Student Number:

Student Name:_____

Mathematics Teacher:_

St Mary's Cathedral College

2004

TRIAL HSC EXAMINATION

Morning Session
Monday 9 August 2004

Mathematics Extension 2

General Instructions

- Reading time - 5 minutes
- Working time - 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks - 120

- Attempt Questions 1 – 8
- All questions are of equal value
- Answer each question in a new booklet

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

Question 1 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Find $\int \frac{1}{\sqrt{7-6x-x^2}}\,dx$. **2**

(b) Evaluate $\int_0^{\frac{\pi}{6}} \sec 4x \tan 4x\,dx$. **2**

(c) (i) Find the real numbers A, B and C such that **2**

$$\frac{8}{(x+2)(x^2+4)} \equiv \frac{A}{x+2} + \frac{Bx+C}{x^2+4}.$$

(ii) Hence find $\int \frac{8}{(x+2)(x^2+4)}\,dx.$ **2**

(d) Use the substitution $t = \tan\frac{x}{2}$ to evaluate $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{1-\cos x}\,dx.$ **4**

(e) Evaluate $\int_0^{\frac{\pi}{6}} x\cos x\,dx$ **3**

Question 2 (15 marks) Use a separate writing booklet. **Marks**

(a) Given the complex number $z = 1 - i\sqrt{3}$, find

(i) $|z| + \dfrac{1}{|z|}$ **1**

(ii) $\arg \bar{z}$ **2**

(b) Write $\left(1+\sqrt{3}i\right)^{-1}$ in modulus/argument form. **2**

(c) Sketch the locus of the following. Draw separate diagrams.

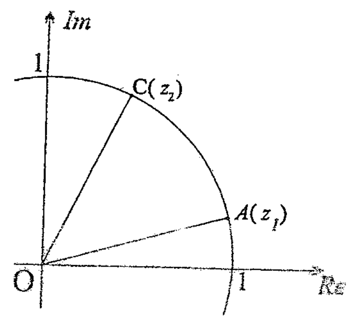
(i) $\arg(z-1-2i) = \dfrac{\pi}{4}$ **1**

(ii) $\arg\left(\dfrac{z-2}{z+2i}\right) = \dfrac{\pi}{2}$ **3**

(d) (i) Find in modulus-argument form all solutions of the equation $z^6 = 1$. Plot and clearly label the roots on an Argand diagram. **3**

(ii) Factorise $z^6 - 1$ completely into linear factors. **1**

(e)



In the Argand diagram above, the two points A and C lie on the circumference of the circle with centre the origin and radius 1. They represent the complex numbers z_1 and z_2 respectively.

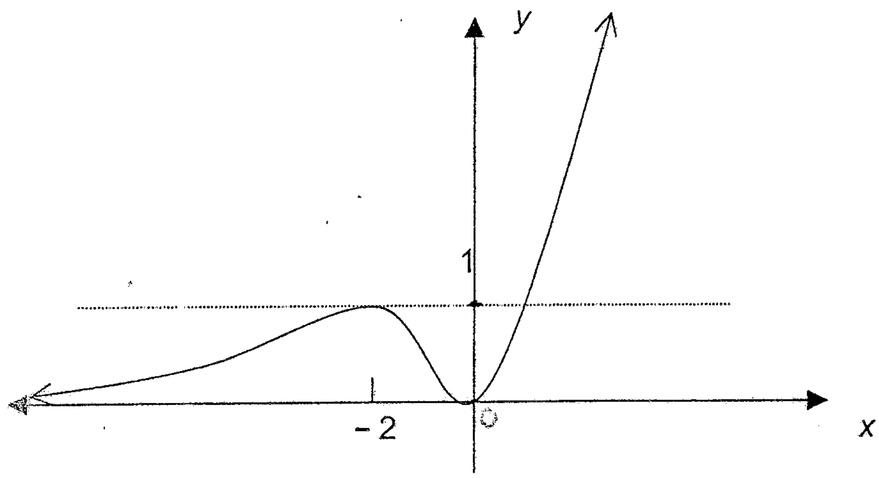
(i) Copy the diagram into your answer booklet and mark on your diagram the position of the point B that represents the complex number $z_1 + z_2$. **1**

(ii) Explain why AC is perpendicular to OB. **1**

Question 3 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) If ω is one of the complex cube roots of unity, write down the other complex cube root. **1**

(b) The diagram shows the graph of $y = f(x)$.



Draw separate sketches of the following. Each diagram should take about 10 lines.

(i) $y = \dfrac{1}{f(x)}$ **2**

(ii) $y = \sqrt{f(x)}$ **2**

(iii) $y = e^{f(x)}$ **2**

(c) Consider the ellipse E with equation $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$.

(i) Find the points of intersection of E with the x and y axes, and the eccentricity and the foci of E. **3**

(ii) Write down the equations of the directrices of E. **1**

(iii) Sketch E. **1**

(d) Prove by induction that $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$ for all integers $n \geq 1$. **3**

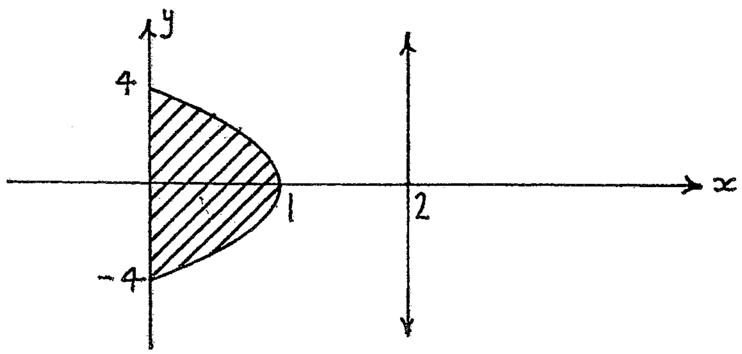
Question 4 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) The area between the curve $y = \sin x$ and the line $y = 1$, from $x = 0$ to $x = \dfrac{\pi}{2}$, is rotated around the line $y = 1$. **4**

Use a slicing technique to find the volume of the resulting solid of revolution.

(b) Use the method of cylindrical shells to find the volume of the solid formed when the region bounded by the curve $y = \log_e x$, the x-axis and the line $x = 4$ is rotated about the y-axis. **4**

(c)



A solid S is formed by rotating the region bounded by the parabola $y^2 = 16(1 - x)$ and the y-axis through $360°$ about the line $x = 2$.

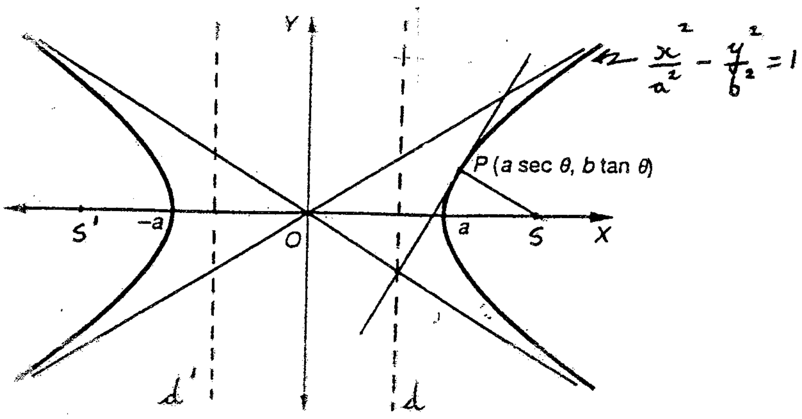
(i) By slicing perpendicular to the axis of rotation, find the exact volume of S. **4**

(ii) The method of cylindrical shells can be used to show that the volume of S is also given by $\int_0^1 16\pi(2 - x)\sqrt{1 - x}\,dx$. **3**

Confirm your answer to part (i) by calculating this definite integral using the substitution $u = 1 - x$.

Question 5 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a)



The point $P(a\sec\theta, b\tan\theta)$ lies on the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$.
The two foci of the hyperbola are S and S', and the two directrices are d and d'. The two asymptotes are also shown.

(i) Show that the equation of the tangent to the hyperbola at the point P is **3**

$$\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1.$$

(ii) The line joining P to the focus S is parallel to an asymptote. Find the gradient of PS and deduce that $\sec\theta+\tan\theta=e$. **2**

(iii) Hence prove that the tangent at P meets the asymptote on the directrix. **2**

(b) (i) By applying de Moivre's theorem and by also expanding $(\cos\theta+i\sin\theta)^4$, show that $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$. **3**

(ii) By letting $\cos 4\theta = \dfrac{1}{2}$, solve the equation $16x^4-16x^2+1=0$. **3**

(iii) Hence show that $\cos^2\dfrac{\pi}{12}\cos\dfrac{5\pi}{12}\cos\dfrac{7\pi}{12}=-\dfrac{1}{16}$. **2**

Question 6 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) An object of mass 20kg is dropped in a medium where the resistance at speed v m/s has a magnitude of $2v$ newtons. The acceleration due to gravity is 10m/s^2.

(i) Draw a diagram to show the forces on the object and show that the equation of motion is $\ddot{x} = \dfrac{100 - v}{10}$. **1**

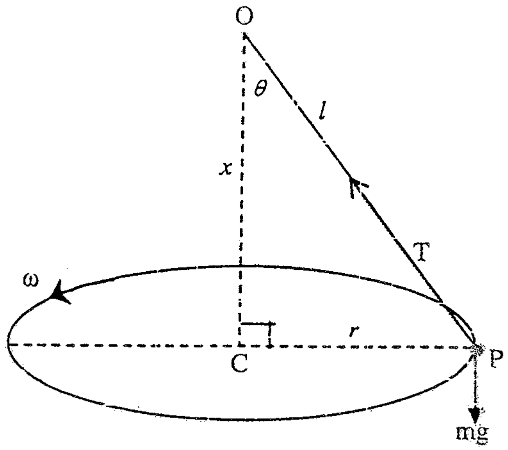
(ii) Show that the velocity, at time t seconds after the object is dropped, is given by $v = 100\left(1 - e^{-\frac{t}{10}}\right)$. **3**

(iii) Find the terminal velocity of the object. **1**

(iv) Show that the distance x metres travelled when the speed is v m/s is given by $x = 1000\log_e\left(\dfrac{100}{100 - v}\right) - 10v$. **3**

(v) Hence find the distance the object has fallen before reaching half its terminal velocity. **1**

(b)



A conical pendulum consists of a bob P of mass m kg on the end of a string l metres long. The bob rotates in a horizontal circle of radius r and centre O at a constant angular velocity ω radians per second. The angle that the string makes with the vertical is θ and $CO = x$ metres. The forces acting on the bob are a gravitational force of mg newtons and a tension in the string of T newtons.

