Name: ..................................................... Maths Class: ......................

# SYDNEY TECHNICAL HIGH SCHOOL

## TRIAL HIGHER SCHOOL CERTIFICATE
## 2004

# Mathematics Extension 2

TIME ALLOWED: 3 hours

***Instructions:***

- Write your name and class at the top of this page, and at the top of each answer sheet.
- At the end of the examination this examination paper must be attached to the front of your answers.
- All questions are of equal value and may be attempted.
- All necessary working must be shown. Marks will be deducted for careless or badly arranged work.
- Marks indicated are a guide only and may be varied if necessary.

**(FOR MARKERS USE ONLY)**

| Q1 | Q2 | Q3 | Q4 | Q5 | Q6 | Q7 | Q8 | TOTAL |
|---|---|---|---|---|---|---|---|---|
| | | | | | | | | |

## QUESTION 1:

2 (a) (i) $\int \frac{dx}{x^2+6x+13}$

2 (ii) $\int_0^4 \frac{dx}{(2x+1)\sqrt{2x+1}}$

3 (b) Show that $\int x^n e^x\,dx = x^n e^x - n\int x^{n-1}e^x\,dx$

Hence find $\int x^2 e^x\,dx$

3 (c) Sketch the locus of the point Z in the Argand plane, which moves so that $\arg(z-1)=\frac{\pi}{2}$

3 (d) (i) Find values of A, B and C so that $\frac{13}{(x^2+4)(x+3)} \equiv \frac{Ax+B}{x^2+4}+\frac{C}{x+3}$

2 (ii) Hence find $\int \frac{13dx}{(x^2+4)(x+3)}$

## QUESTION 2:

(a) If $z = 1+\sqrt{3}i$ find

4 (i) $\bar{z}$ (ii) $|z|$ (iii) arg z (iv) $\arg(iz)$

(b) Express $z = 1+\sqrt{3}i$ in mod-arg form and hence find

3 (i) $\sqrt{z}$ (ii) $z^6$ (in simplest form)

3 (c) On the same set of axes, sketch $y=|x-1|$ and $y=|x+1|$ and then use this graph, or otherwise, to find the value of *k* if $|x-1|+|x+1| \geq k$ for all values of x.

3 (d) (i) The point W represents the complex number $w=a+ib$, and $w=\frac{z}{z+1}$ where $z=x+iy$.
The point Z representing the complex number z moves along the y-axis only.

Show that $a=\frac{y^2}{1+y^2}$ and $b=\frac{y}{1+y^2}$

2 (ii) Find the locus of W both algebraically and geometrically.

## QUESTION 3 :

4 (a) (i) Using the method of Mathematical Induction, prove deMoivre's Theorem

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

2 (ii) Prove that the points on the Argand Diagram with co-ordinates representing $(\text{cis}\frac{\pi}{3})^n$ for n=1,2,3..,6 are the vertices of a regular hexagon inscribed in a circle of radius 1 unit.

(b) If $f(x) = \cos x + i\sin x$,

1 (i) Find f(0)

1 (ii) Show that $\frac{f'(x)}{f(x)} = i$

3 (iii) By integrating both sides of part (ii), deduce that $\cos x + i\sin x = e^{ix}$.
This is EULER's THEOREM.

1 (iv) Using Euler's Theorem from part (iii), prove deMoivre's Theorem

1 (c) (i) Describe the locus of the point z, where $|z - a| = r$

2 (ii) If $|z - a| = r$ and $|z - b| = s$ what is the geometric significance when $|a - b| = r + s$

## QUESTION 4:

3 (a) By using t-results, or otherwise, find $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{d\theta}{\sin\theta}$ leaving your answer in exact form

(b) Sketch the following on different sets of axes showing all important features

**(DO NOT USE CALCULUS)**

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(i) $y = \frac{|x|}{x}$

(ii) $y = \ln(\frac{1}{x^2})$

(iii) $y = |\tan x|$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$

(iv) $y = \sec x$ for $-2\pi \le x \le 2\pi$

4 (c) Use calculus, or otherwise, to sketch $y = \frac{e^x}{x}$ showing all stationary points and asymptotes, if they exist.

