SYDNEY TECHNICAL HIGH SCHOOL

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

2005

Mathematics Extension 2

General Instuctions

- Reading time - 5 minutes
- Working time - 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks - 120

- Attempt Questions 1 – 8
- All questions are of equal value

Name : ____________________________

Teacher : ____________________________

Question 1	Question 2	Question 3	Question 4	Question 5	Question 6	Question 7	Question 8	**Total**

Question 1 (15 marks) **Marks**

(a) Find $\int \frac{2x}{x+1}\,dx$ 2

(b) Find $\int \frac{dx}{\sqrt{8+2x-x^2}}$ 2

(c) Use partial fractions to find $\int \frac{2}{x^2-x}\,dx$ 3

(d) Find $\int \sin 2x \cos^3 x \, dx$ 4

(e) Find $\int \frac{1}{(36+x^2)^{\frac{3}{2}}}\,dx$ using the substitution $x = 6\tan\theta$ 4

Question 2 (15 marks) (Start a new page) **Marks**

(a) Find the gradient of the curve $2x^3 - x^2 y + y^3 = 1$ 3

at the point $(2,-3)$.

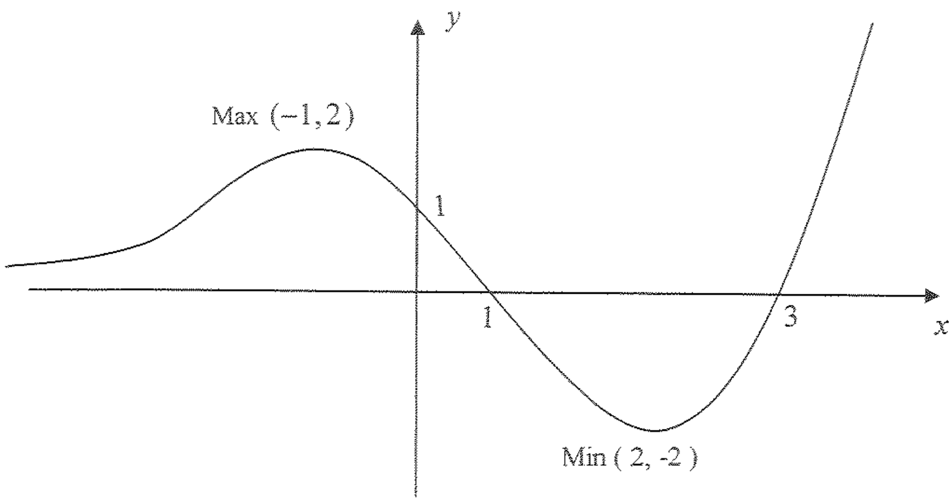
(b) Solve $|x-2| + |x+1| = 3$ 2

QUESTION 2 (Continued)

(c) The base of a solid is the area enclosed by the curve $y = x^2$, the line $x = 2$ and the x axis. Each cross-section of the solid by a plane perpendicular to the x axis is a regular hexagon with one side in the base of the solid. 4

Find the volume of the solid.

(d)



The sketch above shows the function $y = f(x)$.

Sketch possible graphs of the following

(i) $y = \dfrac{1}{f(x)}$ 2

(ii) $y = \int f(x)\,dx$ 2

(iii) $y^2 = f(x)$ 2

Question 3 (15 marks) (Start a new page) **Marks**

(a) Given $z = -3\sqrt{3} + 3i$

(i) express z in modulus argument form. 2

(ii) find the smallest positive integer n such that z^n is real. 1

(b) Evaluate $\text{Im}\left(\frac{4}{1-i}\right)$ 2

(c) Sketch the locus described by $|z+2| = |z-4i|$ 2

(d) (i) Sketch the intersection of the locus described by 3

$|z| \leq 3$ and $-\frac{\pi}{4} \leq \arg(z+3) \leq \frac{\pi}{4}$

(ii) If the complex number ω lies on the boundary of the region 2

sketched in part (i), find the minimum value of $|\omega|$.

(e) OABC is a rectangle on the Argand diagram in which side OC is twice the length of OA, where O is the origin.