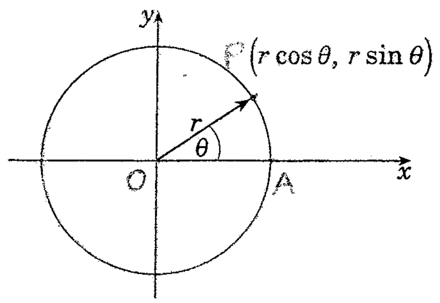
(i) By resolving forces, show that the period of this motion is $2\pi\sqrt{\dfrac{x}{g}}$. **3**

(ii) What is the effect on the motion if the mass is doubled? **1**

(iii) If the number of revolutions per second increases from 2 to 3, find the change in x. (Take $g = 10$ m/s^2) **2**

Question 7 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a)



A particle P moves with constant angular velocity ω on a circle of radius r and centre O. OA is a fixed arbitrary line about O.
At time t, angle $AOP = \theta$.

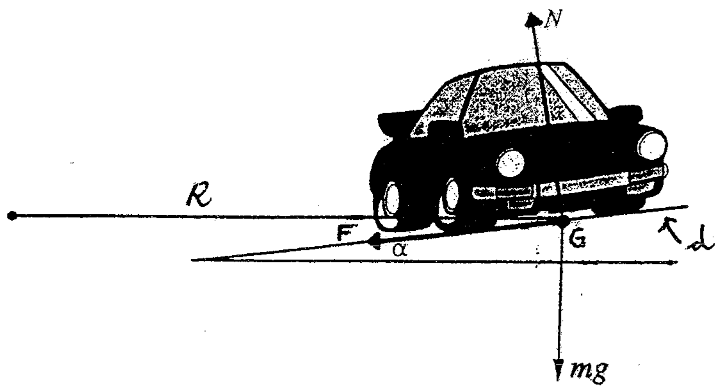
(i) Let s be the length of the arc AP. **1**
Prove that the instantaneous velocity v of the particle is $r\omega$.

(ii) The position of the particle at time t is given by: **4**

$$x = r\cos\theta$$
$$y = r\sin\theta.$$

Derive expressions for the tangential and normal accelerations of the particle at any time t.

(b)



A car of mass m, represented by the point G, is travelling at constant speed v around a curve of radius R on a road of width d. The road slopes at an angle α towards the centre of the curve. The forces acting on the car are the gravitational force mg, a sideways friction force F (acting down the road as drawn) and a normal reaction N, at right angles to the road.

(i) By resolving the vertical and horizontal components of force, and by taking into account the sideways friction force, find expressions for $F\sin\alpha$ and $F\cos\alpha$. **3**

(ii) Suppose that the radius of the bend is 75 m and that the road is banked so that there is no tendency for the car to slip sideways if it travels at a maximum speed of v=60 km/h. Assume that the value of g is $9.8\ \text{m s}^{-2}$. **3**
Find the value of angle α, giving full reasons for your answer.

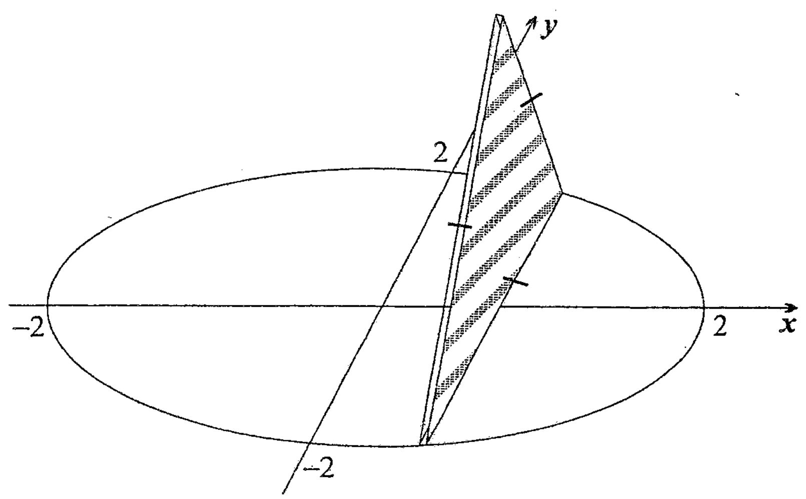
(iii) A second car, of mass M, travels around the same curve but with a greater speed (V m/s). **4**
Show that the sideways friction force (F), exerted by the surface of the road on the wheels of this car, is

$$F = \frac{Mg\left(V^2 - v^2\right)}{\sqrt{v^4 + R^2 g^2}}.$$

Question 8 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) The roots of $x^3 - 5x^2 + 5 = 0$, are α, β, and γ. **2**
Find the polynomial equation whose roots are $\alpha - 1$, $\beta - 1$, and $\gamma - 1$.

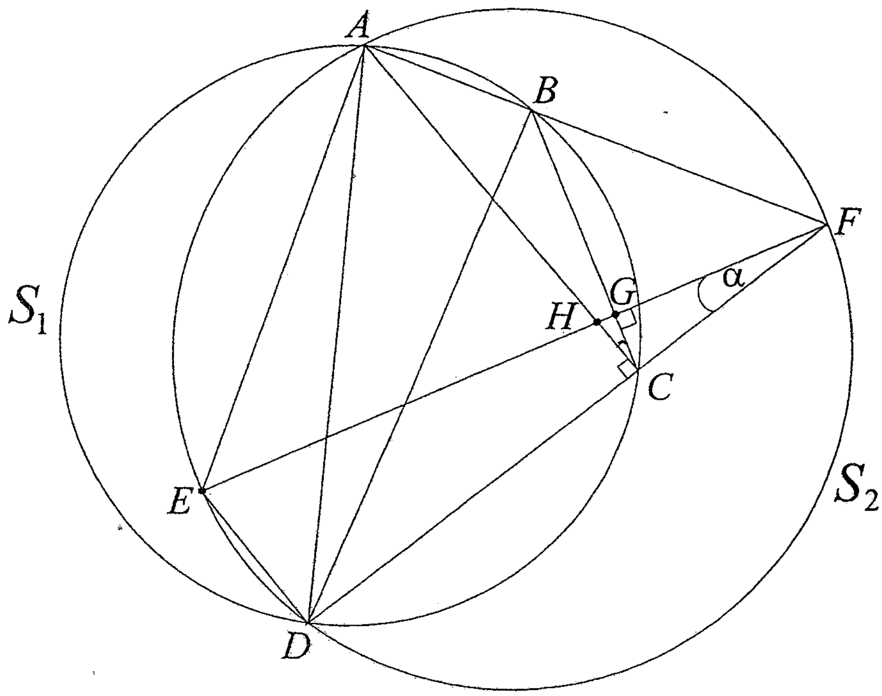
(b)



The diagram above shows a cross-sectional slice of a solid whose base is the region enclosed by the circle $x^2 + y^2 = 4$. Each such cross-section of the solid is an equilateral triangle. Find the volume of the solid. **4**

(c) Use the method of Mathematical Induction to prove that $(1+x)^n - nx - 1$ is divisible by x^2 for all positive integers $n \geq 2$. **4**

(d)



In the diagram above, $ABCD$ is a cyclic quadrilateral inscribed in the circle S_1 , and $AC \perp DC$.

The chords AB and DC produced intersect at F, and S_2 is the circle through A, D and F.

The line through F perpendicular to BC meets BC at G, meets AC at H and meets the circle S_2 at E.

Let $\angle DFE = \alpha$.

(i) Prove that $\angle HCG = \alpha$. **1**

(ii) Prove that $AB \perp DB$. **1**

(iii) Prove that $AE \parallel BD$. **2**

(iv) Prove that E, A, B and G are concylic. **1**

MATHEMATICS EXTENSION 2

2004 TRIAL HSC EXAMINATION

SOLUTIONS AND MARKING SCHEME

Question 1 — Guidelines for marking

(a)
$$\int \frac{1}{\sqrt{7-6x-x^2}}\,dx = \int \frac{1}{\sqrt{7-(x^2+6x)}}\,dx$$
$$= \int \frac{1}{\sqrt{7-(x^2+6x+9)+9}}\,dx$$
$$= \int \frac{1}{\sqrt{16-(x+3)^2}}\,dx$$
• Completes square or correct integral following from incorrect completion of square [illegible]
$$= \sin^{-1}\left(\frac{x+3}{4}\right) + c \; \#$$
• Correct answer (may omit the +c) (2)

(b)
$$\int_0^{\frac{\pi}{6}} \sec 4x \tan 4x\,dx = \left[\tfrac{1}{4}\sec 4x\right]_0^{\frac{\pi}{6}}$$
• Correct primitive (1)
$$= \tfrac{1}{4}\left\{\frac{1}{\cos(4\times\frac{\pi}{6})} - \frac{1}{\cos(4\times 0)}\right\}$$
$$= \tfrac{1}{4}\left\{\frac{1}{\cos\frac{2\pi}{3}} - \frac{1}{\cos 0}\right\}$$
$$= \tfrac{1}{4}\left\{\frac{1}{-\cos\frac{\pi}{3}} - 1\right\}$$
$$= \tfrac{1}{4}\left\{\frac{1}{-\frac{1}{2}} - 1\right\}$$
$$= \tfrac{1}{4}\left\{-2-1\right\}$$
$$= -\tfrac{3}{4} \; \#$$
• Correct answer (2)