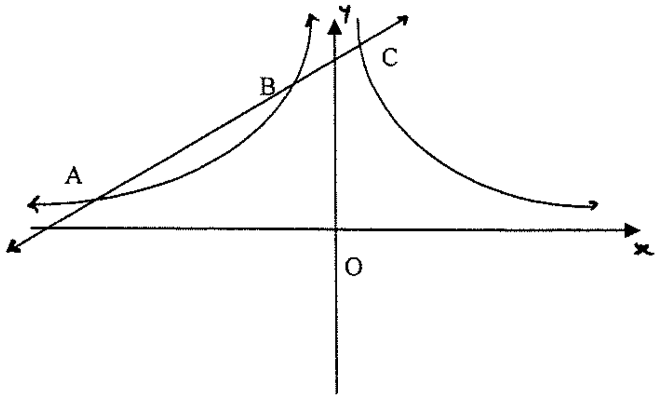
## QUESTION 5:

(a) For the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ find

3 (i) the eccentricity
(ii) the co-ordinates of the foci
(iii) the equations of the directrices

3 (b) Given that $P(x)=3x^3-11x^2+8x+4$ has a double root, fully factorise $P(x)$

(c)




In the diagram above, the points A, B and C represent the points of intersection of the line $y=4x+8$ and the curve $y=\frac{1}{x^2}$. The x-values of A, B and C are $\alpha$, $\beta$ and $\gamma$

1 (i) Show that $\alpha$, $\beta$ and $\gamma$ satisfy $4x^3+8x^2-1=0$

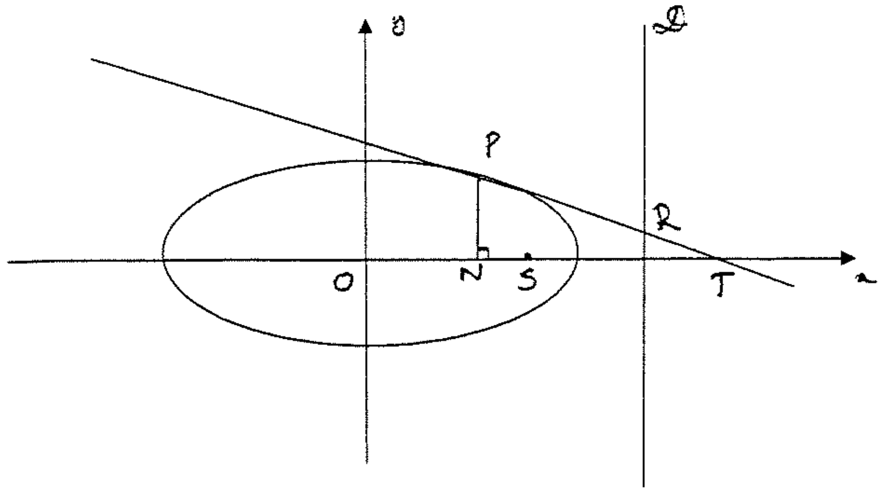
3 (ii) Find a polynomial with roots $\alpha^2$, $\beta^2$ and $\gamma^2$

2 (iii) Find $\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}$

3 (iv) Prove that $OA^2 + OB^2 + OC^2 =$ , 132 where O is the origin.

**QUESTION 6:**

(a)




P(acosθ, bsinθ) is any point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. The tangent at P cuts the major axis in T and the Directrix in R. N is the foot of the perpendicular from P to the major axis, O is the centre and S is the focus.

2 (i) Show that the equation of the tangent at P is $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$

