A polynomial $p(t)$ is called the **minimal polynomial** of T if $p(t)$ is a monic (leading coefficient is 1) polynomial of least positive degree for which $p(T) = T_0$. It follows for $A \in \mathcal{M}_{n\times n}(\mathcal{F})$, if $p(t)$ is a monic polynomial of least positive degree for which $p(A) = O$.