Sydney Technical High School

TRIAL HIGHER SCHOOL CERTIFICATE

2006

MATHEMATICS EXTENSION 2

General Instructions

- Reading time - 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Approved calculators may be used
- All necessary working should be shown in every question
- A table of standard integrals is supplies at the back of this paper
- Start each question on a new page
- Attempt all Questions 1 – 8
- All questions are of equal value
- **Total marks 120**

Name: ______________________

Class: ______________________

Question 1	Question 2	Question 3	Question 4	Question 5	Question 6	Question 7	Question 8	TOTAL

Question 1 **Marks**

a) Find:

(i) $\int \frac{x\,dx}{(1+x^2)^2}$ **2**

(ii) $\int \sin^3 x\,dx$ **2**

(iii) $\int x\sqrt{1-x}\,dx$ **3**

b) (i) Find real numbers a and b such that

$$\frac{5-3x}{(x+1)(x^2+1)} = \frac{a}{x+1} + \frac{bx+c}{x^2+1}$$ **2**

(ii) Hence find $\int \frac{5-3x}{(x+1)(x^2+1)}\,dx$ **2**

c) Evaluate $\int_0^{\pi} \frac{dx}{\cos\frac{x}{2}+\sin\frac{x}{2}+3}$ using the substitution $t = \tan\left(\frac{x}{4}\right)$ **4**

Question 2

a) (i) Express $w = -1 - i$ in modulus – argument form. **2**

(ii) Hence express w^{12} in the form $x + iy$ where x and y are real numbers. **2**

b) Find the equation, in Cartesian form, of the locus of the point z if **2**

$$|z - i| = |z + 3|.$$

c) Sketch the region in the Argand diagram that satisfies the inequality **3**

$$\text{Re}\left(\frac{1}{z}\right) \le \frac{1}{2}$$

d) (i) On the Argand diagram draw a neat sketch of the locus specified by **1**

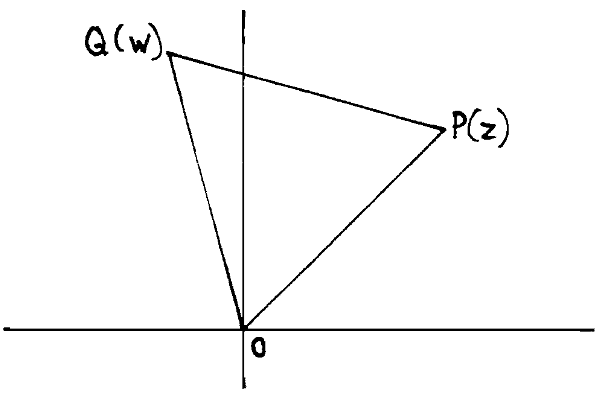
$$\arg(z + 1) = \frac{\pi}{3}$$

(ii) Hence find z so that $|z|$ is a minimum. **2**

e) Points P and Q represent the complex numbers z and w respectively in the Argand Diagram. If ΔOPQ (where O is the origin) is an equilateral triangle

(i) Show why $wz = z^2 cis\frac{\pi}{3}$ **1**

(ii) Prove that $z^2 + w^2 = zw$ **2**



Question 3

a) The hyperbola, H, has a Cartesian equation $\dfrac{x^2}{25} - \dfrac{y^2}{16} = 1$

(i) Find the coordinates of the foci S and S′ **1**

(ii) Show that any point, P, on H can be represented by the coordinates $(5\sec\theta,\ 4\tan\theta)$ and hence, or otherwise, prove that $PS - PS'$ is a constant. **3**

(iii) Show that the equation of the normal at the point P on the hyperbola is **3**

$$\frac{5x}{\sec\theta} + \frac{4y}{\tan\theta} = 41$$

(iv) If this normal meets the x axis at M and the y axis at N, prove that **3**

$$\frac{PM}{PN} = \frac{16}{25}$$

b) Consider the function $y = \cos^{-1}(\cos x)$. Given the domain and range are

D: all real x

R: $0 \le y \le \pi$

(i) State whether the function is even, odd or neither and find its period. **2**

