

Question 1**Marks**

a) Find $\int x^3 e^{x^4+7} dx$

1

b) (i) Express $\frac{x^2+x+2}{(x^2+1)(x+1)}$ in the form

3

$$\frac{Ax+B}{x^2+1} + \frac{C}{x+1} \text{ where A,B and C are constants.}$$

(ii) Hence find $\int \frac{x^2+x+2}{(x^2+1)(x+1)} dx$

2

c) Evaluate $\int_0^1 \tan^{-1} x dx$

3

d) (i) Use the substitution $t = \tan \frac{x}{2}$ to show

$$\int_{\frac{3\pi}{2}}^{\frac{5\pi}{2}} \frac{1}{2+\cos x} dx = \frac{2\pi\sqrt{3}}{9}$$

3

(ii) Using the substitution $u = 4\pi - x$, evaluate

$$\int_{\frac{3\pi}{2}}^{\frac{5\pi}{2}} \frac{x}{2+\cos x} dx$$

3

Question 2

a) $\frac{4+3i}{1+\sqrt{2}i} = a + ib$ for a, b real.

2

Find the exact values of a and b .

b) (i) Solve $z^4 + 1 = 0$, giving your answers in mod-arg form.

3

(ii) Plot these roots on an Argand diagram.

1

(iii) Find the exact area of the quadrilateral that they form.

1

c) $z_1 = 4cis \frac{\pi}{12}$ and $z_2 = 2cis \frac{5\pi}{12}$.

(i) On an Argand diagram, draw the vectors OA, OB, OC representing $z_1, z_2, z_1 + z_2$ respectively (not to scale).

2

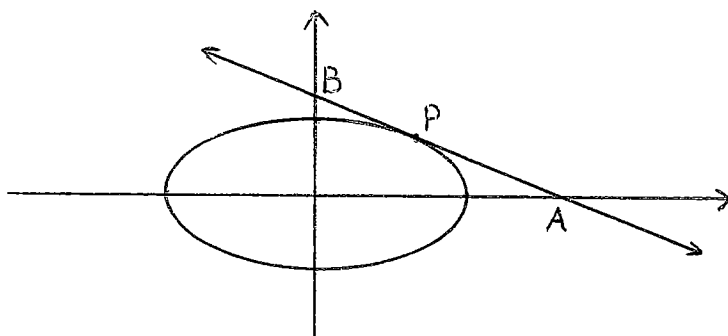
(ii) Hence or otherwise find $|z_1 + z_2|$ in simplest exact form

2

d) $\arg(z - 2) = \arg(z + 2) + \frac{\pi}{4}$ is the locus of the point P representing z on an Argand diagram.

- (i) Show with a diagram why this locus is an arc of a circle 2
- (ii) Find the centre and radius of this circle 2

Question 3



a) In the diagram above, $P(a\cos\theta, b\sin\theta)$ lies on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

The tangent at P cuts the x-axis at A and the y-axis at B.

- (i) Derive the parametric equation of the tangent at P in any form and find the coordinates of A and B in parametric form. 4
- (ii) Show that $\frac{PA}{PB} = \tan^2\theta$ 2

- b) The rectangular hyperbola H has equation $x^2 - y^2 = 8$. Write down:
- (i) The eccentricity 1
 - (ii) The coordinates of the foci 1
 - (iii) The equations of the directrices 1
 - (iv) The equations of the asymptotes 1
 - (v) Sketch the curve showing the above 1

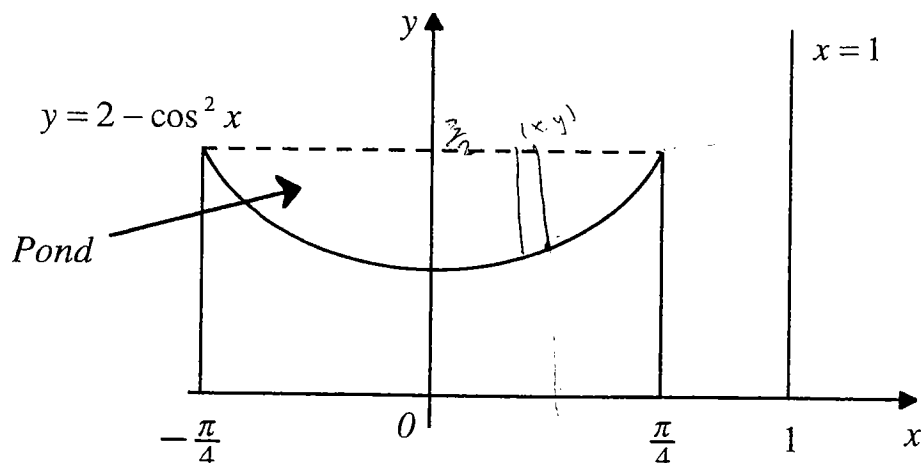
If this curve is rotated through 45° in an anticlockwise direction the equation takes the form $xy = 4$.

