SYDNEY TECHNICAL HIGH SCHOOL

TRIAL HIGHER SCHOOL CERTIFICATE

2008

Mathematics Extension 2

General Instuctions

- Reading time - 5 minutes
- Working time - 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks - 120

- Attempt Questions 1 – 8
- All questions are of equal value

Name :____________________________

Teacher :____________________________

Question 1	Question 2	Question 3	Question 4	Question 5	Question 6	Question 7	Question 8	**Total**

Question 1 (15 marks)

a) By completing the square find $\displaystyle\int \frac{dx}{\sqrt{3+2x-x^2}}$ 2

b) Find $\displaystyle\int \sin^2 x \cos^3 x \, dx$ 2

c) Using integration by parts or otherwise find 3

$$\int \cos^{-1} x \, dx$$

d) Use the substitution $x = 2\cos\theta$ to find $\displaystyle\int \frac{\sqrt{4-x^2}}{x^2} \, dx$ 4

e) Use the substitution $t = \tan\frac{\theta}{2}$ to evaluate 4

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{2+\cos\theta}$$

Question 2 (15 marks) (Start a new page)

a) i) Find all pairs of integers x and y such that 3

$$(x+iy)^2 = -3-4i$$

ii) If $x+iy$ with the values of x and y found in part i) 2

are two of the roots of $w^4 = a+ib$, find the two other roots.

b) Given $z = \sqrt{3}+i$

i) Find w in the form $x+iy$ if $z+\overline{w} = 2\sqrt{3}-2i$ 2

ii) Show that $z\,w$ is purely imaginary 2

and hence write down the value of $\arg\,(z\,w)$

iii) Write z in modulus – argument form 1

iv) If the points on the Argand diagram representing z , zw and the origin

are the vertices of a triangle find the area of this triangle. 2

c) i) On an Argand diagram sketch the locus of the point P representing 2

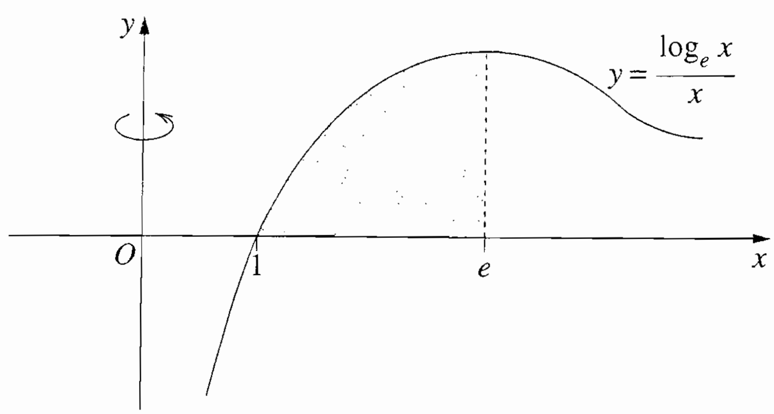
the complex number z such that $\left| z-(\sqrt{3}+i) \right| = 1$

ii) Write down the range of values of $|z|$ if $\left| z-(\sqrt{3}+i) \right| = 1$ 1

Question 3 (15 marks) (Start a new page)

a) The diagram shows the graph of $y = \dfrac{\log_e x}{x}$.

It has a maximum turning point at $x = e$ and a point of inflection at $x = 2e$.



i) For each of the following draw a one-third page sketch showing clearly any intercepts and the co-ordinates of any turning points.

α) $y = \dfrac{x}{\log_e x}$ 2

β) $y = \dfrac{d}{dx}\left[\dfrac{\log_e x}{x}\right]$ 2

δ) $y = \dfrac{\log_e(-x)}{x}$ 2

ii) Find the coordinates of the maximum turning point on the curve 1

$$y = \log_e \left[\frac{\log_e x}{x}\right]$$

Question 3 is continued on page 5

Question 3 (continued)

iii) Use the method of cylindrical shells to find the volume of the solid formed when the region bounded by $y = 0$, $y = \dfrac{\log_e x}{x}$ and $x = e$ is rotated about the y axis. 4

b) The curve $x^2 + xy + y^2 = 12$ is symmetrical about $y = x$.

i) Show that $\dfrac{dy}{dx} = \dfrac{-(y+2x)}{x+2y}$ 1

ii) Find a point (a, b) where $\dfrac{dy}{dx} = 0$ and explain why there is a vertical tangent at (b, a). 3

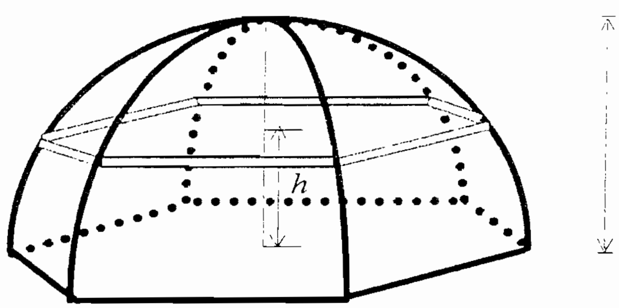
Question 4 (15 marks) (Start a new page)

a) A dome is sitting on a regular hexagonal base of side 20 metres.

The height of the dome is also 20 metres.

