Sydney Technical High School

TRIAL HIGHER SCHOOL CERTIFICATE

2007

MATHEMATICS EXTENSION 2

General Instructions

- Reading time - 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Approved calculators may be used
- All necessary working should be shown in every question
- A table of standard integrals is supplies at the back of this paper
- Start each question on a new page
- Attempt all Questions 1 – 8
- All questions are of equal value
- **Total marks 120**

Name: ______________________

Class: ______________________

Question 1	Question 2	Question 3	Question 4	Question 5	Question 6	Question 7	Question 8	TOTAL

QUESTION 1 (15 Marks) **Marks**

a) Find by using a suitable substitution or otherwise

i) $\int \frac{dx}{\sqrt{9-16x^2}}$ 2

ii) $\int \frac{dx}{\sqrt{x^2+6x+13}}$ 2

iii) $\int \sec^3 x \tan x \, dx$ 2

b) Using the substitution $x = 3\tan\theta$ or otherwise find $\int \frac{dx}{(9+x^2)^{\frac{3}{2}}}$ 4

c) i) Show that $\frac{d}{dx}\left[\frac{1}{2a}\log_e\left(\frac{x-a}{x+a}\right)\right] = \frac{1}{x^2-a^2}$ 2

ii) Hence by using the substitution $x = u^2$ or otherwise find $\int \frac{\sqrt{x}}{x-1}dx$ 3

Question 2 (15 marks)

a) Find d if $(3+2i)(4-di)$ is wholly imaginary 2

b) If $\alpha = -2+2\sqrt{3}\,i$ and $\beta = 1-i$

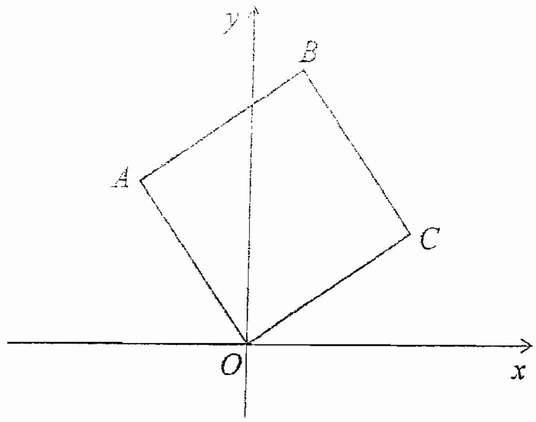
i) Find $\frac{\alpha}{\beta}$ in the form $x+iy$ 1

ii) Express α in modulus – argument form 1

iii) Given $\beta = \sqrt{2}\left(\cos\left(\frac{-\pi}{4}\right)+i\sin\left(\frac{-\pi}{4}\right)\right)$ find the modulus- argument form of $\frac{\alpha}{\beta}$ 2

iv) Hence find the exact value of $\cos\left(\frac{\pi}{12}\right)$ 2 c)

Marks



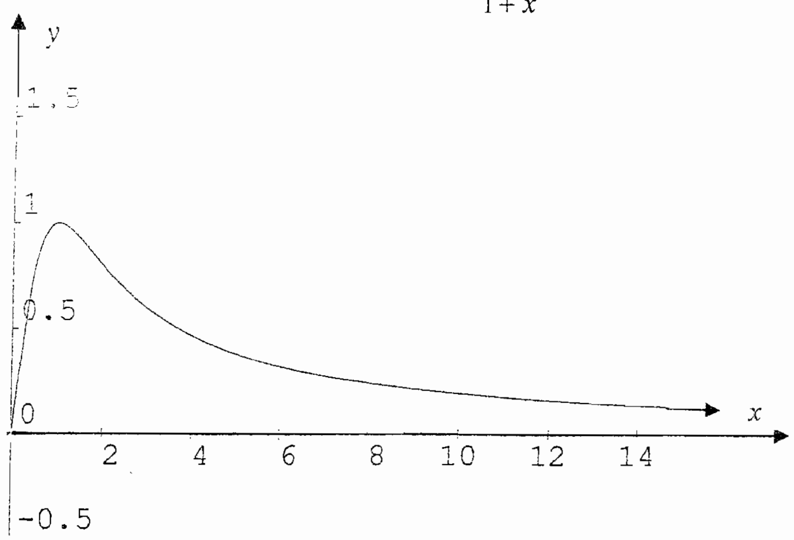
On the Argand diagram above, $OABC$ is a square. If B represents the complex number $4+6i$ find the complex number represented by C. 3

d) i) Sketch the region in the complex number plane where the inequalities $|z-1| \le |z-i|$ and $|z-2-2i| \le 1$ hold simultaneously 2

ii) If P is a point on the boundary of this region representing the complex number z, find the values of z in the form $x+iy$ where $\arg(z-1)=\dfrac{\pi}{4}$ 2