(ii) Hence sketch the graph of the function over $-4\pi \le x \le 4\pi$ **1**

c) Solve for x: **2**

$$\tan^{-1}(3x) - \tan^{-1}(2x) = \tan^{-1}\left(\frac{1}{5}\right)$$

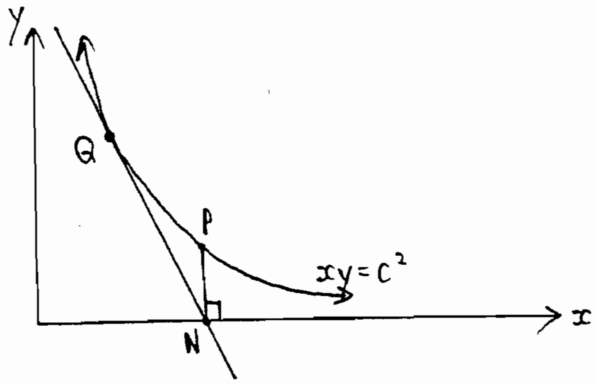
Question 4

a) Find Q which is rational where **2**

$$\sqrt{Q} = \sqrt{2+\sqrt{3}} + \sqrt{2-\sqrt{3}}$$

b) If $f(x) = f(x-1) + x^2$ and $f(3) = 7$, evaluate $f(1)$. **2**

c)



In the diagram above, P $(ct_1, \frac{c}{t_1})$ and Q $(ct_2, \frac{c}{t_2})$ are distinct variable points on the rectangular hyperbola $xy = c^2$. PN is the perpendicular from P to the x axis and the tangent at Q passes through N.

(i) Show that $t_1 = 2t_2$ **3**

(ii) Find the Cartesian equation of the locus of T, the point of intersection of the tangents at P and Q. **3**

d) (i) By solving the equation $z^3 = 1$, find the 3 cube roots of 1. **2**

(ii) Let w be a cube root of 1 where w is not real. **1**

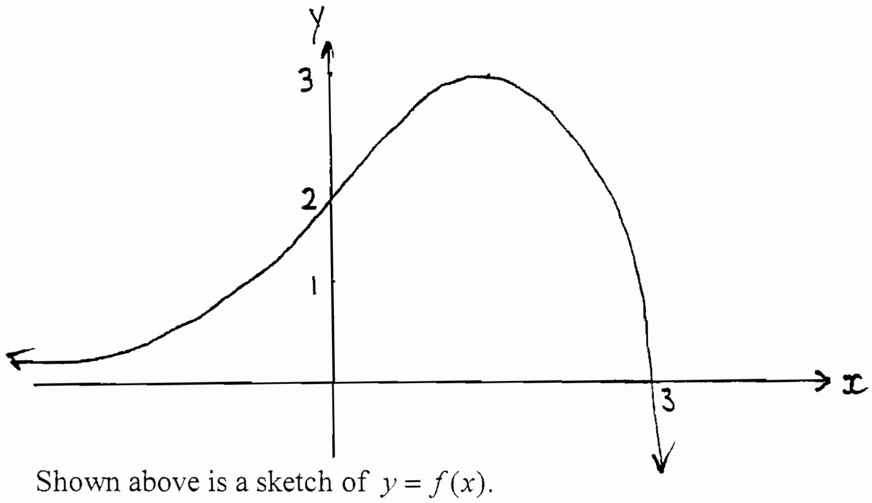
Show that $1 + w + w^2 = 0$

(iii) Find the quadratic equation, with integer coefficients, that has roots **2**

$4 + w$ and $4 + w^2$

Question 5

a)



Shown above is a sketch of $y = f(x)$.

On separate diagrams draw sketches of:

(i) $y = \dfrac{1}{f(x)}$ **2**

(ii) $y = [f(x)]^3$ **2**

(iii) $y = f(|x|)$ **2**

(iv) $y = \log_e[f(x)]$ **2**

b) The deck of a ship was 3m below the level of a wharf at low tide and 1m above the wharf level at high tide. Low tide was at 9:30am and high tide at 4:00pm. Find the first time after low tide when the deck was level with the wharf, if the motion of the tide was simple harmonic. **4**

c) Prove by mathematical induction that, for all integers $n \geq 1$, **3**

$$(\cos\theta - i\sin\theta)^n = \cos(n\theta) - i\sin(n\theta)$$

Question 6

a) Find the integers m and n such that $(x+1)^2$ is a factor of $x^5 + 2x^2 + mx + n$ **2**

b) None of the roots α, β and γ of the equation $x^3 + 3px + q = 0$ is zero.