Each strut of the dome (going from the base to the top vertex)

is a quarter of a circle with its centre at the centre of the hexagonal base.



i) For the slice h metres above the base and parallel to the base, show that the length x of each side of the slice is given by $x = \sqrt{400 - h^2}$ metres. 2

ii) Show that the area of the slice described above is given by 2

$$A = \frac{3\sqrt{3}}{2}(400 - h^2) \text{ square metres.}$$

iii) Hence, or otherwise calculate the volume of the dome. 3

Question 4 is continued on page 7

Question 4 (continued)

b) i) Differentiate $\dfrac{x^2}{25}+\dfrac{y^2}{9}=1$ implicitly 2

ii) Derive the equation of the tangent to the ellipse 2

$\dfrac{x^2}{25}+\dfrac{y^2}{9}=1$ at the point $(x_1\ ,\ y_1)$.

iii) Write down the equations of the directrices of $\dfrac{x^2}{25}+\dfrac{y^2}{9}=1$. 1

iv) If $x_1>0$ and $y_1>0$ find the values of x_1 so that the tangent 3

at $(x_1\ ,\ y_1)$ intersects the nearest directrix below the x axis.

Question 5 (15 marks) (Start a new page)

a) $(x+1)^2$ is a factor of $P(x)=x^4+2x^3+ax^2+bx+4$

i) Find the values of a and b. 3

ii) If $(x-k\,i)$ is also a factor of $P(x)$ 2

find the value of k **and** the other factor.

b) Let $I_n=\int(\log_e x)^n\,dx$

i) Show that $I_n=x(\log_e x)^n-n\,I_{n-1}$ for $n=1,2,3,......$ 3

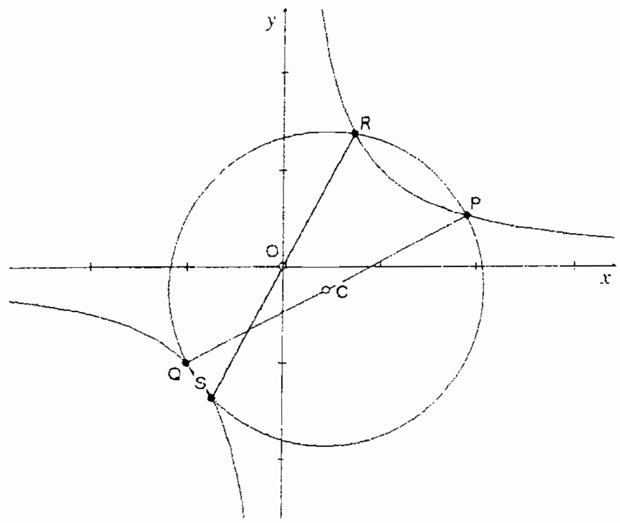
ii) Hence evaluate $\int_1^{e^4}(\log_e x)^2\,dx$ 2

Question 5 is continued on page 8

Question 5 (continued)

c) The circle $(x-h)^2+(y-k)^2=r^2$ and the hyperbola $xy=c^2$ intersect at the points P, Q, R and S.

The x coordinates of P, Q, R and S are α, β, γ and δ respectively.



i) Show that the equation with roots α, β, γ and δ is **2**

$$x^4-2hx^3+(h^2+k^2-r^2)x^2-2c^2kx+c^4=0$$

ii) If the mid point of PQ is the centre of the circle, show that $\alpha+\beta=2h$ **1**

iii) Hence show that the origin is the mid point of RS. **2**

Question 6 (15 marks) (Start a new page)

a) A particle of unit mass is projected vertically upwards with an initial velocity of $2\,m/s$. It experiences an air resistance which is numerically equal to gv^2 where g is the acceleration due to gravity.

i) Explain why the equation of motion is given by $a = -gv^2 - g$ 1

ii) Find an expression in terms of g 3

for the time taken to reach the maximum height.

iii) Find an expression in terms of g for the maximum height reached. 3

b) Given $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, 3

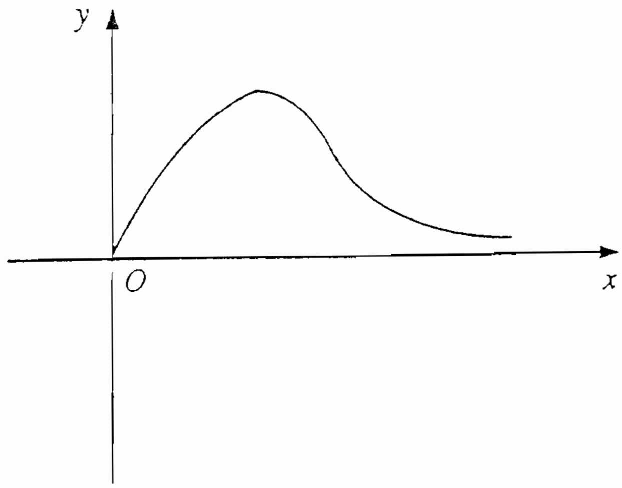
Use the substitution $x = 4\cos\theta$

to find the exact roots of the equation $x^3 - 12x + 8 = 0$.