(c)
$$\frac{8}{(x+2)(x^2+4)} \equiv \frac{A}{x+2} + \frac{Bx+C}{x^2+4}$$
$$\therefore\; 8 = A(x^2+4) + (Bx+C)(x+2).$$
• Obtains $A(x^2+4)+(Bx+C)(x+2)$ and either attempts to equate coeffs. or substitutes appropriate values for x, or equivalent progress (1)

When $x=-2$, $8 = 8A$
$A = 1.$

When $x=0$, $8 = 4A + 2C$
$8 = 4(1) + 2C$
$C = 2.$

$8 = A(x^2+4) + Bx^2 + 2Bx + Cx + 2C$
$8 = Ax^2 + 4A + Bx^2 + 2Bx + Cx + 2C$
$8 = (A+B)x^2 + (2B+C)x + 4A + 2C.$

Equating the coefficient of x gives
$2B + C = 0$
$2B + 2 = 0$
$B = -1$

(ii)
$$\int \frac{8}{(x+2)(x^2+4)}\,dx = \int\left(\frac{1}{x+2} + \frac{-x+2}{x^2+4}\right)dx$$
$$= \int\left(\frac{1}{x+2} + \frac{2-x}{x^2+4}\right)dx$$
$$= \int\frac{1}{x+2}\,dx + \int\frac{2}{x^2+4}\,dx - \int\frac{x}{x^2+4}\,dx$$
$$= \int\frac{1}{x+2}\,dx + 2\int\frac{1}{x^2+4}\,dx - \frac{1}{2}\int\frac{2x}{x^2+4}\,dx$$
• Correct treatment of at least one of $\int\frac{A}{x+2}\,dx$, $\int\frac{Bx}{x^2+4}\,dx$ or $\int\frac{C}{x^2+4}\,dx$ [illegible]
$$= \ln(x+2) + 2\left[\tfrac{1}{2}\tan^{-1}\left(\tfrac{x}{2}\right)\right] - \tfrac{1}{2}\ln(x^2+4) + c$$
$$= \ln(x+2) + \tan^{-1}\left(\tfrac{x}{2}\right) - \tfrac{1}{2}\ln(x^2+4) + c$$
$$= \ln\left(\frac{x+2}{\sqrt{x^2+4}}\right) + \tan^{-1}\left(\tfrac{x}{2}\right) + c \; \#$$
• Correct solution (2)

(d)
$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{1-\cos x}\,dx.$$

Let $t = \tan\left(\frac{x}{2}\right).$

Then $x = 2\tan^{-1}(t).$

So $\frac{dx}{dt} = \frac{2}{1+t^2}\times 1$

$dx = \frac{2}{1+t^2}\,dt$

When $x = \frac{\pi}{3}$, $t = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}.$

When $x = \frac{\pi}{2}$, $t = \tan\frac{\pi}{4} = 1.$

• Obtains expressions $dx = \frac{2}{1[illegible]}$ [illegible], $\cos x = \frac{1-t^2}{1+t^2}$. (1)

(AND)

• Changes limits correctly [illegible]

$$\text{So } \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{1-\cos x}\,dx = \int_{\frac{1}{\sqrt{3}}}^{1} \frac{1}{1-\frac{1-t^2}{1+t^2}} \times \frac{2}{1+t^2}\,dt$$
$$= \int_{\frac{1}{\sqrt{3}}}^{1} \frac{1}{\frac{1+t^2-1+t^2}{1+t^2}} \times \frac{2}{1+t^2}\,dt$$
$$= \int_{\frac{1}{\sqrt{3}}}^{1} \frac{2}{2t^2}\,dt$$
$$= \int_{\frac{1}{\sqrt{3}}}^{1} t^{-2}\,dt$$
$$= \left[\frac{t^{-1}}{-1}\right]_{\frac{1}{\sqrt{3}}}^{1}$$
• Correct primitive (3)
$$= -\left[\frac{1}{t}\right]_{\frac{1}{\sqrt{3}}}^{1}$$
$$= -\left\{1 - \sqrt{3}\right\}$$
$$= \sqrt{3} - 1 \; \#$$
• Correct answer (4)

(e) $\int_0^{\frac{\pi}{6}} x\cos x\,dx = \int_0^{\frac{\pi}{6}} x.\frac{d}{dx}(\sin x)\,dx$

$= [x\sin x]_0^{\frac{\pi}{6}} - \int_0^{\frac{\pi}{6}} \sin x.1.dx$

$= \{(\frac{\pi}{6}\times\sin\frac{\pi}{6}) - 0\} - [-\cos x]_0^{\frac{\pi}{6}}$

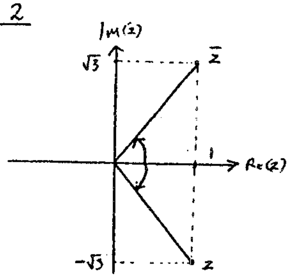
$= \{\frac{\pi}{6}\times\frac{1}{2}\} + \{\cos\frac{\pi}{6} - \cos 0\}$

$= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$ #

- Reasonable attempt to use the method of integration by parts (1)
- Obtains $[x\sin x]_0^{\frac{\pi}{6}} - \int_0^{\frac{\pi}{6}} \sin x.1\,dx$, or a single minor error or equivalent (2)
- Correct solution (3).

Question 2

(a)



(i) $|z| = \sqrt{1+(-\sqrt{3})^2}$

$= 2$

$\therefore |z| + \frac{1}{|z|} = 2\frac{1}{2}$ # ← • Correct answer (1).

(ii) $\bar{z} = 1 + i\sqrt{3}$ ← • Correct conjugate or correct argument following from incorrect conjugate (1)

$\arg(\bar{z}) = \tan^{-1}(\frac{\sqrt{3}}{1})$

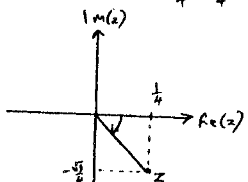
$= \frac{\pi}{3}$ # ← • Correct answer (2)

(b) $(1+\sqrt{3}i)^{-1} = \frac{1}{1+\sqrt{3}i}$

$= \frac{1}{1+\sqrt{3}i} \times \frac{1-\sqrt{3}i}{1-\sqrt{3}i}$

$= \frac{1-\sqrt{3}i}{4}$

$= \frac{1}{4} - \frac{\sqrt{3}}{4}i$



$|z| = \sqrt{(\frac{1}{4})^2 + (-\frac{\sqrt{3}}{4})^2}$

$= \sqrt{\frac{4}{16}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$

$\arg z = -\tan^{-1}(\frac{\frac{\sqrt{3}}{4}}{\frac{1}{4}})$ • Correct modulus or argument (1)

$= -\tan^{-1}(\frac{1}{\sqrt{3}})$

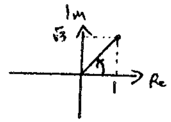
$= -\frac{\pi}{3}$.

$z = r\text{cis}\,\theta = \frac{1}{2}\text{cis}(-\frac{\pi}{3}) = \frac{1}{2}[\cos(-\frac{\pi}{3}) + i\sin(-\frac{\pi}{3})]$

$= \frac{1}{2}(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3})$ # ← • Correct answer or equivalent (2)

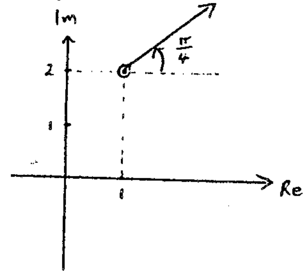
(OR) $(1+\sqrt{3}i)^{-1} = [2\cos(\frac{\pi}{3})]^{-1}$

$= \frac{1}{2}\text{cis}(-\frac{\pi}{3})$ #



(c) (i) $\arg(z-1-2i) = \frac{\pi}{4}$.

$\therefore \arg[z-(1+2i)] = \frac{\pi}{4}$



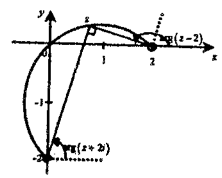
• Correct answer (1)

(ii) $\arg\left(\frac{z-2}{z+2i}\right) = \frac{\pi}{2}$.

$\therefore \arg(z-2) - \arg(z+2i) = \frac{\pi}{2}$.

$\therefore \arg[z-(2+0i)] - \arg[z-(0-2i)] = \frac{\pi}{2}$

$\therefore \arg[z-(2+0i)] = \frac{\pi}{2} + \arg[z-(0-2i)]$.



- Reasonable attempt to show that th… angle between the rays drawn from (2, … and (−2, 0) is $\frac{\pi}{2}$.
- Correct except for not showing semi-circle (2) (OR) not showing that the semi-circle passed through O (2)
- Correct answer (3)

(d) (i) $z^6 = 1$.

$\therefore |z^6| = |1|$

$|z|^6 = 1$

$|z| = 1^{\frac{1}{6}}$

$|z| = 1$.

ie. $r = 1$.

∴ The roots are always 1 unit away from the origin. Hence they lie around the unit circle with centre O.

Let $z = \text{cis}\,\theta$, since $r = 1$.

Then $(\text{cis}\,\theta)^6 = 1$

So $\cos 6\theta + i\sin 6\theta = 1 + 0i$, using de Moivre. • Partial progress (1)

So $\cos 6\theta = 1$, equating real parts,

and $\sin 6\theta = 0$, equating imaginary parts.

$\therefore\ 6\theta = 0, 2\pi, 4\pi, 6\pi, 8\pi, 10\pi, \ldots$

So $\theta = 0, \frac{2\pi}{6}, \frac{4\pi}{6}, \frac{6\pi}{6}, \frac{8\pi}{6}, \frac{10\pi}{6}, \ldots$

But $z^6 = 1$ has 6 roots.

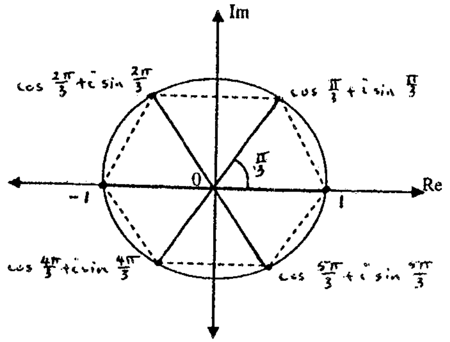
$\therefore\ \theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}$ (other values of θ give repeated roots).