(i) Obtain the monic equation whose roots are $\frac{\beta\gamma}{\alpha}$, $\frac{\alpha\gamma}{\beta}$ and $\frac{\alpha\beta}{\gamma}$ expressing its coefficients in terms of p and q. **4**

(ii) Show that if $\gamma = \alpha\beta$ then $(3p-q)^2 + q = 0$. **2**

c) For the equation $x^3 - 6x^2 + 9x - 5 = 0$

(i) By considering stationary points, show that the equation has only one real root α. **3**

(ii) Determine the two consecutive integers between which α lies. **1**

(iii) By considering the product of the roots of the equation, express the modulus of each of the complex roots in terms of α and deduce that the value of this modulus lies between 1 and $\frac{\sqrt{5}}{2}$. **3**

Question 7

a) (i) Let $I_n = \int_1^e x(\ln x)^n \, dx$, $n = 0, 1, 2, 3 \ldots$ **2**

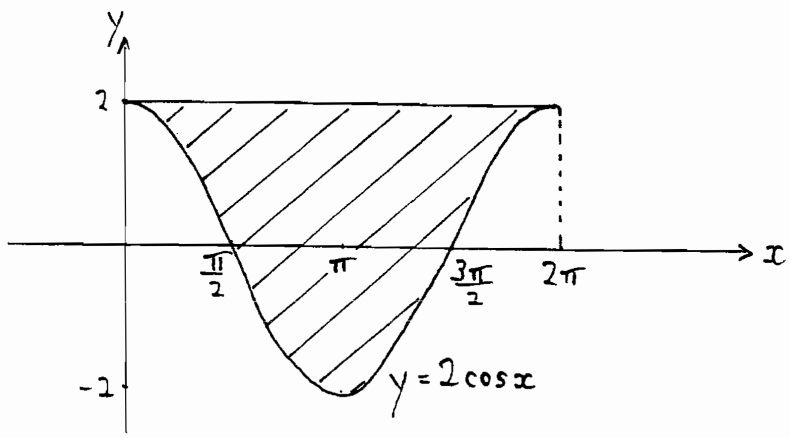
Use integration by parts to show that $I_n = \frac{e^2}{2} - \frac{n}{2} I_{n-1}$, $n = 1, 2, 3 \ldots$

(ii) The area bounded by the curve $y = \sqrt{x}\,(\ln x)^2$, the x axis **3**

and the lines $x=1$ and $x = e$ is rotated about the x axis.

Find the exact value of the volume of the solid of revolution so formed.

b)



The shaded region is rotated about the y axis to obtain a solid of revolution.

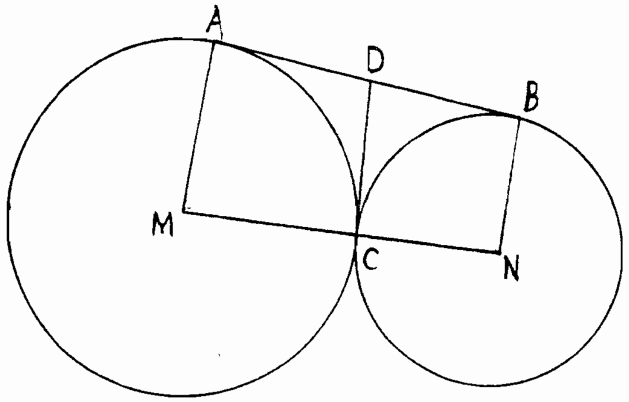
(i) Use the method of cylindrical shells to show that the volume of this solid is given by **2**

$$4\pi \int_0^{2\pi} x(1 - \cos x) \, dx \ .$$

(ii) Hence calculate this volume. **2**

Question 7 (cont.)

c)



In the diagram MCN is a straight line. Circles are drawn with centre M, radius MC and centre N, radius NC. AB is a common tangent to the two circles with points of contact at A and B respectively. CD is the common tangent at C, and meets AB at D.

(i) Explain why AMCD and BNCD are cyclic quadrilaterals. **2**

(ii) Show that Δ ACD ||| ΔCBN **2**

(iii) Show that MD || CB **2**

Question 8

A particle of mass m is projected vertically upwards under gravity. The air resistance to the motion is $\frac{1}{100}mgv^2$ where v is the speed of the particle.

(a) (i) Show that during the upward motion of the particle, if x is the upward vertical displacement of the particle from its projection point at time t, then **1**

$$\ddot{x} = \frac{-1}{100}g\left(100 + v^2\right)$$

(ii) If the initial speed of projection is u, show that the greatest height (above the projection point) reached by the particle is **5**

$$\frac{50}{g}\ln\left(\frac{100 + u^2}{100}\right).$$

(iii) Show that during the downward motion of the particle, if x is the downward vertical displacement of the particle from its highest position at a time t after it begins the downward motion, then **1**

$$\ddot{x} = \frac{1}{100}g(100 - v^2)$$

(iv) Show that the speed of the particle on return to its point of projection is **5**

$$\frac{10u}{\sqrt{100 + u^2}}$$

(v) Find the terminal velocity V of the particle for the downward motion. **1**

(vi) If the initial speed of projection of the particle is V, as found in part (v), show that the speed on return to the point of projection is $\frac{1}{\sqrt{2}}V$. **2**

End of Paper.

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$