SYDNEY TECHNICAL HIGH SCHOOL

TRIAL HIGHER SCHOOL CERTIFICATE

2003

MATHEMATICS EXTENSION 2

General Instructions:

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Approved calculators may be used
- All necessary working should be shown in every question
- A table of standard integrals is supplied at the back of this paper
- Start each question on a new page
- Attempt all Questions 1 – 8
- All questions are of equal value
- **Total marks 120**

Name: ______________________________

Class: ______________________________

Question 1	Question 2	Question 3	Question 4	Question 5	Question 6	Question 7	Question 8	TOTAL

Question 1 **Marks**

a) (i) Find $\int \frac{1}{\cos x + 2}\,dx$ using the substitution $t = \tan\frac{x}{2}$ **3**

Evaluate:

(ii) $\int_2^4 \frac{dx}{x^2 - 4x + 8}$ **3**

(iii) $\int_{-1}^1 \frac{4 + x^2}{4 - x^2}\,dx$ **4**

b) Let n be a positive integer and let

$$I_n = \int_1^2 (\log_e x)^n\,dx$$

(i) Prove that $I_n = 2(\log_e 2)^n - nI_{n-1}$ **2**

(ii) Hence evaluate $\int_1^2 (\log_e x)^4\,dx$ as a polynomial in terms of $\log_e 2$ **3**

Question 2

a) The complex number z and its conjugate $\bar{z}$ satisfy the equation $z\bar{z} + 2iz = 12 + 6i$. Find the possible values of z. **4**

b) On an Argand diagram shade the region containing all points representing complex numbers z such that $\text{Re}(z) \le 1$ *and* $0 \le \arg(z + i) \le \frac{\pi}{4}$ **3**

c) Express $z_1 = \frac{7 + 4i}{3 - 2i}$ in the form $a + ib$ where a and b are real. **1**

d) On an Argand diagram sketch the locus of the point representing the complex number z such that $|z - 3 - i| = \sqrt{10}$. Find the greatest value of $|z|$ subject to this condition. **3**

e) (i) Given that w is a complex root of the equation $x^3 = 1$, show that w^2 is also a root of this equation. **2**

(ii) Show that $1 + w + w^2 = 0$, and $1 + w^2 + w^4 = 0$. **2**

Question 3 **Marks**

The ellipse E has Cartesian equation $\dfrac{x^2}{25}+\dfrac{y^2}{16}=1$

a) Find

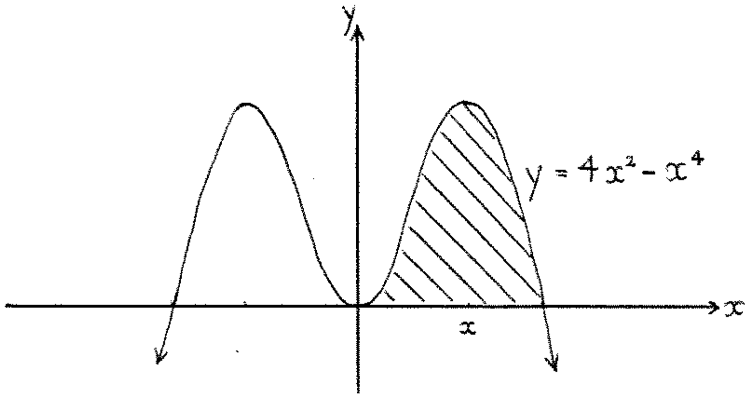
(i) the coordinates of the foci S and S^1 **1**

(ii) Show that any point P on E can be represented by the coordinates $(5\cos\vartheta,\ 4\sin\vartheta)$ and hence or otherwise prove that $PS+PS^1$ is a constant. **3**

(iii) Show that the equation of the normal at the point P on the ellipse is $\dfrac{5x}{\cos\vartheta}-\dfrac{4y}{\sin\vartheta}=9$ **3**

(iv) If this normal meets the x axis at M and the y axis at N, prove that $\dfrac{PM}{PN}=\dfrac{16}{25}$ **4**

b) The region shaded below is rotated about the y-axis to form a solid of revolution.