Question 6 is continued on page 10

Question 6 (continued)

c)



This is the graph of $f(x) = x^n e^{-x}$ for $x > 0$

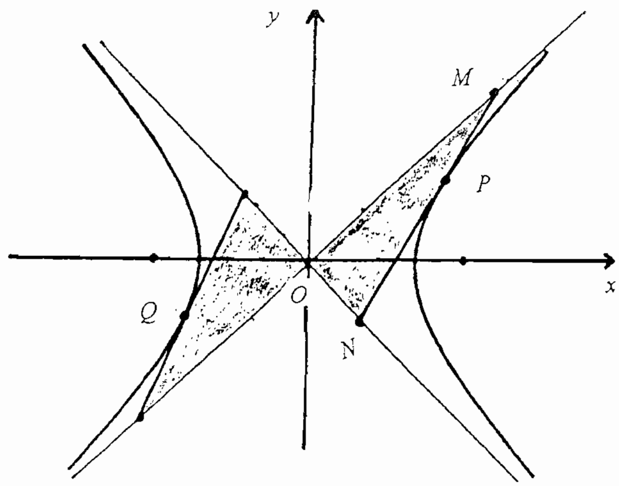
i) Show that the maximum turning point occurs at $x = n$ 2

ii) By considering the values of $f(n)$, $f(n-1)$ and $f(n+1)$ 3

prove that $(1 + \frac{1}{n})^n < e < (1 - \frac{1}{n})^{-n}$

Question 7 (15 marks) (Start a new page)

a) The diagram shows the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ with tangents drawn at P and Q meeting the hyperbola's asymptotes. O is the origin.



The equation of the tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

at $P(a\sec\theta\ ,\ b\tan\theta)$ is given by $\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1$.

The tangent at $P(a\sec\theta\ ,\ b\tan\theta)$ meets the asymptotes at M and N as shown in the diagram.

i) Write down the equations of the asymptotes of $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. 1

ii) Show that the coordinates of M are $\left(\frac{a}{\sec\theta-\tan\theta}\ ,\ \frac{b}{\sec\theta-\tan\theta}\right)$ 2

iii) Find the coordinates of N. 1

iv) Prove that $OM \times ON$ is a constant. 2

v) Explain why the shaded areas are equal. 1

Question 7 is continued on page 12

Question 7 (continued)

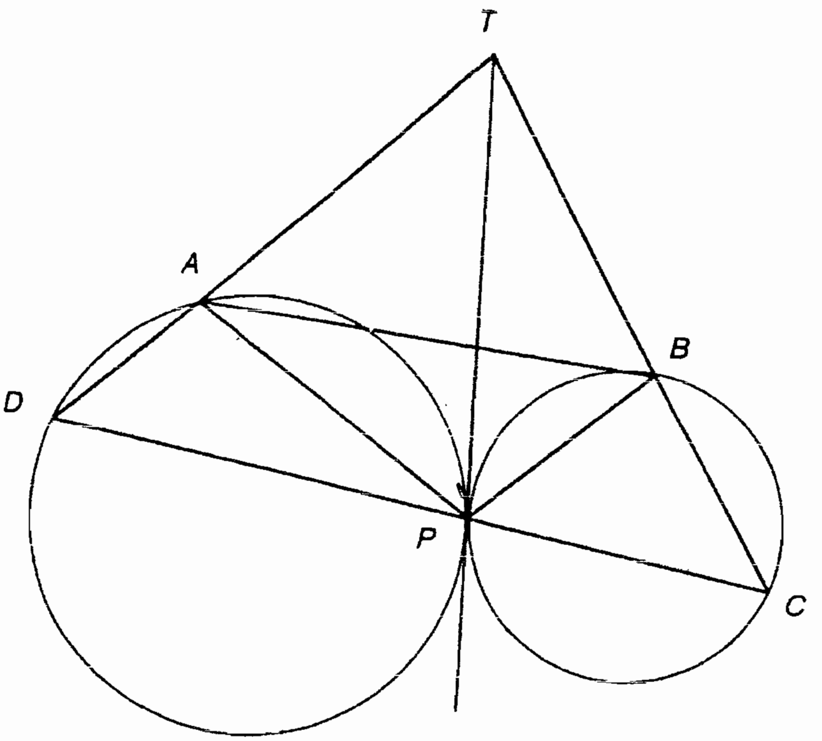
b) i) By solving $\dfrac{z-1}{z+1} = \cos\theta + i\sin\theta$, show that $z = i\cot\frac{\theta}{2}$ 4

ii) Find all the solutions of $w^6 = -1$ 2

iii) Hence find all the solutions of $(z-1)^6 + (z+1)^6 = 0$ 2

Question 8 (15 marks) (Start a new page)

a)



Circles APD and BPC touch at P.

The point D , P and C are collinear.

TP is the common tangent at P.

TC cuts the circle BPC at B while TD cuts the circle APD at A.

i) Explain why $\angle TPB = \angle BCP$. 1

ii) Show that $ATBP$ is a cyclic quadrilateral. 3

iii) Show that $ABCD$ is a cyclic quadrilateral. 2

Question 8 is continued on page 14

Question 8 (continued)

b) i) Show that for all values of x and y 1

$$\sin(x+y)-\sin(x-y)=2\cos x\sin y$$

ii) Use mathematical induction to show that for all positive integers n 4

$$\cos x+\cos 2x+\cos 3x+\cdots\cdots\cdots\cdots+\cos nx=\frac{\sin(n+\frac{1}{2})x-\sin\frac{x}{2}}{2\sin\frac{x}{2}}$$

iii) Hence show that 4

$$\cos 2x+\cos 4x+\cos 6x+\cdots\cdots\cdots\cdots+\cos 16x=8\cos 9x\cos 4x\cos 2x\cos x$$

