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SYDNEY TECHNICAL HIGH SCHOOL

YEAR 12 HSC COURSE

Extension 2 Mathematics

TRIAL HIGHER SCHOOL CERTIFICATE

August 2009

TIME ALLOWED: 120 minutes

READING TIME: 5 minutes

Instructions:

- Write your name and class at the top of this page, and on all your answer sheets.
- Hand in your answers attached to the rear of this question sheet.
- All necessary working must be shown. Marks may not be awarded for careless or badly arranged work.
- All questions are of equal value.
- Marks indicated within each question are a guide only and may be varied at the time of marking
- START ALL QUESTIONS ON A NEW PAGE
- Approved calculators may be used.
- A table of *Standard Integrals* is attached. You may detach this page now.

(FOR MARKERS USE ONLY)

1	2	3	4	5	6	7	8	TOTAL
15 /15	14 /15	11 /15	18 /15	12 /15	11 /15	13 /15	3 /15	~~85~~ /120

90

QUESTION 1:

Marks

(a) Find

2 (i) $\int cos^3x\, dx$

2 (ii) $\int \frac{dx}{x^2-4x+8}$

2 (iii) $\int_1^5 \frac{dx}{(2x-1)\sqrt{2x-1}}$

4 (b) Prove that $secx = \frac{secxtanx+sec^2x}{secx+tanx}$

and hence show that $\int_0^{\frac{\pi}{4}} secx\, dx = \ln(\sqrt{2}+1)$

5 (c) (i) Find values of A, B and C so that

$$\frac{5}{(x^2+4)(x+1)} = \frac{Ax+B}{x^2+4} + \frac{C}{x+1}$$

(ii) Hence find $\int \frac{5\, dx}{(x^2+4)(x+1)}$

QUESTION 2: (Start a new page)

Marks

6 (a) If $z = 1 - i$, find

(i) $\bar{z}$ (ii) $|z|$ (iii) $arg\ z$ (iv) $arg\ iz$ (v) z^6 (in simplest form)

2 (b) (i) Sketch the region where the inequalities

$$|z-2| \leq |z-2i| \quad \text{and} \quad |z-1-2i| \leq 1$$

hold simultaneously.

3 (ii) P is a point on the boundary of the region in part (i) above, and is represented by the complex number z, where $arg\, z = \frac{\pi}{4}$.

Find the 2 possibilities for z (in the form $a+ib$).

4 (c) A plane curve is defined by the equation

$$x^2 + 2xy + y^5 = 4$$

The curve has a horizontal tangent at the point $P(X, Y)$.

By using implicit differentiation, or otherwise, show that X is the unique solution to

$$X^5 + X^2 + 4 = 0$$

QUESTION 3: (Start a new page)

Marks

2 (a) (i) Without using calculus, sketch the curve $y = (x+1)^2(1-x)$

2 (ii) On a separate diagram from above, but using the same scale on the axes, *and also without calculus*, sketch the curve

$$y^2 = (x+1)^2(1-x)$$

In your answer, pay close attention to the shape of the curve as y approaches zero.

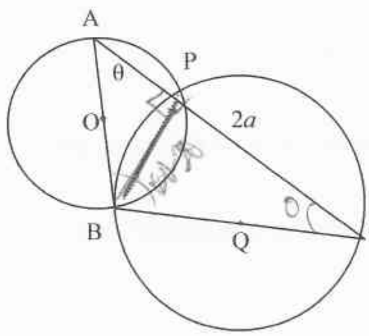
6 (b) Sketch each of the following curves on separate axes for $0 \leq x \leq 2\pi$

(i) $y = sin^2 x$ (ii) $y = |sinx|$

(iii) $y = \sqrt{sin^2 x}$ (iv) $y = \frac{1}{sinx}$

(v) $y = \frac{|sinx|}{sinx}$ (vi) $y = e^{sinx}$

(c) The hypotenuse AC of a right-angled triangle ABC has a length of $2a$ units and makes an angle of θ with one of the shorter sides, as shown below.



AP + PC = 2a

Circles are drawn using the two shorter sides as diameters, intersecting at points B and P. For this diagram, P is NOT on the side AC.

O and Q are the centres of the circles.

(i) Redraw the diagram in your answer book. (No marks)

2 (ii) Prove that the point P lies on AC (you may initially assume that it doesn't)

3 (iii) Show that the length of PB is $asin2\theta$

QUESTION 4: (Start a new page)

Marks

2 (a) Show that $\int_0^{\frac{\pi}{4}} tan\theta d\theta = \frac{1}{2}\ln 2$

2 (b) (i) Prove that, for any complex numbers z_1 and z_2

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2$$

3 (ii) Hence, using the method of Mathematical Induction, prove that

$$\arg(z_1 z_2 \dots\dots z_n) = \arg z_1 + \arg z_2 + \dots\dots\dots + \arg z_n$$

(c) A cubic polynomial is given by $P(x) = x^3 + ax + b$

where a and b are constants.