Using the method of cylindrical shells to calculate the volume of this solid, show that:

(i) The volume δV of a shell at x is given by $\delta V=2\pi(4x^3-x^5)\delta x$ **2**

(ii) Hence find the volume of this solid. **2**

Question 4

a) Let $f(x) = -x^2 + 8x - 12$. On separate diagrams, and without calculus, sketch the following graphs. Indicate clearly any asymptotes and intercepts with the axes.

(i) $y = f(x)$ **2**

(ii) $y = |f(x)|$ **2**

(iii) $y^2 = f(x)$ **2**

(iv) $y = \dfrac{1}{f(x)}$ **2**

(v) $y = e^{f(x)}$, giving the coordinates of any turning points by not using calculus. **3**

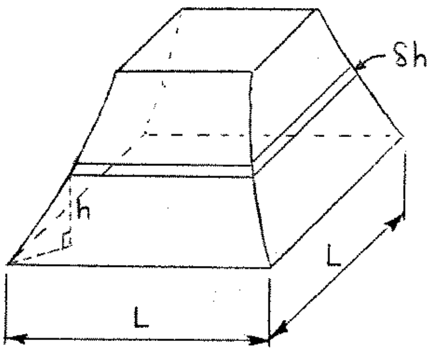
b) Given $p + q \geq 2\sqrt{pq}$ if p and q are positive real numbers

(i) Show that $e^a + e^b \geq 2e^{\frac{a+b}{2}}$ for all real a and b **2**

(ii) Hence find the minimum value of $e^{-2x} + e^{-x} + 1 + e^x + e^{2x}$ for all real x. **2**

Question 5

a)



A stone building of height H metres has the shape of a flat topped square 'pyramid' with curved sides as shown in the figure. The cross-section at height h metres is a square with sides parallel to the sides of the base and of length l, $l = \dfrac{L}{\sqrt{h+1}}$ where L is the side length of the square base in metres.

(i) Write an expression for the volume of a slice at height h metres. **2**

(ii) Hence find the volume of the building in terms of L and H. **2**

Question 5

b) The Fibonacci Sequence, F_n, is defined by:

$F_1 = 1$
$F_2 = 1$
$F_{n+2} = F_{n+1} + F_n$ for all $n \geq 1$

(i) Write down the first 12 terms of the sequence **1**

(ii) Prove, by mathematical induction, that for all positive integers, n, F_{4n} is divisible by 3. **5**

c) Find $\int \frac{\sin^3 \vartheta}{\cos^4 \vartheta} \, d\vartheta$ **3**

d) Consider the function of $y = \tan^{-1}(\tan x)$

(i) What is its period? **1**

(ii) Hence sketch the function for $-2\pi \leq x \leq 2\pi$ **1**

Question 6

The equation $x^3 + 2x - 1 = 0$ has roots α, β, and γ. In each of the following cases, find an equation with integer coefficients having the roots stated below.