So the roots are, in mod/arg form,

$z_1 = \text{cis}\,0$,
$z_2 = \text{cis}\,\frac{\pi}{3}$,
$z_3 = \text{cis}\,\frac{2\pi}{3}$,
$z_4 = \text{cis}\,\pi$,
$z_5 = \text{cis}\,\frac{4\pi}{3}$,
$z_6 = \text{cis}\,\frac{5\pi}{3}$

So the roots are equally spaced around the unit circle at intervals of $\frac{\pi}{3}$.

• Correct roots (2)

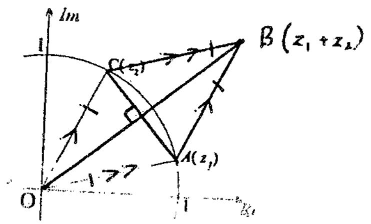


• Correct solution (3)

{ So the roots are ± 1, $\text{cis} \pm\frac{\pi}{3}$, $\text{cis} \pm \frac{2\pi}{3}$ }.

(ii) From (i), $z^6 - 1 = (z-1)(z - \text{cis}\frac{\pi}{3})(z - \text{cis}\frac{2\pi}{3})(z+1)(z - \text{cis}\frac{4\pi}{3})(z - \text{cis}\frac{5\pi}{3})$, over the complex field. • Correct answer (or equivalent) (1)

(e)(i)



• Correct answer (1)

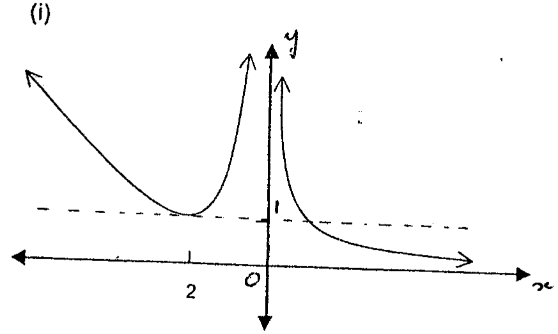
(ii) OABC is a rhombus and hence the diagonals are perpendicular. • Correct answer (1)

Question 3

(a) ω^2 • Correct answer (1)

(b)

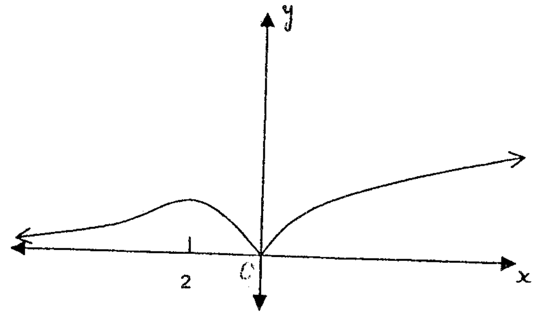
(i)



• Substantially correct, but missing one feature (1)

• Correct graph (2)

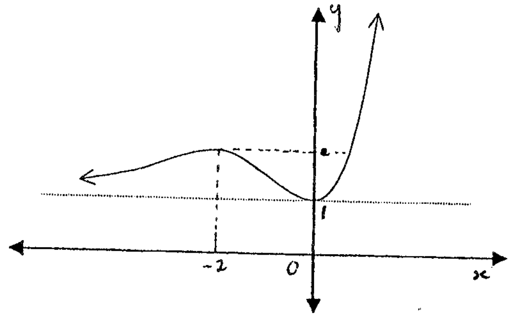
(ii)



• Substantially correct, but missing one feature (1)

• Correct graph (2)

(iii)



• Substantially correct, but missing one feature (1)

• Correct graph (2)

(c) (i) $\frac{x^2}{25} + \frac{y^2}{16} = 1$.

$a = 5,\ b = 4.$

So ellipse intersects x-axis at $(\pm 5, 0)$

and y-axis at $(0, \pm 4)$ #

$b^2 = a^2(1 - e^2)$.

$\therefore\ 16 = 25(1 - e^2)$

$e = \sqrt{1 - \frac{16}{25}}$

$= \sqrt{\frac{9}{25}}$

$\therefore\ e = \frac{3}{5}$ #

So foci are $(\pm 5 \times \frac{3}{5}, 0)$,

i.e. $(\pm 3, 0)$ #

• One mark for each correct answer (3)

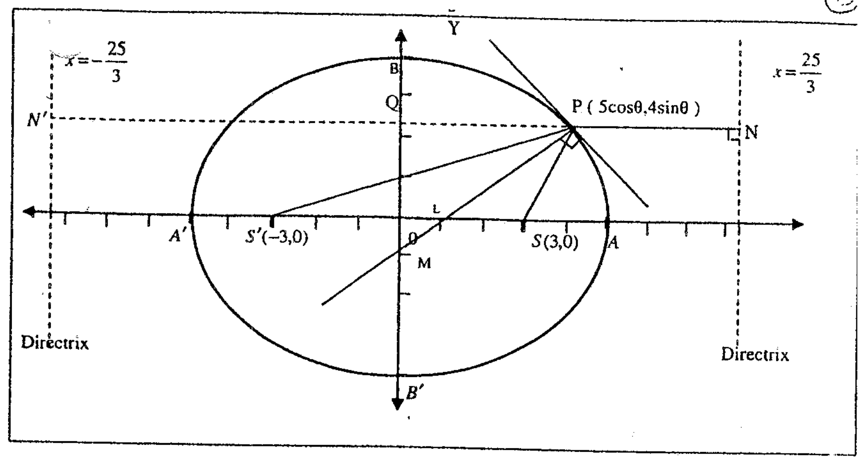
(ii) $x = \pm \frac{a}{e}$

$= \pm \frac{5}{3/5}$

$\therefore\ x = \pm \frac{25}{3}$ #

• Correct answer (1)

(iii)



• Correct graph (1)

(d) When $n = 1$,

$L.H.S. = (\cos\theta + i\sin\theta)^1$

$= \cos\theta + i\sin\theta.$

$R.H.S. = \cos[(1)\theta] + i\sin[(1)\theta]$

$= \cos\theta + i\sin\theta$

$= L.H.S.$

So the statement is true for $n = 1$.

• Includes test for $n = 1$ (1)

Suppose that the statement is true for a positive integer k.

That is, suppose $(\cos\theta + i\sin\theta)^k = \cos(k\theta) + i\sin(k\theta)$. (**)

We prove the statement for $n = k+1$.

That is, we prove $(\cos\theta + i\sin\theta)^{k+1} = \cos[(k+1)\theta] + i\sin[(k+1)\theta]$.

$L.H.S. = (\cos\theta + i\sin\theta)^{k+1}$ (AND)

$= (\cos\theta + i\sin\theta)^k \times (\cos\theta + i\sin\theta)$ • Uses induction assumption (2)

$= [\cos(k\theta) + i\sin(k\theta)](\cos\theta + i\sin\theta)$, by the induction hypothesis,

$= \cos(k\theta)\cos\theta + \cos(k\theta)i\sin\theta + i\sin(k\theta)\cos\theta + i^2\sin(k\theta)\sin\theta$,

$= \cos(k\theta)\cos\theta - \sin(k\theta)\sin\theta + i[\cos(k\theta)\sin\theta + \sin(k\theta)\cos\theta]$

$= \cos[(k\theta)+\theta] + i\sin[(k\theta)+\theta]$,

$= \cos[(k+1)\theta] + i\sin[(k+1)\theta]$,

= R.H.S.

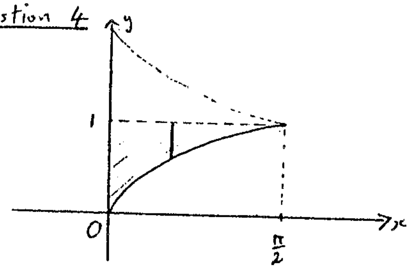
• Applies correct trigonometric identity (3)

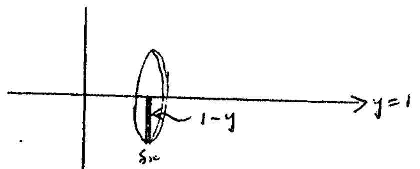
Hence, if the statement is true for the integer k, then it is also true for the next integer $k+1$.

But we know that the statement is true for $n=1$. So it must be true for $n=2$, $n=3$, and so on for all positive integral values of n.

Question 4

(a)





• Attempts to use slicing technique (1)

Area of each circular slice is $\pi(1-y)^2$.

Hence the volume of a circular slice δx thick $= \delta V = \pi(1-y)^2\delta x$
$= \pi(1-\sin x)^2\delta x$.