Question 3 **(15 marks)** **Marks**

a) The diagram shows the graph of $f(x) = \dfrac{2x}{1+x^2}$ *for* $x \geq 0$



For each of the following draw a one-third page sketch:

i) Sketch the graph of $y = \dfrac{2x}{1+x^2}$ for all real x 1

ii) Use your completed graph in (i) to help sketch the graphs of

α) $y = \dfrac{|2x|}{1+x^2}$ 2

β) $y^2 = \dfrac{2x}{1+x^2}$ 2

γ) $y = \log_e\left[\dfrac{2x}{1+x^2}\right]$ 2

iii) Sketch $y = \dfrac{1+x^2}{2x}$ clearly showing and stating the equations of any asymptotes. 2

iv) Find the value(s) of A so that the graphs of

$y = \dfrac{Ax}{1+x^2}$ and $y = \dfrac{1+x^2}{Ax}$ have no points in common. 2

b) The area between the curve $y = \dfrac{2x}{1+x^2}$ and the x-axis for $0 \leq x \leq 1$ is rotated about the y-axis. Use the method of cylindrical shells to find the volume of the resulting solid of revolution 4

Question 4 (15 marks) **Marks**

a) $P(a\cos\theta, b\sin\theta)$ and $Q(a\cos(-\theta), b\sin(-\theta))$ are the extremities of the latus rectum, $x = ae$, of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

i) Draw a neat diagram, marking the points P and Q and clearly showing the angle θ. 1

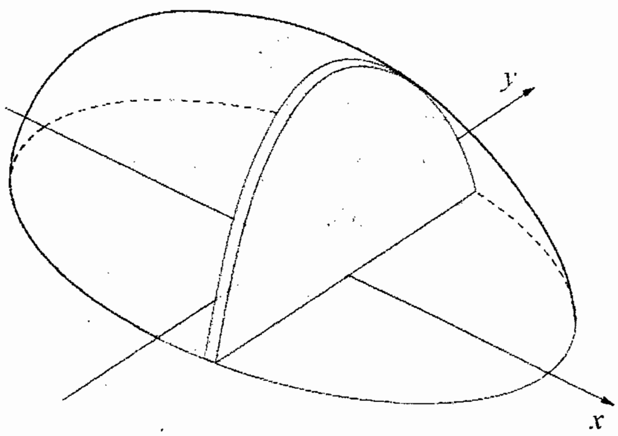
ii) Show that $\cos\theta = e$ 1

iii) Show that the length of PQ is $\frac{2b^2}{a}$ 2

b) Show that the area enclosed between the parabola $x^2 = 4ay$ and its latus rectum is $\frac{8a^2}{3}$ units^2 3

c) A solid figure has as its base, in the xy plane, the ellipse $\frac{x^2}{16} + \frac{y^2}{4} = 1$.

Cross-sections perpendicular to the x-axis are parabolas with latus rectums in The xy plane



i) Show that the area of the cross-section at $x = h$ is $\frac{16 - h^2}{6}$ units^2. 3

[*use your answer to part*(b)]

ii) Hence, find the volume of this solid. 2

d) Over the complex field $P(x) = 2x^3 - 15x^2 + Cx - D$ has a zero $x = 3 - 2i$

i) Determine the other two zeros 2

ii) Find the value of D 1

Question 5 (15 marks)

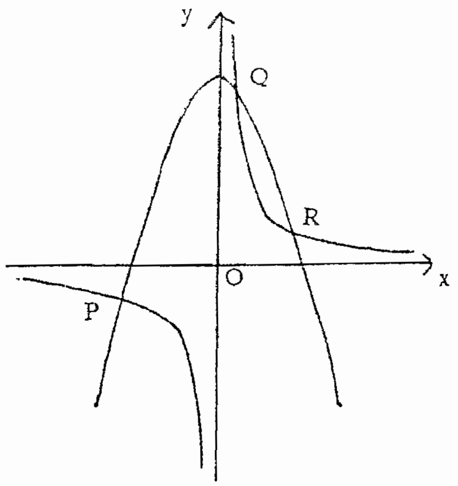
a) The roots of the equation $z^5 - 1 = 0$ are $1,\ w,\ w^2,\ w^3,\ w^4$

i) Mark this information on an Argand diagram 1

ii) Find a real quadratic equation with roots $w + w^4$ and $w^2 + w^3$ 2

iii) Hence find the value of $\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5}$ 2

b)



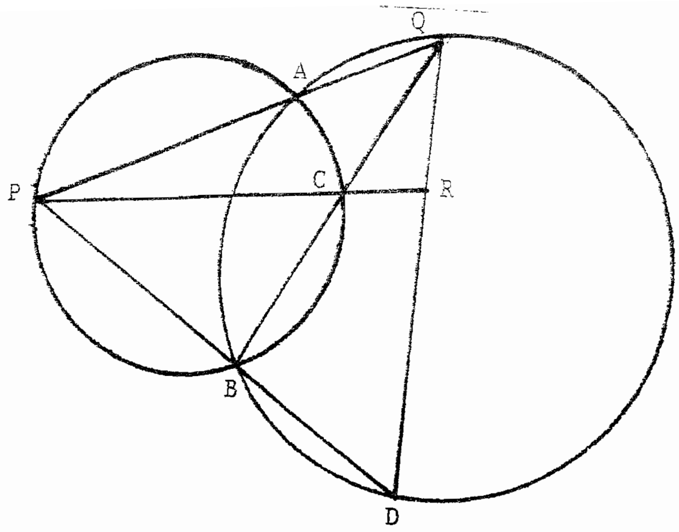
The curves $y = k - x^2$, for some real number k, and $y = \frac{1}{x}$ intersect at the points P, Q and R where $x = \alpha,\ x = \beta$ and $x = \gamma$.