End of Paper

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

QUESTION 1

a) $\displaystyle\int \frac{dx}{\sqrt{3+2x-x^2}}$

$\displaystyle = \int \frac{dx}{\sqrt{4-(x-1)^2}}$

$\displaystyle = \sin^{-1}\frac{(x-1)}{2} + c$

b) $\displaystyle\int \sin^2 x \cos^3 x \, dx$

$\displaystyle = \int \sin^2 x(1-\sin^2 x)\cos x \, dx$

$\displaystyle = \int (\sin^2 x - \sin^4 x)\cos x \, dx$

let $u = \sin x$

$du = \cos x \, dx$

$\displaystyle = \int u^2 - u^4 \, du$

$\displaystyle = \frac{1}{3}u^3 - \frac{1}{5}u^5$

$\displaystyle = \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + c$

c) $\displaystyle\int \cos^{-1} x \, dx$

$\displaystyle = x\cos^{-1}x - \int x \cdot \frac{-1}{\sqrt{1-x^2}} \, dx$

$\displaystyle = x\cos^{-1}x + \int \frac{x}{\sqrt{1-x^2}} \, dx$

$\displaystyle = x\cos^{-1}x - \sqrt{1-x^2} + c$

d) $\displaystyle\int \frac{\sqrt{4-x^2}}{x^2} \, dx$

$x = 2\cos\theta$

$dx = -2\sin\theta \, d\theta$

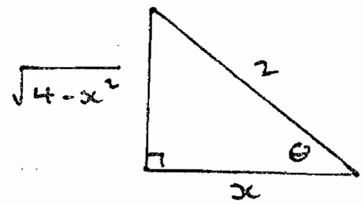
$\displaystyle = \int \frac{\sqrt{4-4\cos^2\theta}}{4\cos^2\theta} \cdot -2\sin\theta \, d\theta$

$\displaystyle = -\int \frac{\sin^2\theta}{\cos^2\theta} \, d\theta$

$\displaystyle = \int 1 - \sec^2\theta \, d\theta$

$= \theta - \tan\theta$

$\displaystyle = \cos^{-1}\frac{x}{2} - \frac{\sqrt{4-x^2}}{x} + c$



$\cos\theta = \frac{x}{2}$

e) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{d\theta}{2+\cos\theta}$

$t = \tan\frac{\theta}{2}$

$d\theta = \frac{2\,dt}{1+t^2}$

$\displaystyle = \int_0^1 \frac{\frac{2}{1+t^2}}{2 + \frac{1-t^2}{1+t^2}} \, dt$

$\displaystyle = \int_0^1 \frac{2\,dt}{2(1+t^2)+1-t^2}$

$\displaystyle = \int_0^1 \frac{2\,dt}{3+t^2}$

$\displaystyle = \left.\frac{2}{\sqrt{3}}\tan^{-1}\frac{t}{\sqrt{3}}\right]_0^1$

$\displaystyle = \frac{2}{\sqrt{3}}\tan^{-1}\frac{1}{\sqrt{3}}$

$\displaystyle = \frac{\pi}{3\sqrt{3}}$

QUESTION 2

a) i) $(x+iy)^2 = -3-4i$

$x^2 - y^2 + i\,2xy = -3-4i$

$\therefore\ x^2 - y^2 = -3$

$xy = -2$

$\therefore\ x = 1,\ y = -2$ or

$x = -1,\ y = 2$

ii) roots evenly spaced around circle

$\therefore$ other roots are

$i(1-2i)$ and $i(-1+2i)$

or

$2+i$ and $-2-i$

b) i) $\sqrt{3}+i+\bar{w} = 2\sqrt{3}-2i$

$\bar{w} = \sqrt{3}-3i$

$\therefore \quad w = \sqrt{3}+3i$

ii) $zw = (\sqrt{3}+i)(\sqrt{3}+3i)$

$= 3+3\sqrt{3}i+\sqrt{3}i-3$

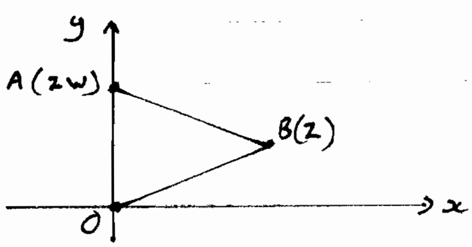
$= 4\sqrt{3}i$

which is purely imaginary

$\therefore \quad \arg(zw) = \frac{\pi}{2}$

iii) $z = 2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$

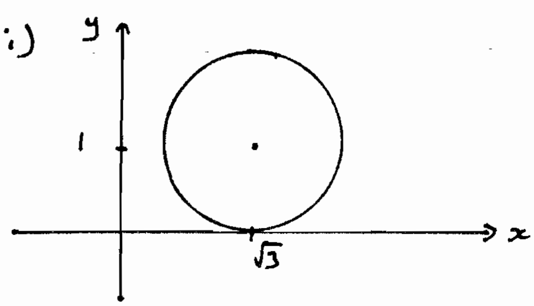
iv)