(i) If A represents the complex number $1+2i$, find the complex numbers represented by B and C given that the argument of the complex number represented by the point C is negative.. 2

(ii) If this rectangle is rotated anticlockwise $\frac{\pi}{3}$ radians about O, find the complex number represented by the new position of A. 1

Question 4 (15 marks) (Start a new page) **Marks**

(a) For the hyperbola with equation $4x^2 - 9y^2 = 36$ find,

(i) the eccentricity 2

(ii) the equation of the asymptotes 1

(b) Given the point $P(a\sec\theta\,,\, b\tan\theta)$ lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

(i) Show that the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

at the point $P(a\sec\theta\,,\, b\tan\theta)$ has equation $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$ 3

(ii) If this tangent in part (i) meets the directrix of the hyperbola 4

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ corresponding to the focus at $S(ae, o)$ at the point Q,

show that $\angle PSQ$ is a right angle.

(c) (i) Show that $(1-\sqrt{x})^{n-1}\sqrt{x} = (1-\sqrt{x})^{n-1} - (1-\sqrt{x})^n$ 1

(ii) If $I_n = \int_0^1 (1-\sqrt{x})^n \, dx$ for $n \geq 0$ 4

show that $I_n = \frac{n}{n+2} I_{n-1}$ for $n \geq 1$.

Question 5 (15 marks) (Start a new page) **Marks**

(a) The region bounded by the curves $y = x^2$ and $y = x + 2$ is rotated about the line $x = 3$. Use the method of cylindrical shells to find the volume of the solid of revolution formed. 4

(b) Solve the equation $8x^4 + 12x^3 - 30x^2 + 17x - 3 = 0$ given that it has a triple root. 4

(c) If α, β, δ are the roots of $x^3 + px^2 + qx + r = 0$, find the polynomial equation with roots $\alpha^2, \beta^2, \delta^2$. 2

(d) The acceleration of a particle moving in simple harmonic motion is given by $\ddot{x} = -n^2 x$ where x is the displacement of the particle from the origin and n is a constant.

(i) Show that the velocity v of the particle is given by $v^2 = n^2(a^2 - x^2)$ where a is the amplitude of the motion. 2

(ii) Given that the speed of the particle is $V\ m/s$ when it is d metres from the origin and that its speed is $\frac{V}{2}\ m/s$ when it is $2d$ metres from the origin, show that :

α) the particle's amplitude is $\sqrt{5}d$ metres. 2

β) the period of the motion is $\frac{4\pi d}{V}$ seconds. 1

Question 6 (15 marks) (Start a new page) **Marks**

(a) Evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{2+\cos x}$ using the substitution $t = \tan\frac{x}{2}$ 4

(b) A particle of mass m is fired vertically upwards with initial velocity V m/s and is subjected to air resistance equal to mkv Newtons where k is a constant and v is the velocity of the particle in metres per second as it moves through the air.

(i) Explain why the equation of motion of the particle is given by 1

$\ddot{x} = -g - kv$ where g is the acceleration due to gravity.

(ii) Show that the maximum height reached by the particle is given by 4

$$H = \frac{V}{k} - \frac{g}{k^2}\ln\left(1+\frac{kV}{g}\right)$$

(c) (i) Find the seven complex roots of the equation $z^7 = 1$. 2

(ii) If ω is the complex root of $z^7 = 1$ 1

with smallest positive argument, find the value of

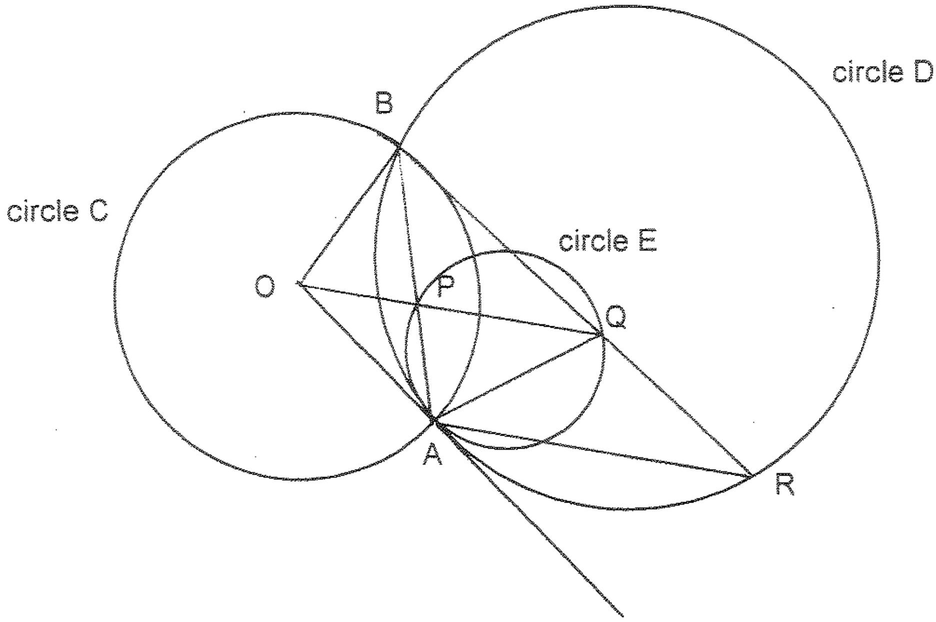
$$1+\omega+\omega^2+\omega^3+\omega^4+\omega^5+\omega^6$$