So the required volume is $V = \lim_{\delta x \to 0} \sum_{x=0}^{\frac{\pi}{2}} \pi(1-\sin x)^2 \delta x$

$= \int_0^{\frac{\pi}{2}} \pi(1-\sin x)^2\,dx$

• Correct definite integral (2)

(OR)

$= \pi\int_0^{\frac{\pi}{2}} (1-2\sin x+\sin^2 x)\,dx$

• Uses correct method but with substantia rrors (2)

$= \pi\int_0^{\frac{\pi}{2}} \{1-2\sin x+\tfrac{1}{2}(1-\cos 2x)\}\,dx$

$= \pi\int_0^{\frac{\pi}{2}} (\tfrac{3}{2}-2\sin x-\tfrac{1}{2}\cos 2x)\,dx$

$= \pi\left[\tfrac{3}{2}x+2\cos x-\tfrac{1}{4}\sin 2x\right]_0^{\frac{\pi}{2}}$

• Correct method but with minor errors (3)

$= \pi\left\{\left[(\tfrac{3}{2}\times\tfrac{\pi}{2})+2\cos\tfrac{\pi}{2}-\tfrac{1}{4}\sin(2\times\tfrac{\pi}{2})\right]-\left[0+2\cos 0-0\right]\right\}$

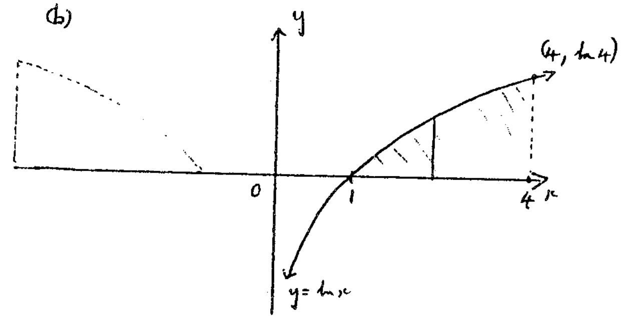
$= \pi\left\{\tfrac{3\pi}{4}+0-0-2\right\}$

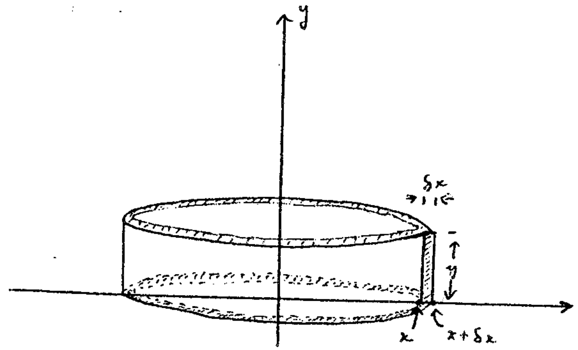
$= \pi\left\{\tfrac{3\pi}{4}-2\right\}$.

So $V = \dfrac{(3\pi-8)\pi}{4}$ units^3 #

• Correct answer, using the slicing method (4)

(b)





• Attempts to use shell method(

Area of base of each cylindrical shell is $\pi(x+\delta x)^2-\pi x^2$.

Hence the volume of a shell δx thick $= \delta V = [\pi(x+\delta x)^2-\pi x^2]y$

$= \pi[x^2+2x\delta x+(\delta x)^2-x^2]y$

$= \pi[2\pi\delta x]y$, ignoring second order magnitudes,

$= 2\pi x y\,\delta x$.

So the volume required is,

$$V = \lim_{\delta x \to 0} \sum_{x=1}^{4} 2\pi x y \, \delta x$$

$$= \int_1^4 2\pi x y \, dx$$

$$= 2\pi \int_1^4 x \log_e x \, dx$$

$$= 2\pi \int_1^4 \ln x \frac{d}{dx}\left(\frac{x^2}{2}\right) dx$$

$$= 2\pi \left\{ \left[\ln x \times \left(\frac{x^2}{2}\right)\right]_1^4 - \int_1^4 \frac{x^2}{2} \times \frac{1}{x} \, dx \right\}$$

$$= 2\pi \left\{ \left[\ln 4 \times \left(\frac{4^2}{2}\right)\right] - \int_1^4 \frac{x}{2} \, dx \right\}$$

$$= 2\pi \left\{ 8 \ln 4 - \frac{1}{2}\left[\frac{x^2}{2}\right]_1^4 \right\}$$

$$= 2\pi \left\{ 8 \ln 4 - \frac{1}{4}(16 - 1) \right\}$$

$$= 2\pi \left\{ 8 \ln 4 - \frac{15}{4} \right\}$$

$$= \left(16 \ln 4 - \frac{15}{2}\right)\pi \text{ units}^3 \ \#$$

- Correct definite integral (2)

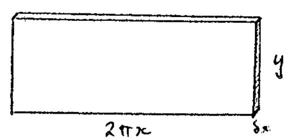
(OR)

- Uses correct method but with substantial errors (2)

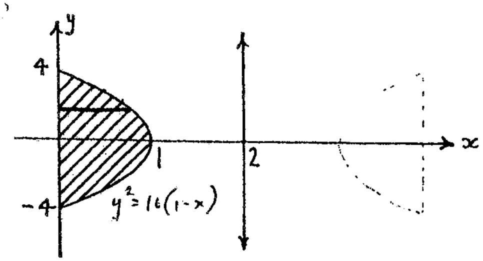
- Correct method but with minor errors (3)

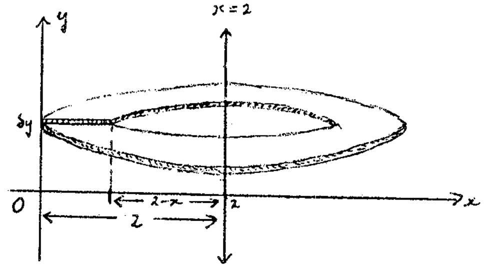
- Correct answer, using the shell method (4)

(OR) stretch the shell into a rectangular sheet



The curved surface of each cylindrical shell is given by $SA = 2\pi x y = 2\pi x \ln x$.
Hence the volume of a shell δx thick is $\delta V = 2\pi x \ln x \, \delta x$
So the required volume is $V = \lim_{\delta x \to 0} \sum_{x=1}^{4} 2\pi x \ln x \, \delta x$.
etc.





Area of each annular slice is $\pi(2)^2 - \pi(2-x)^2$

$$= \pi[4 - (4 - 4x + x^2)]$$

$$= \pi(4x - x^2).$$

Hence the volume of an annular slice δy thick is,

$$\delta V = \pi(4x - x^2)\delta y.$$

So the volume required is,

$$V = \lim_{\delta y \to 0} \sum_{y=-4}^{4} \pi(4x - x^2)\delta y$$

$$= 2\pi \int_0^4 (4x - x^2) \, dy$$

$$= 2\pi \int_0^4 \left[4\left(1 - \frac{y^2}{16}\right) - \left(1 - \frac{y^2}{16}\right)^2\right] dy$$

$$= 2\pi \int_0^4 \left[4 - \frac{y^2}{4} - 1 + \frac{y^2}{8} - \frac{y^4}{256}\right] dy$$

$$= 2\pi \int_0^4 \left(3 - \frac{y^2}{8} - \frac{y^4}{256}\right) dy$$

- Obtains correct expression for the area of an annulus in terms of x (1)

(OR)

- Attempts to use slicing technique (1)

(AND)

- Attempts to use the fact that $x = 1 - \frac{y^2}{16}$ (2)

(OR)

- Uses correct method but with substantial errors (2)

- Correct definite integral (3)

(OR)

- Correct method but with minor errors (3)

$$= 2\pi\left[3y - \frac{y^3}{24} - \frac{y^5}{1280}\right]_0^4$$

$$= 2\pi\left\{\left(12 - \frac{64}{24} - \frac{1024}{1280}\right) - 0\right\}$$

$$= \frac{256\pi}{15} \text{ units}^3 \ \#$$

• Correct answer, using the slicing method (4)

(ii) $\int_0^1 16\pi(2-x)\sqrt{1-x}\,dx$.

Let $u = 1-x$.

Then $\frac{du}{dx} = -1$.

So $du = -dx$.

When $x=0$, $u = 1-0 = 1$.

When $x=1$, $u = 1-1 = 0$.

• Correctly substitutes $u = 1-x$, $x = 1-u$ and $du = -dx$ (1)

OR

• Correctly changes limits of integration (1)

So $\int_0^1 16\pi(2-x)\sqrt{1-x}\,dx = \int_1^0 16\pi[2-(1-u)]\sqrt{u}\,.-du$

$$= -16\pi\int_1^0 (1+u)\sqrt{u}\,du$$

• Does both the above (2)

OR

$$= -16\pi\int_1^0 \left(u^{\frac{1}{2}} + u^{\frac{3}{2}}\right)du$$

• Correctly evaluates the integral apart from failure to change the limits (2)

$$= -16\pi\left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}} + \frac{u^{\frac{5}{2}}}{\frac{5}{2}}\right]_1^0$$

$$= -16\pi\left[\tfrac{2}{3}u^{\frac{3}{2}} + \tfrac{2}{5}u^{\frac{5}{2}}\right]_1^0$$

$$= -16\pi\left\{0 - \left(\tfrac{2}{3} + \tfrac{2}{5}\right)\right\}$$

$$= -16\pi\left\{-\tfrac{16}{15}\right\}$$

$$= \frac{256\pi}{15} \ \#$$

• Correctly evaluates integral showing relevant working (3)

Question 5

(a) (i) $x = a\sec\theta$. $\quad y = b\tan\theta$

$\therefore \frac{dx}{d\theta} = a\sec\theta\tan\theta$. $\quad \therefore \frac{dy}{d\theta} = b\sec^2\theta$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$

$$= b\sec^2\theta \times \frac{1}{a\sec\theta\tan\theta}$$

$$= \frac{b\sec\theta}{a\tan\theta}. \ *$$

• obtains $\frac{dy}{dx} = \frac{b^2}{a^2}\left(\frac{x}{y}\right)$ or equivalent

Equation of tangent is,

$$y - b\tan\theta = \frac{b\sec\theta}{a\tan\theta}(x - a\sec\theta)$$

$$ay\tan\theta - ab\tan^2\theta = bx\sec\theta - ab\sec^2\theta$$

$$bx\sec\theta - ay\tan\theta = ab(\sec^2\theta - \tan^2\theta)$$

$$= ab(1)$$

$$= ab$$

$$\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1 \ \# \quad \text{(dividing by } ab)$$

• Correct solution but with minor errors (2)

• Correct solution (3)

* OR using implicit differentiation,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\frac{d}{dx}\left(\frac{x^2}{a^2}\right) - \frac{d}{dx}\left(\frac{y^2}{b^2}\right) = \frac{d}{dx}(1)$$

$$\frac{2x}{a^2} - \frac{d}{dy}\left(\frac{y^2}{b^2}\right) \times \frac{dy}{dx} = 0$$

$$\frac{2x}{a^2} - \frac{2y}{b^2} \times \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{2x}{a^2} \times \left(-\frac{b^2}{2y}\right)$$

$$= \frac{b^2}{a^2}\left(\frac{x}{y}\right)$$

$$= \frac{b^2}{a^2} \times \frac{a\sec\theta}{b\tan\theta}$$

$$= \frac{b\sec\theta}{a\tan\theta}.$$

etc.