a) (i) $-\alpha, -\beta, -\gamma$ **1**

(ii) $\alpha, -\alpha, \beta, -\beta, \gamma, -\gamma$ **2**

(iii) $\alpha^2, \beta^2, \gamma^2$ **3**

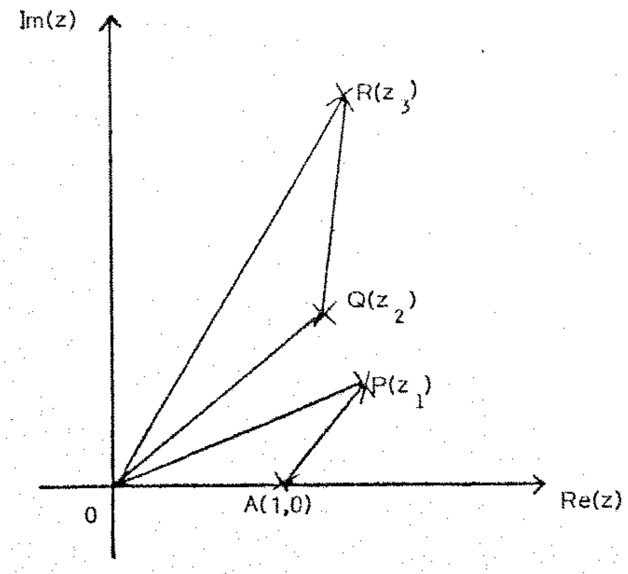
b) (i) Prove that 1 and -1 are both roots of multiplicity 2 of the polynomial $P(x) = x^6 - 3x^2 + 2$ **2**

(ii) Express $P(x)$ as the product of irreducible factors over the field of

(α) rational numbers **1**

(β) complex numbers **1**

c)



In the Argand diagram above, Δ OQR is constructed similar to Δ OAP.

Show that

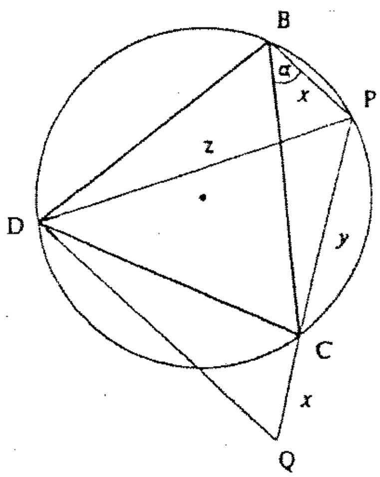
(i) $|z_3| = |z_1|\ |z_2|$ **2**

(ii) $\arg\ z_3 = \arg z_1 + \arg z_2$ **2**

(iii) What is the significance of these results? **1**

Question 7

The figure shows two towns located at B and C. BCD is an equilateral triangle. A road junction is to be placed at P, somewhere on the minor arc BC of the circumscribed circle of the triangle BCD.



Let BP, CP and DP have lengths x, y, z respectively.
The point Q is on the line PC, extended so that BP and CQ have the same length x. Let $< PBC = \alpha$.

Question 7 (cont)

a) (i) Show that <BPD = < CPD =60° **2**

(ii) Find <DCQ in terms of α **1**

(iii) Prove Δ PBD $\equiv \Delta$ QCD. **2**

(iv) Prove ΔDPQ is equilateral **2**

(v) Now show that $z = x + y$ **1**

b) Owing to the tides, the depth of water in an estuary may be assumed to rise and fall with time in simple harmonic motion.
At a certain place there is a danger of flooding when the depth of the water is above 1.25m. One day high tide was 1.5m at 1am and the following low tide was 0.5m at 7:30am.

(i) Find the amplitude in metres and period in minutes of this tidal motion. **2**

(ii) Hence find between what times after 1am was there no danger of flooding. **3**

c) Find $\int \frac{1-x}{1-\sqrt{x}}\,dx$ **2**

Question 8

a) (i) Find the 1st and 2nd derivatives of $P(x) = \frac{x^4}{4} + \frac{x^3}{3} + \frac{x^2}{2} + x + c$ **1**

(ii) Hence or otherwise show that $P(x) = 0$ has not real roots if $c > \frac{7}{12}$ **3**

b) (i) Write down in mod-arg form, the five roots of $z^5 - 1 = 0$ **3**

(ii) By combining appropriate pairs of these roots, show that for $z \neq 1$, **4**

$$\frac{z^5 - 1}{z - 1} = (z^2 - 2z\cos\frac{2\pi}{5} + 1)\,(z^2 - 2z\cos\frac{4\pi}{5} + 1)$$

(iii) Deduce that $\cos\frac{2\pi}{5}$ and $\cos\frac{4\pi}{5}$ are the roots of the equation **4**

$$4x^2 + 2x - 1 = 0$$

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$