It is given that the polynomial equation $P(x) = 0$ has three roots, α, β, and γ

1 (i) Find the value of $\alpha + \beta + \gamma$

2 (ii) Show that $\alpha^2 + \beta^2 + \gamma^2 = -2a$

3 (iii) If the polynomial has a double root, show that this double root is $\frac{-3b}{2a}$

2 (iv) If the polynomial has 3 distinct roots, show that $4a^3 + 27b^2 < 0$

QUESTION 5: (Start a new page)

Marks

5 (a) Given the hyperbola $16x^2 - 9y^2 = 144$, find

(i) the length of the major axis

(ii) the eccentricity

(iii) the co-ordinates of the foci

(iv) the equations of the directrices

(v) the equations of the asymptotes

(b) The parametric co-ordinates of a point P on the curve $y^2 = x^3$ are $x = t^2$ and $y = t^3$

2 (i) Show that the equation of the tangent to this curve at P is

$$t^3 - 3tx + 2y = 0$$

1 (ii) Explain why there can be no more than 3 distinct tangents to $y^2 = x^3$ drawn from any remote point (x_1, y_1), which is not on the curve.

2 (iii) Show that if the tangents to the curve at the points on it having parameters t_1, t_2 and t_3 all pass through the remote point(x_1, y_1), then

$$t_1^2 + t_2^2 + t_3^2 = 6x_1$$

5 (c) The area under the curve $y = x^2$, above the x-axis and between the lines $x = 1$ and $x = 2$, is rotated through 2π radians about the line $x = 2$.

Using the method of cylindrical shells, show that the volume of the solid so formed is $\frac{11\pi}{6}$ cubic units.

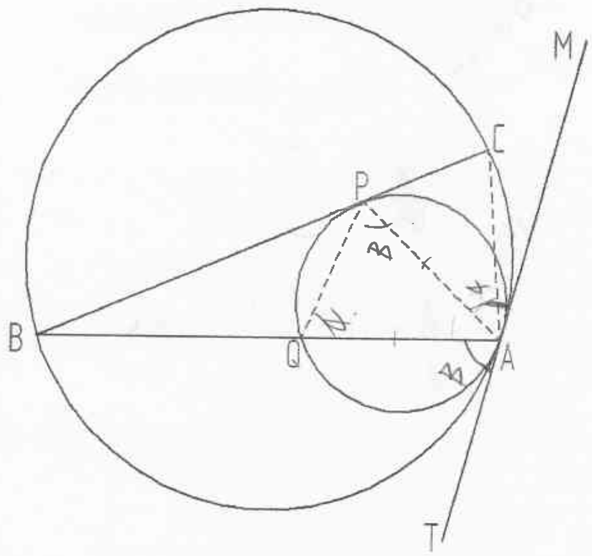
QUESTION 6: (Start a new page)

Marks

(a) Two circles touch internally at a point A and have a common tangent TAM as shown below.

A tangent to the inner circle through a point P (which is not the centre of either circle) meets the outer circle at B and C.

AB cuts the inner circle at Q.



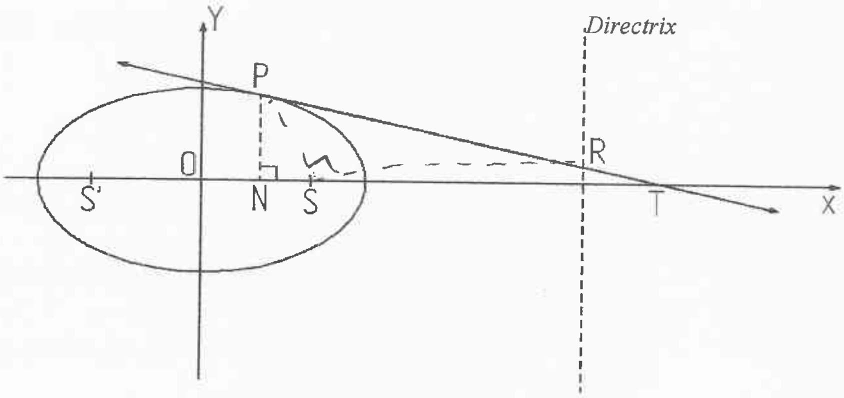
(i) Redraw the diagram neatly onto your answer page *(no marks).*

5 (ii) Giving all appropriate reasons, prove that AP bisects the angle BAC.

QUESTION 6 continues over the page.....)

QUESTION 6 continued…….)

(b) P$(a\cos\theta, b\sin\theta)$ is any point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



The tangent at P cuts the major axis of the ellipse at T and the Directrix at R, while N is the foot of the perpendicular from P to the x-axis.