Area $= \frac{1}{2}\times 4\sqrt{3}\times\sqrt{3}$

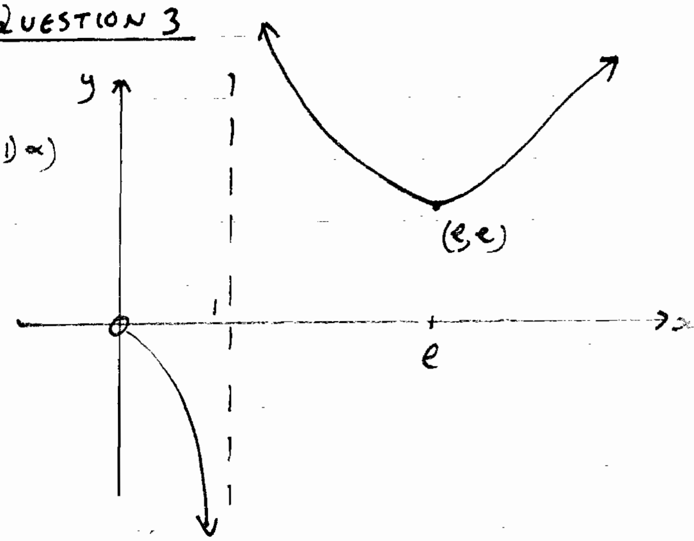
$= 6$ sq units

c) i)

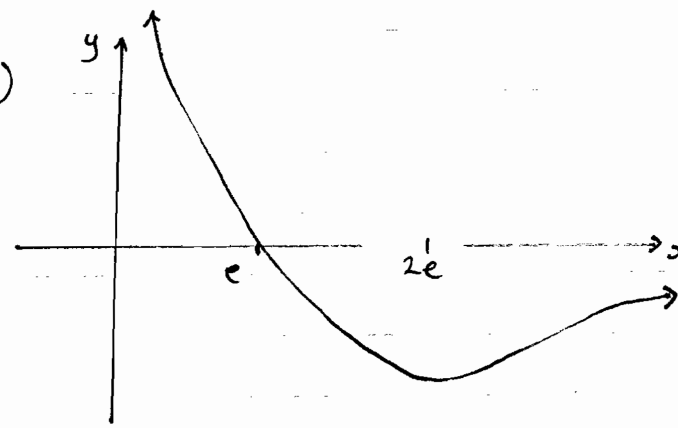


QUESTION 3

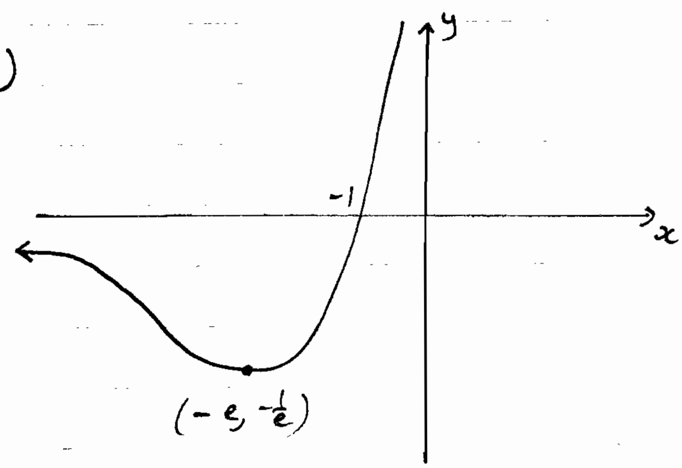
a) i) α)



β)



δ)



ii) max occurs at $x = e$

$\therefore \quad y = \log_e\left(\frac{\log_e e}{e}\right)$

$= -1$

$\therefore$ max at $(e, -1)$

iii) $\Delta V = 2\pi x \cdot y \cdot \Delta x$

$= 2\pi x \dfrac{\log_e x}{x} \Delta x$

$\therefore V = 2\pi \displaystyle\int_1^e \log_e x \, dx$

$= 2\pi \left[x \log_e x - x \right]_1^e$

$= 2\pi \left[(e - e) - (0 - 1) \right]$

$= 2\pi$ cubic units

b) i) $x^2 + xy + y^2 = 12$

$2x + x\dfrac{dy}{dx} + y + 2y\dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx}(x + 2y) = -2x - y$

$\dfrac{dy}{dx} = \dfrac{-(2x+y)}{x+2y}$

ii) $\dfrac{dy}{dx} = 0$

$\therefore 2x + y = 0$

$y = -2x$

$\therefore x^2 - 2x^2 + 4x^2 = 12$

$3x^2 = 12$

$x^2 = 4$

$x = \pm 2$

$\therefore (2, -4)$ or $(-2, 4)$

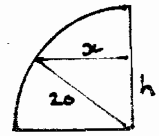
symmetric about $y = x$ $\therefore$ function is its own inverse

$\therefore f'(b) = \dfrac{1}{f'(a)}$ $\left(\dfrac{dy}{dx} \times \dfrac{dx}{dy} = 1 \right)$

$= \dfrac{1}{}$

QUESTION 4

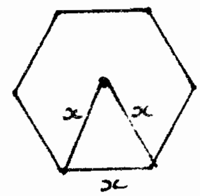
a) i)



$x^2 + h^2 = 20^2$

$\therefore x = \sqrt{400 - h^2}$

ii)



$A = 6 \times \left(\frac{1}{2} \times x \times x \times \sin\frac{\pi}{3} \right)$