(ii) $P(a\sec\theta, b\tan\theta)$ $\quad S(ae, 0)$

$$\text{Gradient of } PS = \frac{b\tan\theta - 0}{a\sec\theta - ae} = \frac{b\tan\theta}{a(\sec\theta - e)}.$$

• obtains gradient of PS (1)

Equation of asymptote is $y = -\frac{b}{a}x$.
So gradient of asymptote is $-\frac{b}{a}$.
Since PS is parallel to the asymptote,

then $$\frac{b\tan\theta}{a(\sec\theta - e)} = -\frac{b}{a}$$

$$ab\tan\theta = -ab(\sec\theta - e)$$

$$\tan\theta = -\sec\theta + e.$$

So $\sec\theta + \tan\theta = e$ #

• Correct solution (2)

(iii) $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$ — ①
$y = -\frac{b}{a}x$ — ② } Meet.

• Attempts to solve eqn. of tangent and eqn. of asymptote simultaneously (1)

(OR)

• Substitutes $x = \frac{a}{e}$ into eqn. of tangent and attempts to solve for y (1)

Substitute ② into ①,

$$\frac{x\sec\theta}{a} - \frac{\left(-\frac{b}{a}x\right)\tan\theta}{b} = 1$$

$$\frac{x\sec\theta}{a} + \frac{x\tan\theta}{a} = 1$$

$$x(\sec\theta + \tan\theta) = a$$

$$x(e) = a$$

• Correct solution (2)

$x = \frac{a}{e}$, which is equation of directrix #

(b) (i) $(\cos\theta + i\sin\theta)^4 = \cos 4\theta + i\sin 4\theta$, by de Moivre.

Also, $(\cos\theta + i\sin\theta)^4 = \cos^4\theta + 4\cos^3\theta(i\sin\theta) + 6\cos^2\theta(i\sin\theta)^2 + 4\cos\theta(i\sin\theta)^3 + (i\sin\theta)^4$,
using Pascal's triangle or the Binomial Thm.,

1
1 1
1 2 1
1 3 3 1
4 6 4 1

$= \cos^4\theta + 4\cos^3\theta\sin\theta . i - 6\cos^2\theta\sin^2\theta - 4\cos\theta\sin^3\theta . i + \sin^4\theta$

$= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta + i(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta)$.

• Use de Moivre and equate real parts (or equivalent progress) (1)

(AND)

Equating real parts,

$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$$
$$= \cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2$$
$$= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta$$
$$= 8\cos^4\theta - 8\cos^2\theta + 1 \text{ #}$$

• Uses $\sin^2\theta = 1-\cos^2\theta$ but do[es] not obtain the correct expressi[on] (2

• Correct answer (3)

(ii) Let $\cos 4\theta = \frac{1}{2}$.
Then $8\cos^4\theta - 8\cos^2\theta + 1 = \frac{1}{2}$.
So $8\cos^4\theta - 8\cos^2\theta + \frac{1}{2} = 0$
and $16\cos^4\theta - 16\cos^2\theta + 1 = 0$.

• Partial progress (1)

Let $x = \cos\theta$.
Then $16x^4 - 16x^2 + 1 = 0$.
So the solutions of the polynomial are the cosines of the first 4 roots of $\cos 4\theta = \frac{1}{2}$.

• Correct solution but with minor errors (2)

$\cos 4\theta = \frac{1}{2}$
$\therefore 4\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}$.
$\therefore \theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}$.

So the roots are $x = \cos\frac{\pi}{12}, \cos\frac{5\pi}{12}, \cos\frac{7\pi}{12}, \cos\frac{11\pi}{12}$ #

• Correct solution (3).

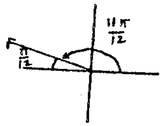
(OR) $16x^4 - 16x^2 + 1 = 0$.
So $8x^4 - 8x^2 + \frac{1}{2} = 0$.
ie. $8x^4 - 8x^2 + 1 = \frac{1}{2}$.
Let $x = \cos\theta$.
Then $8\cos^4\theta - 8\cos^2\theta + 1 = \frac{1}{2}$.
Hence $\cos 4\theta = \frac{1}{2}$.
So $4\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}$.
$\theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}$.
ie. $x = \cos\frac{\pi}{12}, \cos\frac{5\pi}{12}, \cos\frac{7\pi}{12}, \cos\frac{11\pi}{12}$ #

(iii) $\cos\frac{\pi}{12} \times \cos\frac{5\pi}{12} \times \cos\frac{7\pi}{12} \times \cos\frac{11\pi}{12} = \alpha\beta\gamma\delta$

$= \frac{1}{16}$

$\cos\frac{\pi}{12} \times \cos\frac{5\pi}{12} \times \cos\frac{7\pi}{12} \times \left(-\cos\frac{\pi}{12}\right) = \frac{1}{16}$

$\cos^2\frac{\pi}{12} \times \cos\frac{5\pi}{12} \times \cos\frac{7\pi}{12} = -\frac{1}{16}$ #

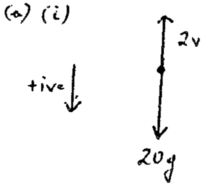


• Uses the product of the roots ($= \frac{e}{a}$) but does not obtain the correct solution (1)

• Correct solution (2)

Question 6

(a) (i)



Force = mass × acceleration (Newton's 2nd law)

So $20g - 2v = 20\ddot{x}$.

Hence $\ddot{x} = \frac{20g - 2v}{20}$

$= \frac{20(10) - 2v}{20}$

$= \frac{200 - 2v}{20}$

$= \frac{100 - v}{10}$ #

• Correct solution (1)

(ii) $\ddot{x} = \frac{100-v}{10}$.

So $\frac{dv}{dt} = \frac{100-v}{10}$. {Note. $\ddot{x} = \frac{dv}{dt}$ – relates v and t}

So $\frac{dt}{dv} = \frac{10}{100-v}$.

So $dt = \frac{10}{100-v}\,dv$.

So $\int dt = -10\int \frac{-1}{100-v}\,dv$

So $t = -10\ln(100-v) + c$.

When $t = 0$, $v = 0$.

Hence $0 = -10\ln 100 + c$.

So $c = 10\ln 100$.

So $t = -10\ln(100-v) + 10\ln 100$

So $t = 10\ln 100 - 10\ln(100-v)$

$= 10\ln\left(\frac{100}{100-v}\right)$.

• Gives the equation of motion in terms of $\frac{dv}{dt}$ (1)

AND

• Separates the variables (2)

So $\frac{t}{10} = \ln\left(\frac{100}{100-v}\right)$

So $e^{\frac{t}{10}} = \frac{100}{100-v}$.

So $(100-v)e^{\frac{t}{10}} = 100$.

So $100 - v = \frac{100}{e^{\frac{t}{10}}}$.

So $v = 100 - \frac{100}{e^{\frac{t}{10}}}$.

So $v = 100 - 100e^{-\frac{t}{10}}$

So $v = 100\left(1 - e^{-\frac{t}{10}}\right)$ #

• Shows that the velocity at time t seconds is as given (3)

(iii) Terminal velocity attained when either $t \to \infty$ or $\ddot{x} = 0$.

Hence the terminal velocity is 100 m/s #

• Correct answer (1)

(iv) Now $\ddot{x} = \frac{100-v}{10}$.

So $v\frac{dv}{dx} = \frac{100-v}{10}$.

So $\frac{dv}{dx} = \frac{100-v}{10v}$.

So $\frac{dx}{dv} = \frac{10v}{100-v}$.

So $dx = \frac{10v}{100-v}\,dv$.

So $\int dx = \int \frac{10v}{100-v}\,dv$

So $x = \int \frac{-10(100-v) + 1000}{100-v}\,dv$.

So $x = -10\int dv + \int \frac{1000}{100-v}\,dv$.

So $x = -10v - 1000\int \frac{-1}{100-v}\,dv$.

So $x = -10v - 1000\ln(100-v) + c$.

But $x = 0$ when $v = 0$.

So $c = 1000\ln 100$ and from (iii) $v < 100$.

So $x = 1000\ln 100 - 1000\ln(100-v) - 10v$.

So $x = 1000\ln\left(\frac{100}{100-v}\right) - 10v$ #

• Gives the equation of motion in terms of $v\frac{dv}{dx}$ (1).

AND

• Separates the variables (2)

• Shows that the distance travelled at speed v is as given (3)

(v) Let $v = 50$.

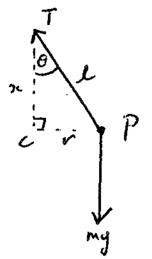
Then $x = 1000 \ln\left(\frac{100}{100-50}\right) - 10(50)$.

So $x = 1000 \ln 2 - 500$

So $x \doteqdot 193.15$.

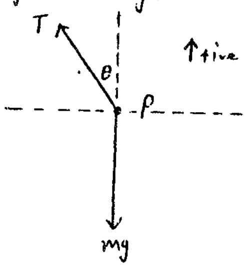
Hence the object has fallen approximately 193.15 m # • Correct answer (1)

(b)



(i) Forces on the bob are T, tension in the string, and mg down.

Resolving vertically

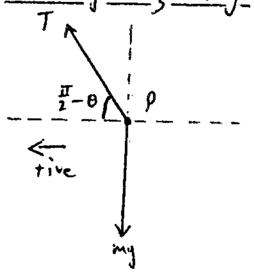


Force = mass × acceleration.

So $T\cos\theta - mg = m\ddot{x}$

So $T\cos\theta - mg = 0$ (no vertical movement) — ①

Resolving horizontally



Force = mass × acceleration.

So $T\cos\left(\frac{\pi}{2} - \theta\right) = m\ddot{x}$.

So $T\sin\theta = mr\omega^2$ (towards C) — ②

Dividing ② by ①,

$$\frac{T\sin\theta}{T\cos\theta} = \frac{mr\omega^2}{mg}.$$

• Attempts to resolve forces vertically and horizontally (1)

So $\tan\theta = \frac{r\omega^2}{g}$.

