ST. IVES HIGH SCHOOL

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

2002

MATHEMATICS

YEAR 12
Extension 2

Time allowed - Three hours

DIRECTIONS:

- Attempt ALL questions.
- All necessary working should be shown in every question. Full marks may not be awarded for careless or badly arranged work.
- Begin each question on a new page. Papers should be stapled together in the top left hand corner.

QUESTION 1

Marks

(a) (i) Find real constants A, B and C such that **3**

$$\frac{x^2 + 5x + 2}{(x^2 + 1)(x + 1)} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x + 1}$$

(ii) Hence find $\int \frac{x^2 + 5x + 2}{(x^2 + 1)(x + 1)}\, dx$. **2**

(b) Evaluate $\int_1^5 x\sqrt{2x - 1}\, dx$. **3**

(c) By using the substitution $x = e^y$ and integrating by parts, or otherwise, evaluate $\int_e^{e^2} (\ln x)^2\, dx$. **4**

(d) (i) Simplify $\sin(A - B) + \sin(A + B)$. **1**

(ii) Hence find $\int \sin 5x.\cos 3x\, dx$. **2**

QUESTION 2 *(Begin on a new page)*

(a) Given the two complex numbers $A = 3-4i$ and $B = 4+3i$, find the complex number in the form $a+ib$ represented by:

(i) $A + B$. **1**

(ii) $\dfrac{A}{B}$ **1**

(iii) A^2 **1**

(b) Find a and b if $(a + ib)^2 = 3 - 4i$ where a and b are real and $a > 0$. **2**

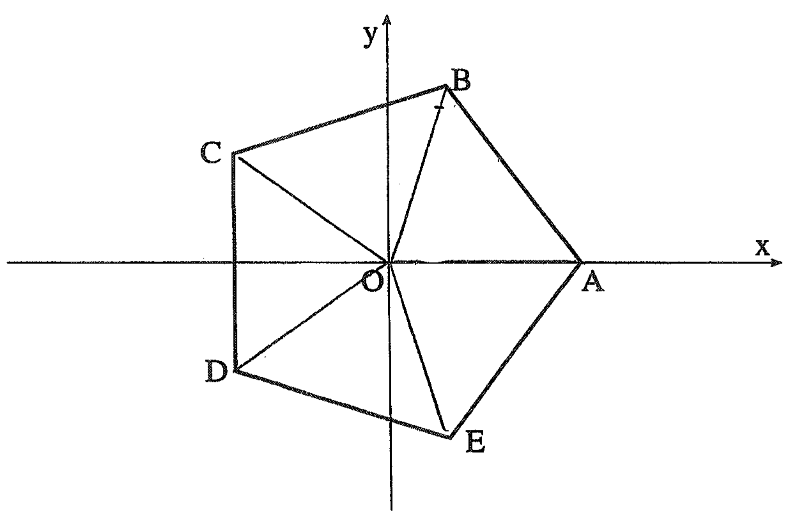
(c) Consider the region defined by $|z - 4i| \leq 3$.

(i) Sketch the region. **1**

(ii) Determine the maximum value of $|z|$. **1**

(iii) Determine the maximum value of $\arg z$, where $-\pi < \arg z \leq \pi$. **2**

(d)



In the diagram above, the complex numbers z_0, z_1, z_2, z_3, z_4 are represented by the vertices of a regular polygon with centre O and vertices A, B, C, D, E respectively.

Given that $z_0 = 2$:

(i) Express z_2 in modulis-argument form. **2**

(ii) Find the value of $z_2^{\,5}$. **2**

(iii) Show that the perimeter of the pentagon is $20\sin\dfrac{\pi}{5}$. **2**

QUESTION 3 *(Begin on a new page)*

(a) Solve the equation $4x^3 - 24x^2 + 23x + 18 = 0$ given that the ~~roots~~ form an arithmetic series. (Leave the roots in exact form) **3**

(b) Let $P(x) = x^4 + ax^3 + 36x^2 - 35x + b$ where a and b are real numbers.

It is known that $x = 5$ and $x = \dfrac{1 - i\sqrt{3}}{2}$ are zeros of $P(x)$.