i) Show that the monic cubic equation with coefficients in terms of k whose roots are $\alpha^2,\ \beta^2$ and γ^2 is given by $x^3 - 2kx^2 + k^2x - 1 = 0$ 3

ii) Find the monic cubic equation with coefficients in terms of k whose roots are $\frac{1}{\alpha^2},\ \frac{1}{\beta^2}$ and $\frac{1}{\gamma^2}$ 2

iii) Hence show that $OP^2 + OQ^2 + OR^2 = k^2 + 2k$, where O is the origin 2 c)

Marks

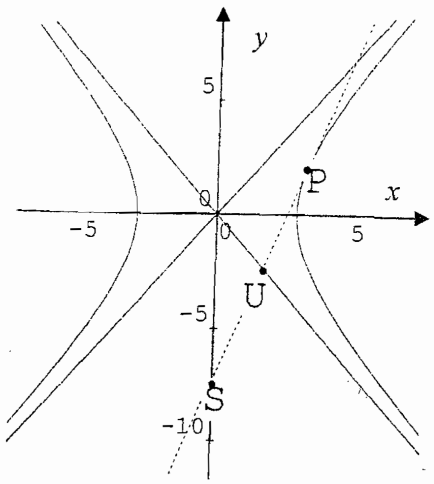


i) Copy the diagram onto your page.

ii) Prove $BCRD$ is a cyclic quadrilateral (Hint: let $\angle D = \theta$) 3

Question 6 (15 marks)

a)



Consider the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$

i) Write down the equation of each asymptote 1

ii) By differentiation find the gradient of the tangent to the hyperbola at $P(3\sec\theta,\ 4\tan\theta)$ 1

iii) Show that the equation of the tangent at P is $4x = 3\sin\theta y + 12\cos\theta$ 2

iv) Find the x-coordinate of U, the point where the tangent meets the asymptote (as shown on the diagram). 2

v) Using the x-values only, find the value for θ such that U is the mid point of PS. 2

Marks

b) i) Show that $\int_0^{\frac{\pi}{4}} \tan\theta \, d\theta = \frac{1}{2}\log_e 2$ 2

ii) If $I_n = \int_0^{\frac{\pi}{4}} \tan^n\theta \, d\theta$ show that for $n \geq 2$ 3

$$I_n + I_{n-2} = \frac{1}{n-1}$$

iii) Hence evaluate $\int_0^{\frac{\pi}{4}} \tan^5\theta \, d\theta$ 2

Question 7 (15 marks)

a) i) Show that $\tan^{-1}(3) - \tan^{-1}(\frac{1}{2}) = \frac{\pi}{4}$ 2

ii) Prove by mathematical induction that

$$\sum_{r=1}^{n} \tan^{-1}\left(\frac{1}{2r^2}\right) = \tan^{-1}(2n+1) - \frac{\pi}{4}$$ 4

is true for all integral values of n for $n \geq 1$

b) A particle is moving in a straight line. After time t seconds it has displacement x metres from a fixed point *0* on the line, velocity $v = \frac{1-x^2}{2}\ ms^{-1}$ and acceleration $a\ ms^{-2}$. Initially the particle is at *0*.

i) Find an expression for a in terms of x 1

ii) Show that $\frac{2}{1-x^2} = \frac{1}{1+x} + \frac{1}{1-x}$ and hence find an expression for x in terms of t. 3

iii) Describe the motion of the particle, explaining whether it moves to the left or right of *0*, whether it slows down or speeds up, and where its limiting position is. 2

c) i) Differentiate $x^3 + y^3 = 6xy$ to find $\frac{dy}{dx}$. 1

ii) Find the x value(s) of the point(s) where $\frac{dy}{dx} = 0$ 2

Question 8 **(15 marks)** **Marks**

a) i) If $S = 1 - x + x^2 - x^3 + \ . \ . \ .$ where $|x| < 1$, find an expression for S, the limiting sum, of the series. 1

ii) By integrating both sides of this expression and then making a substitution for x show that $\log_e 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ . \ . \ .$ 2

b) i) Show that $\int x \tan^{-1} x \, dx = \frac{1}{2}(x^2 + 1) \ \tan^{-1} x - \frac{1}{2} x + c$ 3

ii) If $I_n = \int_0^1 x^n \tan^{-1} x \, dx$ for $n \geq 2$ show that

$$I_n = \frac{\pi}{2(n+1)} - \frac{1}{n(n+1)} - \frac{n-1}{n+1} I_{n-2}$$ 4

c) i) Write the general solution to $\cos 5\theta = \cos A$ 1

ii) Hence or otherwise find the total number of solutions to the equation $\cos 5\theta = \sin\theta$ for $0 \leq \theta \leq 10\pi$ 4

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$