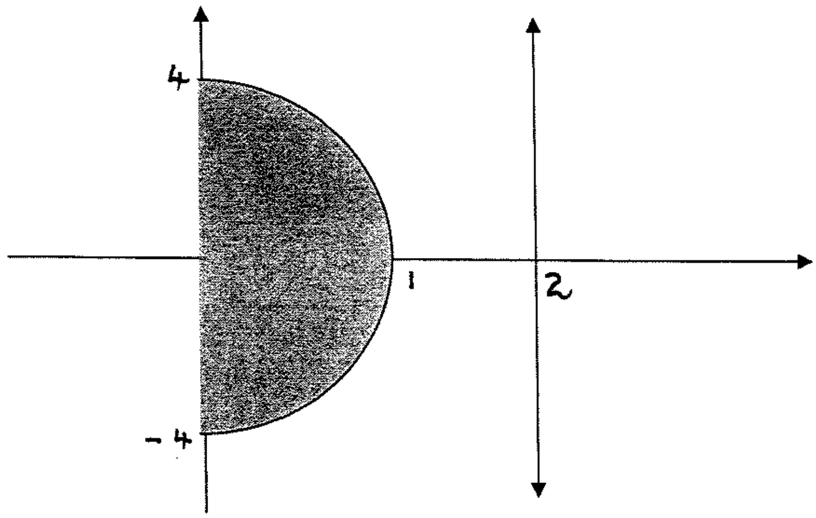
2 (ii) Show that ON.OT = $a^2$

5 (iii) Showing all steps carefully, prove that $\angle PSR = 90°$

*QUESTION 6 continues on the next page...)*

***QUESTION 6 continued...)***

(b)




A solid $S$ is formed by rotating the region bounded by the parabola $y^2 = 16(1-x)$ and the y-axis through $2\pi$ about the the line x=2.

**3** (i) Use the method of cylindrical shells to show that the volume of $S$ is given by

$$\int_0^1 16\pi(2-x)\sqrt{1-x}\,dx$$

**3** (ii) Calculate this definite integral by using the substitution $u=1-x$ (or otherwise)

## QUESTION 7:

(a) If $1, w_1$ and $w_2$ are the cube roots of unity, prove that

2 (i) $w_1 = \overline{w_2} = w_2^2$

1 (ii) $w_1 + w_2 = -1$

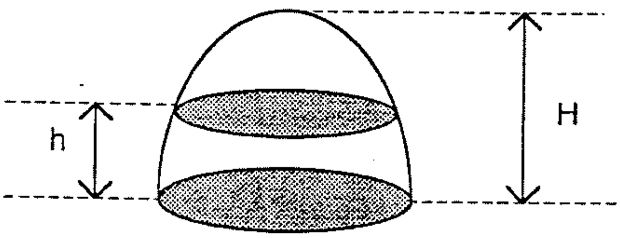
1 (iii) $w_1 w_2 = 1$

(b)

3 (i) By using the substitution $x = a\sin\theta$ or otherwise, verify that $\int_0^a \sqrt{a^2 - x^2}\,dx = \frac{1}{4}\pi a^2$

2 (ii) Deduce that the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\pi ab$

6 (iii)




The diagram above shows a mound of height H. At height h above the horizontal base, the horizontal cross section of the mound is elliptical in shape, with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \lambda^2 \quad \text{where} \quad \lambda = 1 - \frac{h^2}{H^2}$$

and x and y are co-ordinates in the plane of cross section.

Show that the volume of the mound is $\frac{8\pi abH}{15}$

## QUESTION 8:

2 (a) (i) Show that the equation of the tangent to the Hyperbola $xy = c^2$ at the point $P(cp, \frac{c}{p})$ is $x + p^2 y = 2cp$

2 (ii) If the tangents at the points P and $Q(cq, \frac{c}{q})$ meet at the point $R(x_1, y_1)$ prove that

$$(\alpha) \quad pq = \frac{x_1}{y_1}$$

and that $(\beta)$ $p + q = \frac{2c}{y_1}$

2 (iii) If the length of the chord PQ is d units, show that

$$d^2 = c^2 (p - q)^2 \left\{ 1 + \frac{1}{p^2 q^2} \right\}.$$

3 (iv) Further, if d in part (iii) above remains constant, deduce that the locus of R is given by

$$4c^2(x^2 + y^2)(c^2 - xy) = x^2 y^2 d^2$$

(b) If n is a positive integer and $f(x) = e^{-x}(1 + x + \frac{x^2}{2!} + .......... + \frac{x^n}{n!})$, $x \geq 0$

3 (i) Show that f(x) is a decreasing function

**NOTE:** ***n! = 1x2x3x4...........(n-1)n***

3 (ii) Deduce that for x>0 and n any positive integer,

$$e^x \geq 1 + x + \frac{x^2}{2!} + .......... + \frac{x^n}{n!}$$

***End of Examination***