$= \dfrac{3\sqrt{3}x^2}{2}$

$= \dfrac{3\sqrt{3}}{2}(400 - h^2)$

iii) $V = \dfrac{3\sqrt{3}}{2} \displaystyle\int_0^{20} 400 - h^2 \, dh$

$= \dfrac{3\sqrt{3}}{2} \left[400h - \frac{1}{3}h^3 \right]_0^{20}$

$= 8000\sqrt{3}$ cubic units

b) i) $\dfrac{2x}{25} + \dfrac{2y}{9} \cdot \dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx} = \dfrac{-\frac{2x}{25}}{\frac{2y}{9}}$

$\dfrac{dy}{dx} = \dfrac{-9x}{25y}$

ii) $m_T = \dfrac{-9x_1}{25y_1}$

$\therefore y - y_1 = \dfrac{-9x_1}{25y_1}(x - x_1)$

$\dfrac{yy_1}{9} + \dfrac{xx_1}{25} = \dfrac{x_1^2}{25} + \dfrac{y_1^2}{9}$

$\dfrac{yy_1}{9} + \dfrac{xx_1}{25} = 1$

iii) $b^2 = a^2(1-e^2)$

$\therefore \; e^2 = 1 - \frac{9}{25}$

$= \frac{16}{25}$

$e = \frac{4}{5}$

$\therefore \quad x = \pm \frac{5}{\frac{4}{5}}$

$x = \pm \frac{25}{4}$

iv) Solve simultaneously $\frac{x x_1}{25} + \frac{y y_1}{9} = 1$

and $x = \frac{25}{4}$

$\therefore \quad \frac{x_1}{4} + \frac{y_1 y}{9} = 1$

$y = \frac{9}{y_1}\left(\frac{4-x_1}{4}\right)$

we require $y < 0$ and $x_1, y_1 > 0$

$\frac{9}{y_1}\left(\frac{4-x_1}{4}\right) < 0$

$\therefore \; 4 - x_1 < 0$

$x_1 > 4$

$\therefore \quad 4 < x_1 < 5$

QUESTION 5

i) $x = -1$ is a double root

$\therefore \; P(-1) = 0$ and $P'(-1) = 0$

$P(x) = x^4 + 2x^3 + ax^2 + bx + 4$

$P'(x) = 4x^3 + 6x^2 + 2ax + b$

$P(-1) = 0 \Rightarrow a - b = -3$

$P'(-1) = 0 \Rightarrow 2a - b = 2$

ii) If $x - ki$ is a factor

so is $x + ki$

$\therefore$ roots are $-1, -1, ki, -ki$

using product of roots

$-1 \times -1 \times ki \times -ki = 4$

$k^2 = 4$

$k = \pm 2$

other factor $(x + 2i)$ if $k = 2$.

b) i) $I_n = \int (\log_e x)^n \, dx$

$u = (\log_e x)^n \qquad u' = \frac{n}{x}(\log_e x)^{n-1}$

$v = x \qquad v' = 1$

$\therefore \quad I_n = x(\log_e x)^n - \int n(\log_e x)^{n-1} dx$

$= x(\log_e x)^n - n I_{n-1}$

ii) $I_2 = \int_1^{e^2} (\log_e x)^2 \, dx$

$= \left[x(\ln x)^2\right]_1^{e^4} - 2 I_1$

$= 16e^4 - 2\left[x \ln x\Big|_1^{e^4} - I_0\right]$

$= 16e^4 - 8e^4 + 2(e^4 - 1)$

$= 10e^4 - 2$

c) i) Solve simultaneously

$(x-h)^2 + (y-k)^2 = r^2$ and

$y = \frac{c^2}{x}$

$(x-h)^2 + \left(\frac{c^2}{x} - k\right)^2 = r^2$

$x^2 - 2hx + h^2 + \frac{c^4}{x^2} - \frac{2kc^2}{x} + k^2 = r^2$

$$x^4 - 2hx^3 + (h^2+k^2-r^2)x^2 - 2c^2kx + c^4 = 0$$

ii) mid point is centre of circle

$$\therefore \frac{\alpha+\beta}{2} = h$$

$$\alpha+\beta = 2h$$

iii) sum of roots

$$\alpha+\beta+\gamma+\delta = 2h$$

$$2h+\gamma+\delta = 2h$$

$$\gamma+\delta = 0$$

$$\therefore \frac{\gamma+\delta}{2} = 0$$

$\therefore$ x coordinate of mid point QR is 0.

QR passes through the origin

$\therefore$ midpoint is $(0,0)$.

QUESTION 6

a) i) only acceleration acting on particle is gravity and resistance both acting against the motion

