Name: ____________________ Teacher: ________________

SYDNEY TECHNICAL HIGH SCHOOL

Trial Higher School Certificate

August 2010

Mathematics-Extension 2

Time allowed : 180 minutes

Reading Time: 5 minutes

Instructions

- Use black or blue pen.
- Approved calculators may be used.
- All necessary working must be shown. Marks may not be awarded for careless or badly arranged work.
- Marks awarded are shown on each question.
- Total marks — 120
- Attempt all questions.
- Start each question on a new page.
- A table of *Standard Integrals* is attached.

Question	1	2	3	4	5	6	7	8	Total /120
Marks /15									

Question 1 **Marks 15**

a) Find $\int x^2\cos(x^3 - 1)dx$ **2**

b) Using integration by parts, or otherwise, evaluate $\int \cos^{-1}x\, dx$ **3**

c) Using the substitution $u^2 = e^x + 1$, or otherwise, find $\int \frac{e^{2x}dx}{\sqrt{e^x + 1}}$ **4**

d) Using partial fractions, or otherwise, find $\int \frac{dx}{4x^2 - 1}$ **3**

e) Evaluate $\int_3^4 \frac{x^2 + x - 4}{x - 2}dx$ **3**

Question 2 **Marks 15**

a) Let $A = 2 - i$ and $B = 3 + 4i$.

Find, in the form $x + iy$

(i) $A - iB$ **1**

(ii) $\bar{A}B$ **1**

(iii) $\frac{5}{A}$ **2**

b) If $z = \sqrt{3} + i$

(i) Express z in modulus–argument form **2**

(ii) Hence find z^4 in $x + iy$ form. **2**

c) On an Argand diagram, clearly show the region where the inequalities $2 < |z| \leq 4$ and $\frac{-\pi}{4} \leq arg\ z \leq \frac{\pi}{2}$ hold simultaneously. **3**

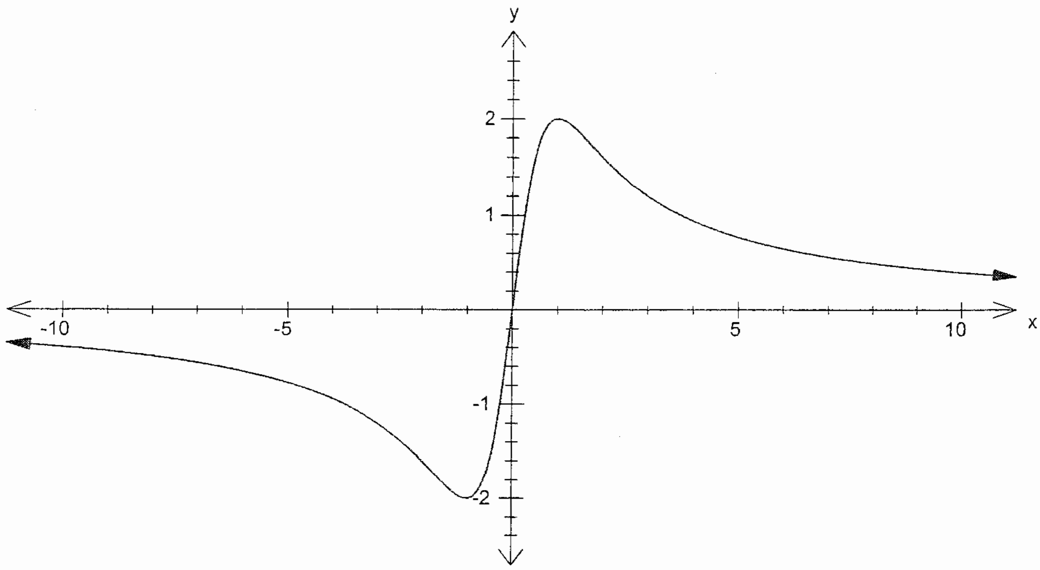
d) (i) With the aid of a diagram, describe the locus of Z on the Argand diagram if **2**

$arg\,(z - 2k) - arg\,(z) = \frac{\pi}{2}$, $k > 0, k \in R(real\ numbers)$.

(ii) What is the Cartesian equation of this locus? **2**

Question 3 **Marks 15**

a) The diagram below shows the graph of $y = f(x)$, which is an odd function.