So $\tan\theta = \frac{v^2}{rg}$ since $v = r\omega$.

Since $\tan\theta = \frac{r}{x}$ (from the triangle),

then $\frac{v^2}{rg} = \frac{r}{x}$. • Obtains $\frac{v^2}{rg} = \frac{r}{x}$ (2)

So $x = \frac{r^2 g}{v^2}$.

So $x = \frac{g}{\omega^2}$

So $\omega = \sqrt{\frac{g}{x}}$.

So $P = \frac{2\pi}{\sqrt{\frac{g}{x}}}$.

So $P = 2\pi\sqrt{\frac{x}{g}}$ #

• Shows that the period of motion is as given (3)

(ii) The mass cancelled out of the equations of motion.* Hence there is no effect on the motion if there's any change in the mass of the particle #

* $\tan\theta = \frac{r\omega^2}{g}$, which is independent of the mass.

• Correct answer (1)

(iii) From (i), $x = \frac{g}{\omega^2}$.

2 revs/s = 4π rads/s.

So $x = \frac{g}{16\pi^2}$.

3 revs/s = 6π rads/s.

So $x = \frac{g}{36\pi^2}$.

So particle rises by $\frac{g}{16\pi^2} - \frac{g}{36\pi^2} = \frac{g}{\pi^2}\left(\frac{1}{16} - \frac{1}{36}\right)$

$\doteqdot 0.035$ m.

Hence the particle rises by about 3.5 cm # • Correct answer (2)

• Changes units correctly (1)

OR

• Substitutes into $x = \frac{g}{\omega^2}$ (or equivalent) but does not obtain the correct answer (1)

Question 7

(a) (i) Using the arc length formula,

$$s = r\theta$$

So $\frac{d}{dt}(s) = \frac{d}{dt}(r\theta)$

So $\frac{ds}{dt} = r\frac{d\theta}{dt}$

So $v = r\omega$ #

• Correct proof (1)

(ii) Position of particle at any time t is given by

$x = r\cos\theta$, $y = r\sin\theta$.

So the velocities of the particle at any time t are

$$\frac{dx}{dt} = \frac{d}{dt}(r\cos\theta) = r \times \frac{d}{d\theta}(\cos\theta)\times\frac{d\theta}{dt} = -r\sin\theta.\dot{\theta}.$$

$$\frac{dy}{dt} = \frac{d}{dt}(r\sin\theta) = r\times\frac{d}{dt}(\sin\theta)\times\frac{d\theta}{dt} = r\cos\theta.\dot{\theta}.$$

{Note: alternative notations for $\frac{d\theta}{dt}$ are $\dot{\theta}$, ω}.

So the accelerations of the particle at any time t are

$$\ddot{x} = \frac{d^2x}{dt^2} = \frac{d}{dt}(-r\sin\theta.\dot{\theta})$$
$$= -r\sin\theta\times\frac{d}{dt}(\dot{\theta}) + \dot{\theta}\times\frac{d}{d\theta}(-r\sin\theta)\times\frac{d\theta}{dt}$$
$$= -r\sin\theta.\ddot{\theta} - r\cos\theta.(\dot{\theta})^2$$

• Obtains values for $\dot{x}$ and $\dot{y}$ components of velocity (1)

{Note: alternative notations for $\frac{d}{dt}(\dot{\theta})$ are $\ddot{\theta}$, $\dot{\omega}$, $\frac{d\omega}{dt}$}.

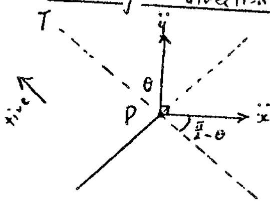
And $$\ddot{y} = \frac{d^2y}{dt^2} = \frac{d}{dt}(r\cos\theta.\dot{\theta})$$
$$= r\cos\theta\times\frac{d}{dt}(\dot{\theta}) + \dot{\theta}\times\frac{d}{d\theta}(r\cos\theta)\times\frac{d\theta}{dt}$$
$$= r\cos\theta.\ddot{\theta} - r\sin\theta(\dot{\theta})^2.$$

(AND)

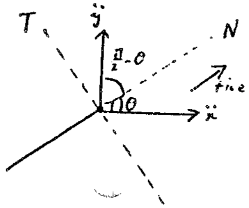
• Obtain values for $\ddot{x}$ and $\ddot{y}$ components of accln. (2)

Now, these accelerations are resolved in the direction of the tangent PT (say) and of the outward normal PN (say)

Resolving in direction of tangent



Resolving in direction of outward normal



(AND)

$$\text{Tangential accel}^n. = \ddot{y}\cos\theta - \ddot{x}\cos(\tfrac{\pi}{2}-\theta)$$
$$= \ddot{y}\cos\theta - \ddot{x}\sin\theta$$
$$= [r\cos\theta.\ddot{\theta} - r\sin\theta.(\dot{\theta})^2]\cos\theta - [-r\sin\theta.\ddot{\theta} - r\cos\theta.(\dot{\theta})^2]\sin\theta$$
$$= r\cos^2\theta.\ddot{\theta} - r\sin\theta\cos\theta.(\dot{\theta})^2 + r\sin^2\theta.\ddot{\theta} + r\sin\theta\cos\theta.(\dot{\theta})^2$$
$$= r\ddot{\theta}(\cos^2\theta + \sin^2\theta)$$
$$= r\ddot{\theta} \quad (\text{or } r\dot{\omega}, \text{ or } r\frac{d^2\theta}{dt^2}, \text{ or } r\frac{d\omega}{dt}) \#$$

• Makes a reasonable attempt to resolve tangentially and normally or equivalent

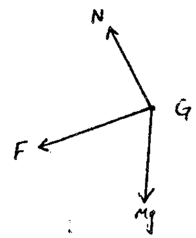
{Note: Because ω is constant, $\dot{\omega} = \ddot{\theta} = 0 \Rightarrow$ tangential accln. = 0}

$$\text{Normal accel}^n. = \ddot{y}\cos(\tfrac{\pi}{2}-\theta) + \ddot{x}\cos\theta$$
$$= \ddot{y}\sin\theta + \ddot{x}\cos\theta$$
$$= [r\cos\theta.\ddot{\theta} - r\sin\theta.(\dot{\theta})^2]\sin\theta + [-r\sin\theta.\ddot{\theta} - r\cos\theta.(\dot{\theta})^2]\cos\theta$$
$$= r\cos\theta\sin\theta.\ddot{\theta} - r\sin^2\theta.(\dot{\theta})^2 - r\sin\theta\cos\theta.\ddot{\theta} - r\cos^2\theta(\dot{\theta})^2$$
$$= -r(\dot{\theta})^2(\sin^2\theta + \cos^2\theta)$$
$$= -r(\dot{\theta})^2 \ (\text{or } -r\omega^2).$$

Hence it is directed towards the centre #

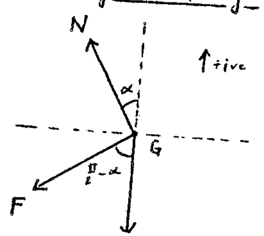
• Correct solution (4)

(b) (i)

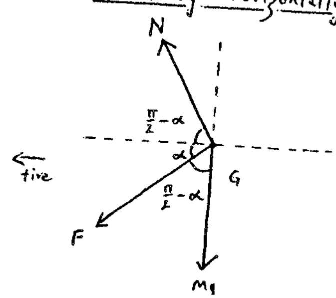


Forces on the car are gravitational force mg down, normal reaction force N and friction force F.

Resolving vertically



Resolving horizontally



Force = mass × acceleration

$N\cos\alpha - F\cos(\frac{\pi}{2}-\alpha) - mg = m(0)$ (no vertical movement)

So $N\cos\alpha - F\sin\alpha - mg = 0$.

So $F\sin\alpha = N\cos\alpha - mg$ # —①

Force = mass × acceleration

$N\cos(\frac{\pi}{2}-\alpha) + F\cos\alpha = m \times \frac{v^2}{R}$ (towards centre)

So $N\sin\alpha + F\cos\alpha = \frac{mv^2}{R}$

So $F\cos\alpha = \frac{mv^2}{R} - N\sin\alpha$ # —(

- Gives a diagram showing the resolution of components into 2 different dims. (1)

(OR)

- Introduces the term $\frac{mv^2}{R}$ into an equation involving horizontal forces (1)
- Gives either $F\sin\alpha$ or $F\cos\alpha$ by resolving components (2)
- Gives correct expressions for $F\cos\alpha$ and $F\sin\alpha$ by resolving components (3)

(ii) Since there is no tendency for the car to slip sideways, then $F = 0$.

So $0 = N\cos\alpha - mg$.

Thus $N\cos\alpha = mg$ — (1a)

Also $0 = \frac{mv^2}{R} - N\sin\alpha$.

Thus $N\sin\alpha = \frac{mv^2}{R}$ — (2a)

Dividing (2a) by (1a) gives

$\tan\alpha = \frac{mv^2}{mRg}$.

So $\tan\alpha = \frac{v^2}{Rg}$.

But $R = 75$m when $v = 60$ km/h.

Now $60\text{km/h} = \frac{60 \times 1000}{60 \times 60}$ m/s

$= \frac{50}{3}$ m/s.

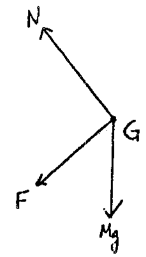
- Implies that $F = 0$. (1)
- Obtains $\tan\alpha = \frac{v^2}{Rg}$ (2)

So $\tan\alpha = \frac{(\frac{50}{3})^2}{75 \times 9.8}$

So $\alpha \doteqdot 20° 42'$ #

- Correct answer (3)

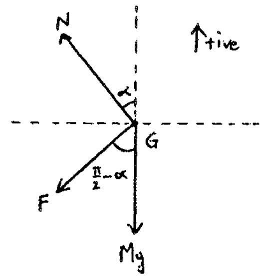
(iii)



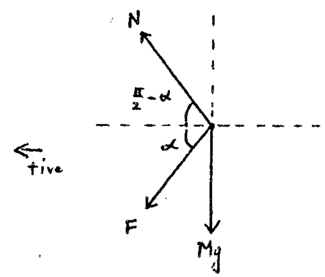
Same track, but mass M and velocity V.