(iii) Find the cubic equation whose roots are 3

$$\omega+\omega^6 \ , \ \omega^2+\omega^5 \ , \ \omega^3+\omega^4$$

Question 7 (15 marks) (Start a new page) **Marks**

(a)



In the diagram above, OA is a radius of a circle C with centre O, and two circles D and E are drawn touching the line OA at A as shown. The larger circle D meets circle C again at B, and the line AB meets the smaller circle E again at P. The line OP meets circle E again at Q, and the line BQ meets the circle D again at R.

(i) Let $\angle OAP = \theta$. Explain why $\angle PQA = \theta$. 1

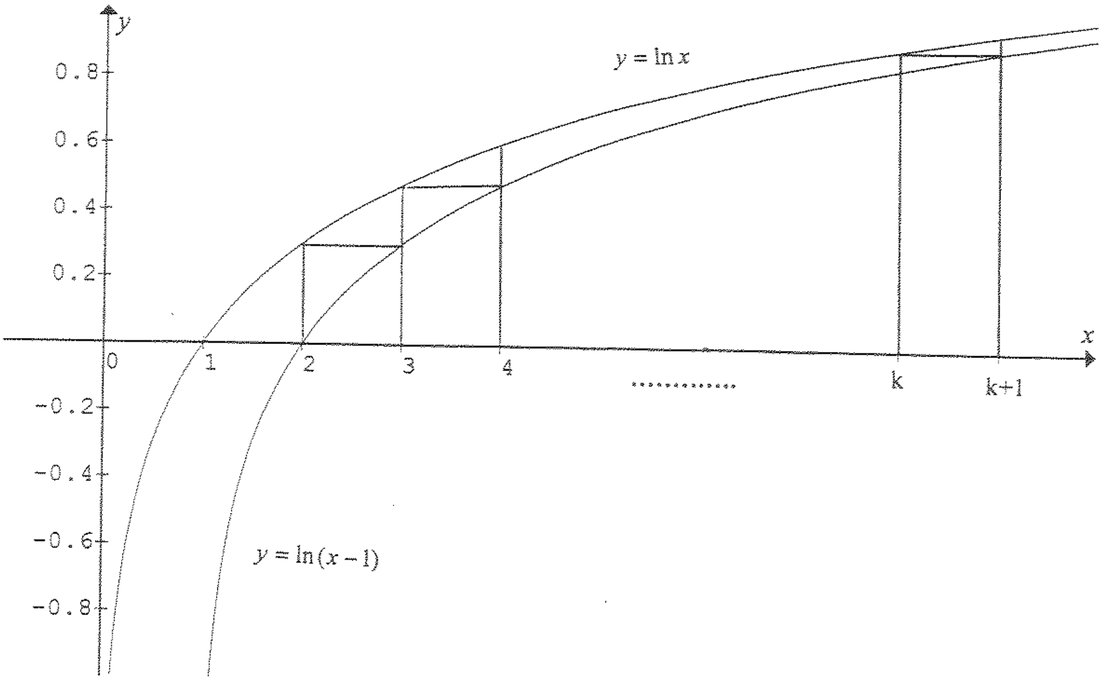
(ii) Prove that the points O , B , Q and A are concyclic. 2

(iii) Prove that OQ bisects $\angle BQA$. 2

(iv) Prove that $OQ // AR$ 2

QUESTION 7 (Continued)

(b)



In the diagram above, the curves $y = \ln x$ and $y = \ln(x-1)$ are sketched

and $k-1$ rectangles are constructed between $x = 2$ and $x = k+1$ where $k \geq 2$.