Draw neat separate sketches showing all necessary detail of the following:

(i) $y = f(-x)$ **1**

(ii) $y = [f(x)]^2$ **1**

(iii) $y = \frac{1}{f(x)}$ **2**

(iv) $y = x + f(x)$, showing any asymptotes. **2**

(v) $y = f'(x)$ **2**

b) Sketch the graph of $y = \frac{x-2}{x^2-4}$, clearly indicating any special features. **3**

c) Consider the function $y = tan^{-1}x - x + \frac{1}{3}x^3$.

(i) Show that $\frac{dy}{dx} > 0$ for all values of $x > 0$. **2**

(ii) Show that $tan^{-1}x > x - \frac{1}{3}x^3$ for all values of $x > 0$. **2**

Question 4 **Marks 15**

a) An ellipse, E can be described as the locus of a point moving so that the sum of its distances from two fixed points (foci) is constant.

(i) If the two fixed points are $S(4,0)$ and $S'(-4,0)$ and the sum of the distances of $P(x,y)$ from these points is 10 *units*, show that the equation of E is given by $\frac{x^2}{25}+\frac{y^2}{9}=1$ *[You may use the standard ellipse equation]* 2

(ii) Verify that $x = 5\cos\theta$ and $y = 3\sin\theta$ are the parametric equations of E. 1

(iii) Find the equation of the normal to E at the point where $\theta = \frac{\pi}{6}$. 3

(iv) Determine the eccentricity of E and, hence, the equations of the directrices. 2

b) Given that α, β and γ are the roots of $3x^3+4x^2-2x-1=0$, find the values of:

(i) $\alpha+\beta+\gamma$ 1

(ii) $\alpha\beta+\alpha\gamma+\beta\gamma$ 1

(iii) $\alpha^2+\beta^2+\gamma^2$ 1

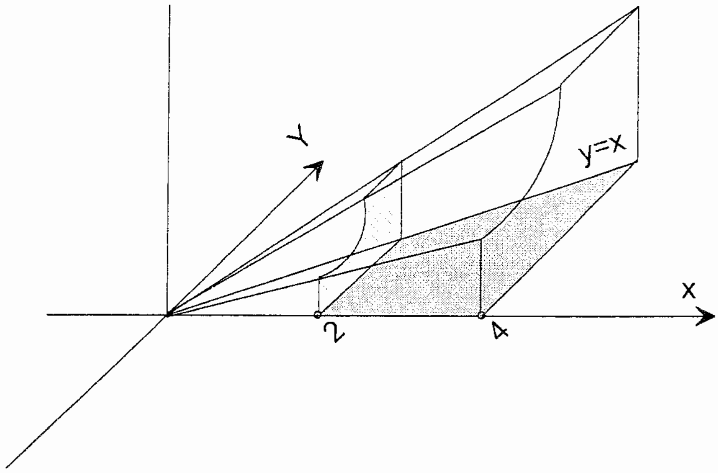
(iv) $\alpha^3+\beta^3+\gamma^3$ 2

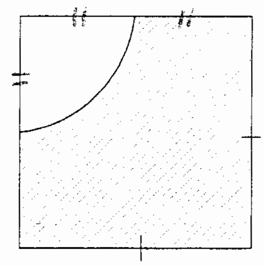
c) Factorise x^4-2x^2-15 over the rational and complex fields. 2

Question 5 **Marks 15**

a) The solid below has its base defined by the x-axis, the line $y = x$ and the lines $x = 2$ and $x = 4$ *(metres)*. Cross-sections consist of a square with a quarter circle (quadrant) removed (as shown). The radius of the circle is half of the side length of the square. **3**

Using the slicing technique, calculate the volume of this solid to the nearest cubic metre.





b) (i) Show that, if $y = px + q$ is a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then $p^2a^2 - b^2 = q^2$ **3**

(ii) Hence or otherwise, find the equations of the tangents from the point $(1,3)$ to the hyperbola $\frac{x^2}{4} - \frac{y^2}{15} = 1$. **3**