Forces on this car are gravitational force Mg down, normal reaction force N (AND) Friction force F (car wants to slide out of the curve, so friction force F acts AGAINST the direction of motion.

Resolving vertically



Resolving horizontally



Force = mass × acceleration

So $N\cos\alpha - F\cos(\frac{\pi}{2}-\alpha) - Mg = 0$

So $N\cos\alpha - F\sin\alpha - Mg = 0$.

So $N = \frac{Mg + F\sin\alpha}{\cos\alpha}$ —①

Force = mass × acceleration

So $N\cos(\frac{\pi}{2}-\alpha) + F\cos\alpha = \frac{MV^2}{R}$

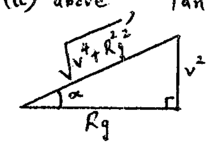
So $N\sin\alpha + F\cos\alpha = \frac{MV^2}{R}$ —②

Substituting ① into ② gives

$\frac{(Mg + F\sin\alpha)\sin\alpha}{\cos\alpha} + F\cos\alpha = \frac{MV^2}{R}$ —③

From (ii) above $\tan\alpha = \frac{V^2}{Rg}$.

So



Thus $\sin\alpha = \frac{V^2}{\sqrt{V^4+R^2g^2}}$ (and) $\cos\alpha = \frac{Rg}{\sqrt{V^4+R^2g^2}}$

Substituting into ③ gives

$$\frac{\left[Mg + F\left(\frac{v^2}{\sqrt{v^4+R^2g^2}}\right)\right] \times \frac{v^2}{\sqrt{v^4+R^2g^2}}}{\frac{Rg}{\sqrt{v^4+R^2g^2}}} + F \times \frac{Rg}{\sqrt{v^4+R^2g^2}} = \frac{MV^2}{R}$$

$$\frac{v^2}{Rg}\left(Mg + \frac{Fv^2}{\sqrt{v^4+R^2g^2}}\right) + \frac{FRg}{\sqrt{v^4+R^2g^2}} = \frac{MV^2}{R}$$

$$\frac{v^2M}{R} + \frac{Fv^4}{Rg\sqrt{v^4+R^2g^2}} + \frac{FRg}{\sqrt{v^4+R^2g^2}} = \frac{MV^2}{R}$$

$$v^2Mg\sqrt{v^4+R^2g^2} + Fv^4 + FR^2g^2 = MV^2g\sqrt{v^4+R^2g^2}$$

$$F(v^4+R^2g^2) = MV^2g\sqrt{v^4+R^2g^2} - v^2Mg\sqrt{v^4+R^2g^2}.$$

So $F = \dfrac{Mg\sqrt{v^4+R^2g^2}\,(V^2-v^2)}{v^4+R^2g^2}$

So $F = \dfrac{Mg\,(V^2-v^2)}{\sqrt{v^4+R^2g^2}}$ #

- Attempts to eliminate N (1) (AND)
- Obtains correct expressions for $\sin\alpha$ and $\cos\alpha$ (2) (AND)
- Substitutes into an equation for F (3)
- Correct derivation (4)

OR

$F\sin\alpha = N\cos\alpha - Mg$ — ①

$F\cos\alpha = \frac{MV^2}{R} - N\sin\alpha$ — ②

$\Rightarrow \dfrac{N\sin\alpha}{N\cos\alpha} = \dfrac{\frac{MV^2}{R} - F\cos\alpha}{Mg + F\sin\alpha}$

So $\tan\alpha = \dfrac{\frac{MV^2}{R} - F\cos\alpha}{Mg + F\sin\alpha}$

So $\tan\alpha\,(Mg + F\sin\alpha) = \frac{MV^2}{R} - F\cos\alpha$

$\Rightarrow F(\cos\alpha + \tan\alpha\sin\alpha) = \frac{MV^2}{R} - Mg\tan\alpha$

$\Rightarrow F\left(\frac{Rg}{\sqrt{v^4+R^2g^2}} + \frac{v^2}{Rg} \times \frac{v^2}{\sqrt{v^4+R^2g^2}}\right) = \frac{MV^2}{R} - Mg \times \frac{v^2}{Rg}$

$\Rightarrow F\left(\frac{R^2g^2+v^4}{\sqrt{v^4+R^2g^2}}\right) = Rg\left(\frac{MV^2-Mv^2}{R}\right)$

$\Rightarrow F = \dfrac{Mg\,(V^2-v^2)}{\sqrt{v^4+R^2g^2}}$ #

Question 8

(a) Let $y = x-1$.

Then $x = y+1$.

Substituting gives

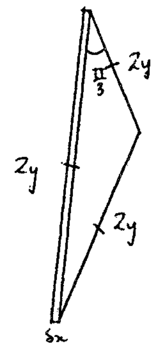
$(y+1)^3 - 5(y+1)^2 + 5 = 0$

$y^3 + 3y^2 + 3y + 1 - 5y^2 - 10y - \quad +5 = 0.$

So $y^3 - 2y^2 - 7y + 1 = 0$ #

- Uses a correct method but obtains incorrect answer (1)
- Obtains correct equation (2)

(b)



Area of each cross-sectional slice is $\frac{1}{2}(2y)^2\sin\frac{\pi}{3} = \sqrt{3}\,y^2$.

Hence the volume of a slice δx thick is $\delta V = \sqrt{3}\,y^2\delta x$
$= \sqrt{3}\,(4-x^2)\,\delta x.$

So the required volume is $V = \lim_{\delta x \to 0} \sum_{x=-2}^{2} \sqrt{3}\,(4-x^2)\,\delta x$

$= \int_{-2}^{2} \sqrt{3}\,(4-x^2)\,dx$

$= 2\sqrt{3}\int_0^2 (4-x^2)\,dx$

$= 2\sqrt{3}\left[4x - \frac{x^3}{3}\right]_0^2$

$= 2\sqrt{3}\left\{\left(8 - \frac{8}{3}\right) - 0\right\}$

$= 2\sqrt{3}\left\{\frac{16}{3}\right\}$

$= \frac{32}{3}\sqrt{3}$ units^3 #

- Reasonable attempt to get area of a slice (1)
- Obtains correct expression for the area of a slice (2)
- Correct definite integral (3)
- Correct answer (4)

(c) When $n = 2$,

$(1+x)^2 - 2x - 1 = 1 + 2x + x^2 - 2x - 1$

$= x^2$, • Includes test for $n = 2$ (1)

which is divisible by x^2.

So the statement is true for $n = 2$.

Suppose that the statement is true for a positive integer k.
That is, suppose $(1+x)^k - kx - 1 = x^2 . P$ for some polynomial P.

We prove the statement for $n = k+1$.
That is, we prove $(1+x)^{k+1} - (k+1)x - 1$ is divisible by x^2.

$(1+x)^{k+1} - (k+1)x - 1 = (1+x)^k(1+x) - (k+1)x - 1$

$= (x^2 P + kx + 1)(1+x) - (k+1)x - 1$, by the induction hypothesis,

$= x^2 P + x^3 P + kx + kx^2 + 1 + x - kx - x - 1$,

$= x^2 P + x^3 P + kx^2$,

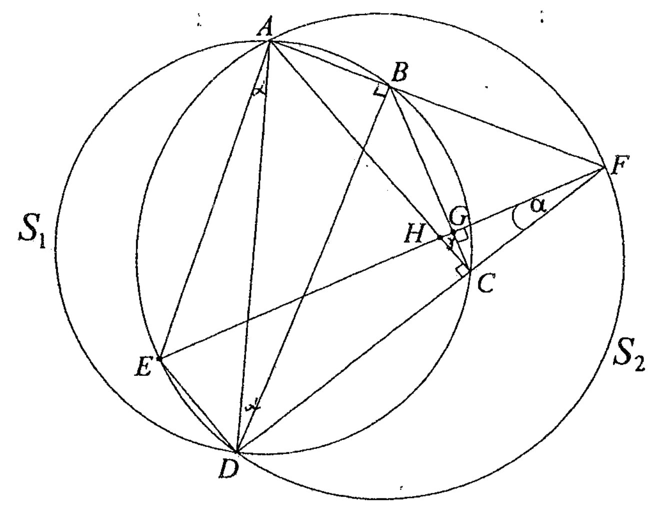
$= x^2(P + xP + k)$, which is divisible by x^2.

Hence, if the statement is true for the integer k, then it is also true for the next integer $k+1$.

But we know the statement is true for $n = 2$.
So it must be true for $n = 3$, $n = 4$, and so on for all positive integral values of $n \geq 2$.

• Uses induction assumption (2)
• Obtains $x^2(P + xP + k)$ or equivalent (3)
• Correct proof (4)

(d)



(i) $\angle GCD = \frac{\pi}{2} + \alpha$ (exterior angle of ΔGCF equals sum of the interior opposite angles).

$\angle GCD = \frac{\pi}{2} + \angle HCG$.

Hence $\angle HCG = \alpha$ # • Correct proof (1)

(ii) $\angle ABD = \angle ACD = \frac{\pi}{2}$ (angles in the same segment)

Hence $AB \perp DB$ # • Correct proof (1)

(iii) $\angle EAD = \alpha$ (angles in the same segment)

$\angle ADB = \alpha$ (angles in the same segment)

So $\angle BAD = \frac{\pi}{2} - \alpha$ (angle sum of $\Delta BAD = \pi$) • Partial progress (1)

Hence $\angle BAE = \alpha + \frac{\pi}{2} - \alpha = \frac{\pi}{2}$

Hence $AB \perp AE$.

So $AE \parallel BD$ (cointerior angles are supplementary) # • Correct proof (2)

(iv) $\angle BAE + \angle BGE = \frac{\pi}{2} + \frac{\pi}{2} = \pi$ ((iii) and given $FH \perp BC$).

Hence E, A, B and G are concyclic as the opposite angles are supplementary # • Correct proof (1)