(i) Explain why $x^2 - x + 1$ must be a factor of $P(x)$. **2**

(ii) Find a and b . **2**

(c) Let α , β and γ be the roots of

$$x^3 - 7x^2 + 18x - 7 = 0$$

(i) Find a cubic equation that has roots $1 + \alpha^2, 1 + \beta^2, 1 + \gamma^2$. **2**

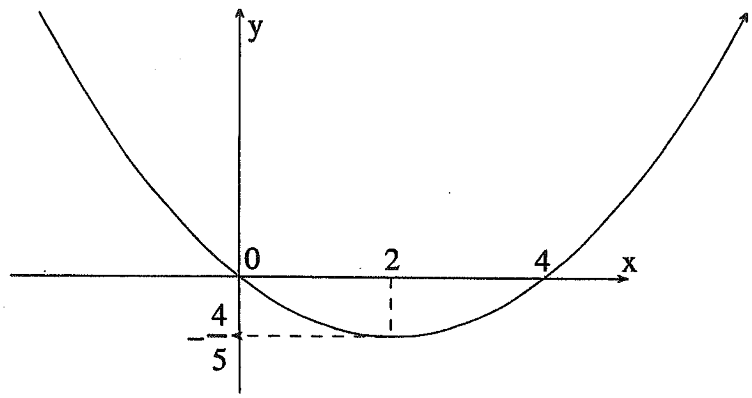
(ii) Hence, or otherwise, find the value of $(1 + \alpha^2)(1 + \beta^2)(1 + \gamma^2)$. **1**

(d) The polynomial $A(x) = ax^n + bx^{n-1} + 1$ is divisible by $(x - 1)^2$. Find a and b in terms of n. **3**

(e) Prove that $1 - x + \dfrac{x^2}{2!} - \dfrac{x^3}{3!} + \ldots\ldots \dfrac{x^n}{n!} = 0$ has no multiple roots for any $n > 1$. **2**

QUESTION 4 *(Begin on a new page)*

(a)



The sketch above shows the parabolic curve $y = f(x)$ where

$$f(x) = \frac{x^2 - 4x}{5} = \frac{x}{5}(x-4)$$

Without the use of calculus, draw sketches of the following, showing intercepts, asymptotes and turning points:

(i) $y = |f(x)|$. **1**

(ii) $y = \frac{1}{f(x)}$ **2**

(iii) $y = \frac{x}{5}|x - 4|$ **2**

(iv) $y = \tan^{-1}(f(x))$ **2**

(b) A technique which can be used to differentiate a rather nasty quotient is to firstly take the logarithm of each side. Then by using the rules of logarithms and implicit differentiation, the derivative can be found. **3**

Use the function $y = \frac{f(x)}{g(x)}$ to demonstrate the above method.

(c) Consider the curve $x^2 - xy + y^2 = 3$.

(i) Show that $\frac{dy}{dx} = \frac{2x - y}{x - 2y}$. **2**

(ii) Hence find the two stationary points on the curve. **2**

(iii) Find the values of x where there are vertical tangents. **1**

QUESTION 5 *(Begin on a new page)*

(a) An ellipse has equation $\frac{x^2}{16} + \frac{y^2}{9} = 1$.

(i) Show that $x = 4\cos\theta$ and $y = 3\sin\theta$ are parametric equations of this ellipse. **2**

(ii) Find the equations of the tangent and normal to the ellipse at the point $\left(2, \frac{3\sqrt{3}}{2}\right)$. **4**

(iii) Calculate the eccentricity of the ellipse. **1**

(iv) Hence find the coordinates of the foci and the equations of the directrices. **2**

(b) (i) Show that the normal at the point $P\left(cp, \frac{c}{p}\right)$ to the rectangular hyperbola $xy = c^2$ is given by **2**

$$p^3x - py = c(p^4 - 1).$$

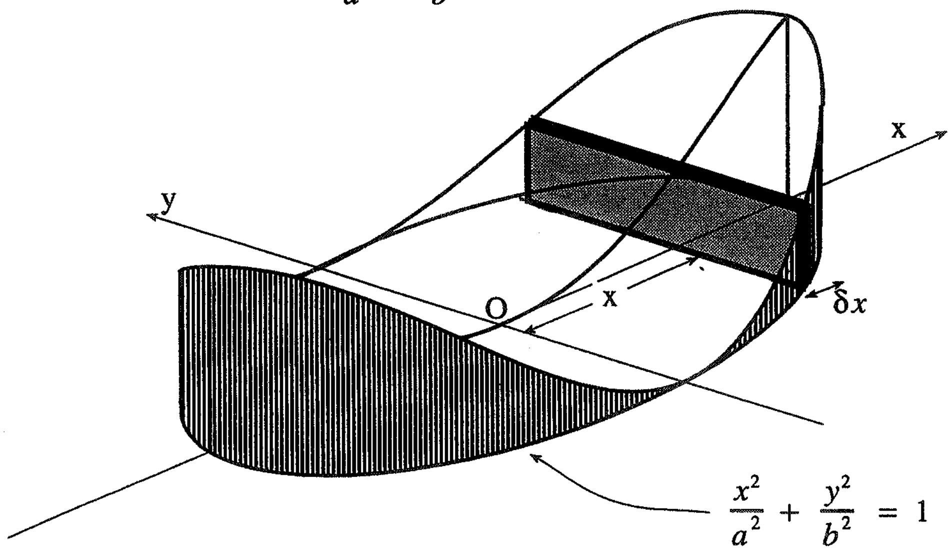
(ii) If this normal meets the hyperbola again at $Q\left(cq, \frac{c}{q}\right)$, show that **2**