O is the centre of the ellipse, while S and S' are the foci.

3 (i) Show that the equation of the tangent at P is $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$
(Show all working)

2 (ii) Show that ON.OT = a^2

5 (iii) Showing all steps carefully, prove that PR subtends a right angle at S.

QUESTION 7: (Start a new page)

Marks

3 (a) Using the substitution $x = a\tan\theta$, or otherwise, find $\int \frac{dx}{(a^2+x^2)^{\frac{3}{2}}}$

(b) You are given the complex polynomial $P(z) = z^5 - 1$

The roots of $P(z) = 0$ are $1, \omega_1, \omega_2, \omega_3, \omega_4$ which are in cyclic order around the unit circle.

3 (i) Prove the following:

(α) $\omega_1 = \overline{\omega_4}$ and $\omega_2 = \overline{\omega_3}$

(β) $\omega_1 + \omega_2 + \omega_3 + \omega_4 = -1$

(γ) $\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$

2 (ii) Using the sum of the products of the roots taken in pairs, or otherwise, show that

$$4\cos\frac{2\pi}{5}\cos\frac{4\pi}{5} + 1 = 0$$

1 (iii) Deduce that $\cos\frac{2\pi}{5}$ and $\cos\frac{4\pi}{5}$ are solutions to $4x^2 + 2x - 1 = 0$

4 (c) (i) If $I_n = \int_0^{\frac{\pi}{4}} \sec^n\theta\, d\theta$,

show that $(n-1)I_n - (n-2)I_{n-2} = (\sqrt{2})^{n-2}$, for $n \geq 2$

2 (ii) Using part (i) above, evaluate $\int_0^{\frac{\pi}{4}} \sec^4\theta\, d\theta$

QUESTION 8: (Start a new page)

Marks

(a) In the right triangular prism shown,

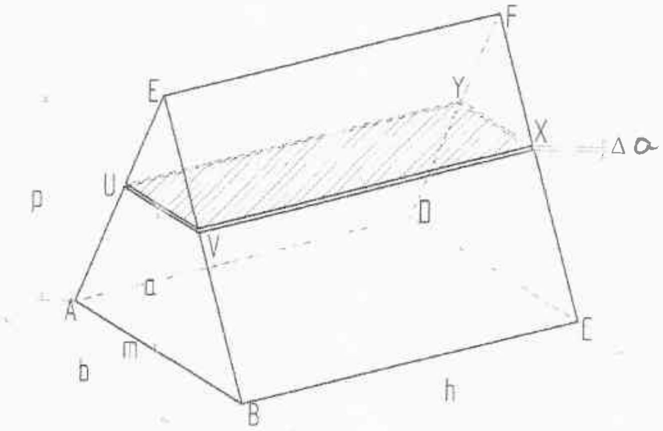
AB=DC= b units

AE=BE=DF=CF

M is the midpoint of AB

EM = p units

BC=AD=EF= h units



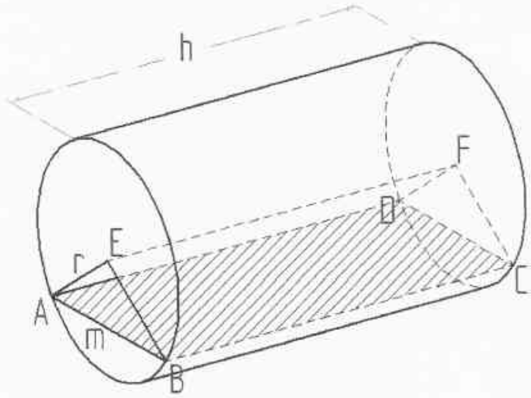
A "slice" UVXY of thickness Δa is taken a units above the base ABCD and parallel to it.

3 (i) Show that the volume of the rectangular slice is given by

$$\Delta V = \left(\frac{p-a}{p}\right) bh\Delta a$$

2 (ii) Hence, show that the volume of the triangular prism is given by $V = \frac{1}{2}pbh$

4 (iii) The triangular prism above is fitted into a right circular cylinder, of base radius r units and height h units, as shown below, where the points E and F are the centres of the circular bases.



Taking the angle AEB as $\frac{2\pi}{n}$, verify that the volume of the cylinder is $\pi r^2 h$

(In your proof you may use the result $\lim_{x \to 0} \tan x = x$)

QUESTION 8 continues over…….)

6 (b) A particle P moves in the x,y-plane and its co-ordinates (x, y) satisfy the equations

$$\frac{d^2x}{dt^2} = -n^2x \quad \text{and} \quad \frac{d^2y}{dt^2} = -n^2y \text{, where } n \text{ is a constant}$$

Initially (t=0), it is given that $x = 4$, $y = 0$, $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 3n$

Show that, as t varies, x and y describe the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$

END OF EXAMINATION PAPER