Let $S = \ln 2 + \ln 3 + \ln 4 + \dots\dots + \ln k$.

(i) Explain why S represents the sum of the areas of the $k-1$ rectangles. 1

(ii) Use an appropriate integration method to show that

$$\int_2^{k+1} \ln(x-1)\, dx = k \ln k - k + 1$$ 4

(iii) Hence show that $k^k < k!e^{k-1} < \frac{1}{4}(k+1)^{k+1}$ where $k \geq 2$ 3

(note $n! = n(n-1)(n-2)\dots\dots\dots 3 \times 2 \times 1$)

Question 8 (15 marks) (Start a new page) **Marks**

(a) Find all x such that $\sin x = \cos 5x$ and $0 < x < \pi$. 3

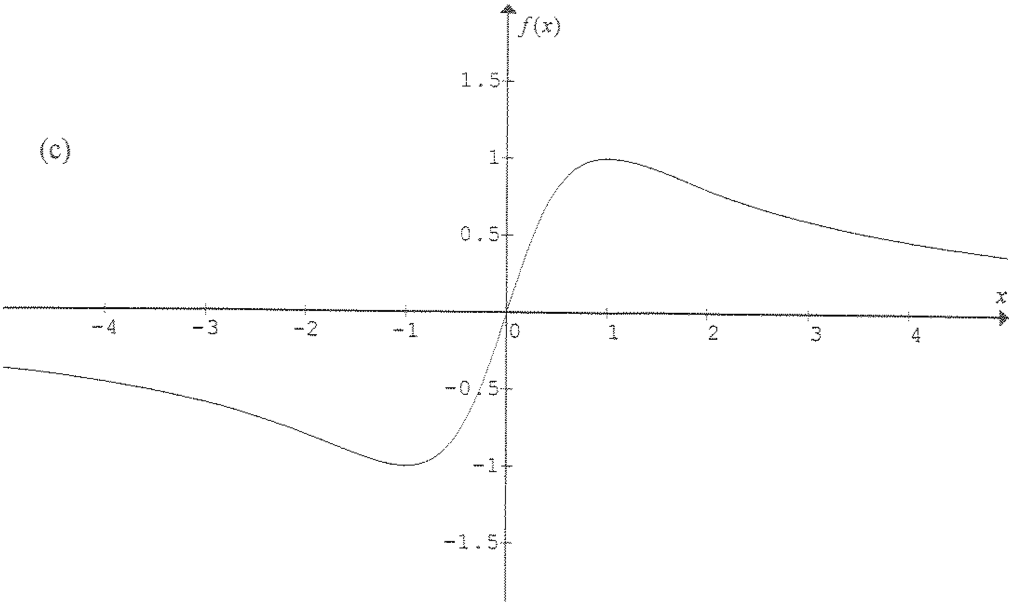
(b) If z is a complex number for which 2

$|z| = 1$ and $\arg(z) = \theta$, $0 \le \theta \le \frac{\pi}{2}$,

find the value of $\arg\left(\frac{2}{1-z^2}\right)$ in terms of θ.

Question 8 continued on next page.

(c)



The curve $f(x)=\dfrac{2x}{1+x^2}$ is sketched above. It has a maximum turning point at $(1,1)$ and minimum turning point at $(-1,-1)$.

(i) State the range of $f(x)=\dfrac{2x}{1+x^2}$ 1

(ii) Let x_0 be a real number not equal to 1 or -1 and consider the sequence

of real numbers defined by $x_{n+1}=f(x_n)$ for $n=0,1,2,\ldots\ldots$

(α) Given $x_1=g(r)$ and $g(x)=\dfrac{e^{2x}-1}{e^{2x}+1}$ express r in terms of x_1. 2

(β) Hence deduce that there exists a real number r such that $x_1=g(r)$ 1

where $g(x)=\dfrac{e^{2x}-1}{e^{2x}+1}$.

(δ) Show that $\dfrac{2\,g(x)}{1+(g(x))^2}=g(2x)$. 2

(γ) Hence, using the above results and Mathematical Induction 4

show that $x_n=g(2^{n-1}\,r)$ for $n=1,2,3,\ldots\ldots$

End of Paper