$$\therefore a = -g - gv^2$$

ii) $$\frac{dv}{dt} = -g(1+v^2)$$

$$\int \frac{dv}{1+v^2} = \int -g\,dt$$

$$\tan^{-1} v = -gt + c$$

when $t=0$ $v=2$

$$\Rightarrow c = \tan^{-1} 2$$

$$\therefore \tan^{-1} v - \tan^{-1} 2 = -gt$$

maximum height when $v=0$

$$\therefore -\tan^{-1} 2 = -gt$$

$$t = \frac{\tan^{-1} 2}{g}$$

iii) $$v\frac{dv}{dx} = -g(1+v^2)$$

$$\int \frac{v\,dv}{1+v^2} = \int -g\,dx$$

$$\tfrac{1}{2}\ln(1+v^2) = -gx + c$$

when $x=0$ $v=2$

$$\tfrac{1}{2}\ln 5 = c$$

$$\therefore \tfrac{1}{2}\ln(1+v^2) - \tfrac{1}{2}\ln 5 = -gx$$

$$x = -\frac{1}{2g}\ln\left(\frac{1+v^2}{5}\right)$$

when $v=0$

$$x = -\frac{1}{2g}\ln\frac{1}{5}$$

b) $$x^3 - 12x + 8 = 0$$

$$(4\cos\theta)^3 - 12(4\cos\theta) + 8 = 0$$

$$64\cos^3\theta - 48\cos\theta + 8 = 0$$

$$8\cos^3\theta - 6\cos\theta + 1 = 0$$

$$4\cos^3\theta - 3\cos\theta = -\tfrac{1}{2}$$

$$\cos 3\theta = -\tfrac{1}{2}$$

$$3\theta = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}$$

$$\theta = \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{8\pi}{9}$$

$$\therefore x = 4\cos\frac{2\pi}{9}, 4\cos\frac{4\pi}{9}, 4\cos\frac{8\pi}{9}$$

c) i) $f(x) = x^n e^{-x}$

$f'(x) = nx^{n-1}e^{-x} + x^n(-e^{-x})$

$= e^{-x}(nx^{n-1} - x^n)$

$= x^{n-1}e^{-x}(n-x)$

st pt when $f'(x)=0$

$n-x=0$

$x=n$

$\therefore$ from graph maximum when $x=n$.

ii) from graph

$f(n-1) < f(n)$

$(n-1)^n e^{-(n-1)} < n^n e^{-n}$

$\left(\frac{n-1}{n}\right)^n < \frac{e^{-n}}{e^{-(n-1)}}$

$\left(\frac{n-1}{n}\right)^n < e^{-1}$

$\left(1-\frac{1}{n}\right)^{-n} > e$

also $f(n+1) < f(n)$

$(n+1)^n e^{-n-1} < n^n e^{-n}$

$\left(\frac{n+1}{n}\right)^n < \frac{e^{-n}}{e^{-n-1}}$

$\left(\frac{n+1}{n}\right)^n < e$

$\therefore \left(1+\frac{1}{n}\right)^n < e < \left(1-\frac{1}{n}\right)^{-n}$

QUESTION 7

a) i) $y = \pm\frac{b}{a}x$

ii) $y = \frac{b}{a}x$

$\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$

Solve simultaneously

$\therefore \frac{x\sec\theta}{a} - \frac{bx\tan\theta}{ab} = 1$

$x(\sec\theta - \tan\theta) = a$

$x = \frac{a}{\sec\theta - \tan\theta}$

$\therefore y = \frac{b}{a}\cdot\frac{a}{\sec\theta - \tan\theta}$

$= \frac{b}{\sec\theta - \tan\theta}$

$\therefore M\left(\frac{a}{\sec\theta - \tan\theta}, \frac{b}{\sec\theta - \tan\theta}\right)$

iii) sub $y = \frac{-b}{a}x$ into tangent

$\frac{x\sec\theta}{a} + \frac{bx\tan\theta}{ab} = 1$

$x = \frac{a}{\sec\theta + \tan\theta}$

$\therefore y = \frac{-b}{\sec\theta + \tan\theta}$

$\therefore N\left(\frac{a}{\sec\theta + \tan\theta}, \frac{-b}{\sec\theta + \tan\theta}\right)$

$\therefore OM \times ON$

$$= \frac{\sqrt{a^2+b^2}}{\sec\theta + \tan\theta} \times \frac{\sqrt{a^2+b^2}}{\sec\theta - \tan\theta}$$

$$= \frac{a^2+b^2}{\sec^2\theta - \tan^2\theta}$$

$$= a^2 + b^2$$

which is constant

v) Area $= \frac{1}{2} ab \sin C$

a, b are intercepts OM, ON

$\therefore$ ab is constant

included angle equal – vertically opposite

$\therefore$ areas equal.

b) i) $\frac{z-1}{z+1} = \cos\theta + i\sin\theta$

$$z - 1 = (z+1)(\cos\theta + i\sin\theta)$$

$$z - 1 = z\cos\theta + iz\sin\theta + \cos\theta + i\sin\theta$$

$$z(1 - \cos\theta - i\sin\theta) = 1 + \cos\theta + i\sin\theta$$

$$\therefore z = \frac{1+\cos\theta + i\sin\theta}{1 - \cos\theta - i\sin\theta}$$

$$= \frac{1+\cos\theta + i\sin\theta}{1-\cos\theta - i\sin\theta} \times \frac{1-\cos\theta + i\sin\theta}{1-\cos\theta + i\sin\theta}$$

$$= \frac{1 - \cos^2\theta + i\sin\theta - i\sin\theta\cos\theta + i\sin\theta + i\sin\theta\cos\theta - \sin^2\theta}{(1-\cos\theta)^2 + \sin^2\theta}$$

$$= \frac{1 - (\cos^2\theta + \sin^2\theta) + i\,2\sin\theta}{1 - 2\cos\theta + \cos^2\theta + \sin^2\theta}$$

$$= \frac{2i\sin\theta}{2(1-\cos\theta)}$$

$$= \frac{i\,2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{1 - (1 - 2\sin^2\frac{\theta}{2})}$$

$$= \frac{i\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}$$

$$= i\cot\frac{\theta}{2}$$

ii) $w_1 = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$

$w_2 = i$

$w_3 = \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}$

$w_4 = \cos\frac{5\pi}{6} - i\sin\frac{5\pi}{6}$

$w_5 = -i$

$w_6 = \cos\frac{\pi}{6} - i\sin\frac{\pi}{6}$

iii) $(z-1)^6 + (z+1)^6 = 0$

$$\left(\frac{z-1}{z+1}\right)^6 = -1$$

$\therefore z = i\cot\frac{\theta}{2}$

where $\theta = \frac{\pi}{6}, \frac{3\pi}{6}, \frac{5\pi}{6}, \frac{-\pi}{6}, \frac{-3\pi}{6}, \frac{-5}{6}$