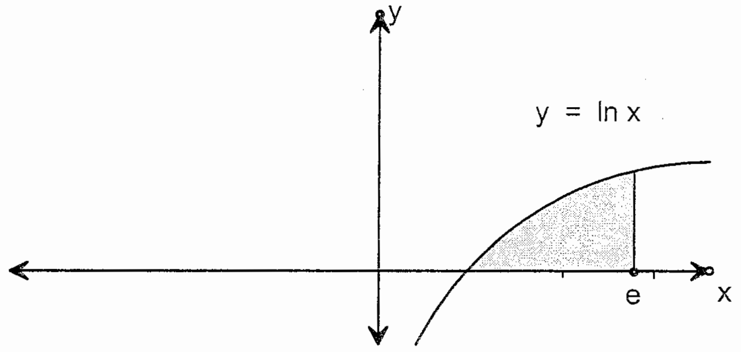
c) If $I_n = \int_0^1 x^n e^{-x} dx$

(i) Show that $I_n = -\frac{1}{e} + nI_{n-1}$ **3**

(ii) Hence find the exact value of $\int_0^1 x^3 e^{-x} dx$. **3**

Question 6 **Marks 15**

a) The region bounded by $y = \ln x$, $x = e$ and the x-axis is rotated about the y-axis. **4**
Use the cylindrical shells method to find the volume of the solid formed.

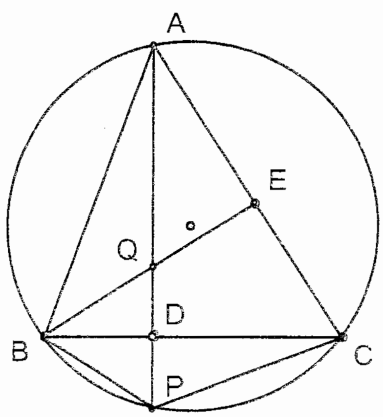


b) The angles A, B and C are consecutive terms in an arithmetic sequence.

(i) Show that $A + C = 2B$ **1**

(ii) Hence, show that $\cos A \cos C - \cos^2 B = \sin A \sin C - \sin^2 B$. **2**

(iii) ABC is an acute angled triangle inscribed in a circle. AP is perpendicular to BC.
Q is the point on AP such that $DQ = DP$. BQ produced meets AC at E.



(i) Copy the diagram showing the above information. **1**

(ii) Show that $\Delta BDP \equiv \Delta BDQ$. **2**

(iii) Show that $BDEA$ is a cyclic quadrilateral. **4**

(iv) Show that BE is perpendicular to AC. **1**

Question 7 **Marks 15**

a) A particle is allowed to fall under gravity from rest in a medium which exerts a resistance proportional to the speed (v) of the particle.

(i) Show that the particle reaches a terminal velocity, T given by $T = \frac{g}{k}$ (where k is a positive constant). **2**

(ii) Show that the distance fallen to reach half its terminal velocity $(\frac{T}{2})$ is given by **4**

$$x = \frac{T^2}{g} \ln 2 - \frac{T^2}{2g}$$

(iii) Determine an expression for the time taken to reach a speed of $\frac{T}{2}$. **3**

b) Consider the curve given by the equation $x^2 - y^2 + xy + 5 = 0$.

(i) Show that $\frac{dy}{dx} = \frac{2x+y}{2y-x}$ **2**

(ii) Hence or otherwise, find the coordinates of the points on the curve where the tangent to the curve is parallel to the line $y = x$. **2**

c) By taking the logarithms of both sides of $y = U(x).V(x)$, verify the Product Rule for differentiation. **2**

Question 8 **Marks 15**

a) (i) If α is a double root of a polynomial $P(x)$, show that α is a zero of $P'(x)$. **2**

(ii) Find the integers m and n such that $(x+1)^2$ is a factor of $x^5 + 2x^2 + mx + n$. **3**

b) (i) Find the five roots of $z^5 = 1$ and write them in mod-arg form. **3**

(ii) Show these roots on an Argand diagram and find the area (in exact form) of the pentagon formed by them. **2**

(iii) Factorise $z^5 - 1$ over the real field. **2**

c) The lengths of the sides of a triangle are the first three terms of an arithmetic sequence, with the first term equal to 1 and the common difference d. **3**
Find the set of possible values of d.

End of paper