$$p^3q = -1.$$

(iii) Hence find the area, in terms of p, of the triangle PQR where R is the point of intersection of the tangent at P with the y-axis. **2**
You may assume the the equation of the tangent is given by $x + p^2y = 2cp$.

QUESTION 6 *(Begin on a new page)*

(a) A solid has as its base, the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.



Each vertical cross-section taken x units from the centre O of the ellipse and perpendicular to the major axis is a rectangle with height equal to the square of its distance from the centre of the ellipse (ie x^2).

(i) Show that the volume of the slice taken x units from the centre with thickness δx is given by **2**

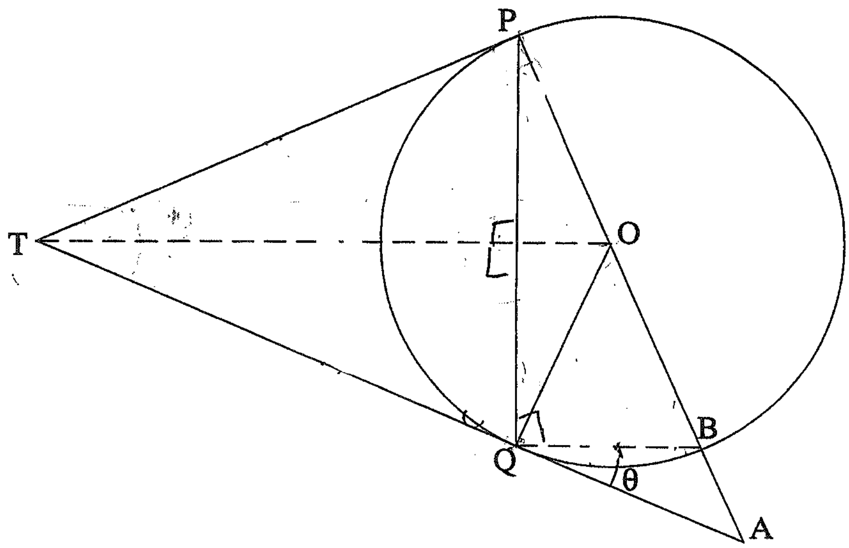
$$\delta V = \frac{2b}{a} x^2 \sqrt{a^2 - x^2} . \delta x .$$

(ii) Find the volume of the solid. **4**

(b) The area enclosed by the parabola $y = 8x - x^2 - 12$ and the x-axis is rotated around the y-axis. By using the method of cylindrical shells, calculate the volume of the solid shape formed. **4**

QUESTION 7 *(Begin on a new page)*

(a)



From an external point T. tangents are drawn to a circle with centre O, touching the circle at P and Q. Angle PTQ is acute.
The diameter PB produced meets the tangent TQ at A. Let $\theta = \angle AQB$.

Copy the above diagram onto youranswer page.

(i) Prove that $\angle PTQ = 2\theta$. **2**

(ii) Prove that ΔPBQ aand ΔTOQ are similar. **2**

(iii) Hence show that $BQ \times OT = 2(OP)^2$. **2**

(b) Points A, B and C are collinear (C not between A and B).
D is a point such that DC is perpendicular to AC.
AB, BC and DC are g, a and y units respectively and $\angle ADB = \theta$.

(i) Sketch this information. **1**

(ii) Show that $\tan\theta = \dfrac{gy}{a(g+a)+y^2}$. **3**

(iii) Find $\dfrac{d\theta}{dy}$ and hence write an expression for y which would maximise θ given that g and a are constants. **5**

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \; x \neq 0 \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln \left\{ x + \sqrt{(x^2 - a^2)} \right\}, \quad |x| > |a|$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln \left\{ x + \sqrt{(x^2 + a^2)} \right\}$$

NOTE: $\ln x = \log_e x, \quad x > 0$