$\therefore z = \pm i\cot\frac{\pi}{12}$,

$\pm i\cot\frac{3\pi}{12}$,

$\pm i\cot\frac{5\pi}{12}$ or

$z = \pm i\cot\frac{\pi}{12}, \pm i, \pm i\cot\frac{5\pi}{12}$

Question 8

a) i) $T\hat{P}B = B\hat{C}P$ (Alternate segment theorem)

ii) Similarly $T\hat{P}A = A\hat{D}P$.

$A\hat{T}B + A\hat{D}P + B\hat{C}P = 180°$ ($\angle$ sum ΔDCT is $180°$)

$\therefore A\hat{T}B + T\hat{P}A + T\hat{P}B = 180°$

$\therefore \quad A\hat{T}B + A\hat{P}B = 180°$ ($A\hat{P}B$ is sum of adjacent $\angle$s $T\hat{P}A$, $T\hat{P}B$)

$\therefore ATBP$ is a cyclic quadrilateral (one pair of opposite $\angle$s supplementary)

(iii)

$T\hat{B}A = T\hat{P}A$ ($\angle$s at circumference standing on same arc AT of circle $ATBP$)

$\therefore T\hat{B}A = A\hat{D}P$ ($T\hat{P}A = A\hat{D}P$ proved in (ii))

$\therefore ABCD$ is cyclic (exterior angle equal to interior opposite angle)

b) i)

	$\sin(x+y) - \sin(x-y) = \sin x \cos y + \cos x \sin y - (\sin x \cos y - \cos x \sin y)$ $= 2\cos x \sin y$	1

ii)	If $n = 1$ LHS $= \cos x$	4

$$\text{RHS} = \frac{\sin\left(\frac{3x}{2}\right) - \sin\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)}$$

Using i) above
$$\text{RHS} = \frac{2\cos x \sin\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)}$$

$$= \cos x = \text{LHS}$$

$\therefore$ true for $n = 1$

Assume true for $n = k$

i.e
$$\cos x + \cos 2x + \cos 3x ...... + \cos kx = \frac{\sin\left(k + \frac{1}{2}\right)x - \sin\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)}$$

When $n = k + 1$

$$\cos x + \cos 2x + \cos 3x ...... + \cos kx + \cos(k+1)x = \frac{\sin\left(k + \frac{1}{2}\right)x - \sin\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)} + \cos(k+1)x$$

$$= \frac{\sin\left(k + \frac{1}{2}\right)x - \sin\left(\frac{x}{2}\right) + \cos(k+1)x . 2\sin\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)}$$

$$= \frac{\sin\left(k + \frac{1}{2}\right)x - \sin\left(\frac{x}{2}\right) + \sin\left((k+1)x + \frac{x}{2}\right) - \sin\left((k+1)x - \frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)}$$ using i) above

$$= \frac{\sin\left(k+\frac{1}{2}\right)x - \sin\left(\frac{x}{2}\right) + \sin\left(k+\frac{3}{2}\right)x - \sin\left(k+\frac{1}{2}\right)x}{2\sin\left(\frac{x}{2}\right)}$$

$$= \frac{\sin\left((k+1)+\frac{1}{2}\right)x - \sin\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)}$$

$\therefore$ True for $n = k+1$

$\therefore$ Since true for $n = 1$, by induction is true for all positive integral values of $k \geq 1$

iii)	$\cos 2x + \cos 4x + \cos 6x ...... + \cos 16x = \cos(2x) + \cos 2(2x) + \cos 3(2x) ...... + \cos 8(2x)$	4

$$= \frac{\sin\left(8+\frac{1}{2}\right)2x - \sin\left(\frac{2x}{2}\right)}{2\sin\left(\frac{2x}{2}\right)}$$

$$= \frac{\sin 17x - \sin x}{2\sin x}$$

$$= \frac{\sin(9+8)x - \sin(9-8)x}{2\sin x}$$

$$= \frac{2\cos 9x \sin 8x}{2\sin x}$$ Using i) above

$$= \frac{2\cos 9x . 2\sin 4x \cos 4x}{2\sin x}$$ Using double angle on $\sin 8x$

$$= \frac{4\cos 9x . 2\sin 2x \cos 2x \cos 4x}{2\sin x}$$ Using double angle on $\sin 4x$

$$= \frac{8\cos 9x . 2\sin x \cos x \cos 2x \cos 4x}{2\sin x}$$ Using double angle on $\sin 2x$

$$= \frac{8\cos 9x . \cancel{2\sin x} \cos x \cos 2x \cos 4x}{\cancel{2\sin x}}$$

$= 8\cos 9x \cos 4x \cos 2x \cos x$