- (vi) Find the equation of the normal to this hyperbola at the point $(2p, \frac{2}{p})$. 2
- (vii) If this normal meets the hyperbola again at $(2q, \frac{2}{q})$, prove that $q = \frac{-1}{p^3}$ 2

Question 4

- a) Consider the function $y = \cos^{-1}(\sin x)$. Given the domain and range are
D: all real x
R: $0 \leq y \leq \pi$
- (i) Find the period of this function. 1
 - (ii) Sketch the graph of the function over the domain $-2\pi \leq x \leq 2\pi$ 2
- b) Given the derivative of the function $f(x) = x - \frac{3\sin x}{2+\cos x}$ is $f'(x) = \left[\frac{1-\cos x}{2+\cos x} \right]^2$, 3
show that $x > \frac{3\sin x}{2+\cos x}$ for $x > 0$.

c)



A mould for an annular fish pond is made by rotating the region bounded by the curve $y = 2 - \cos^2 x$ and $y = \frac{3}{2}$ between $x = -\frac{\pi}{4}$ and $x = \frac{\pi}{4}$ through one complete revolution about the line $x = 1$. All measurements are in metres.

(i) Use the method of cylindrical shells to show that the volume of the fish pond is 3

given by $V = \pi \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (1 - x) \cos 2x dx$

(ii) Find the capacity of the fish pond correct to the nearest litre. 3

d)

The deck of a boat was 2.4m below the level of a wharf at low tide and 0.6m above the wharf at high tide. Low tide was at 11.30am and high tide at 5.35pm. Find when the deck was level with the wharf assuming the tidal motion is simple harmonic. Give your answer correct to the nearest minute. 3

Question 5

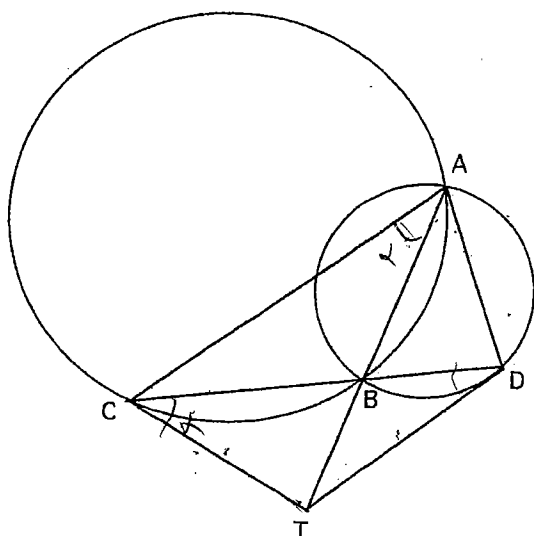
- a) When a polynomial $P(x)$ is divided by $(x - 2)$ and $(x - 4)$ the remainders are 11 and 15 respectively. Determine the remainder when $P(x)$ is divided by $(x - 2)(x - 4)$. 2
- b) (i) If α is a multiple root of $P(x) = 0$, prove that $P'(\alpha) = 0$. 2
(ii) If $ax^4 + 4bx + c = 0$ has a double root, prove that $ac^3 - 27b^4 = 0$. 3
- c) If $P(x) = x^4 - 2x^3 - x^2 + 6x - 6$ has a zero $1-i$, factorise $P(x)$ fully over the real field. 2
- d) (i) If $I_n = \int \operatorname{cosec}^n x dx$ for $n \geq 0$ show that 3
$$I_n = \frac{-\cot x \operatorname{cosec}^{n-2} x}{n-1} + \frac{n-2}{n-1} I_{n-2}$$
given $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$.
(ii) Hence show that if $I_n = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \operatorname{cosec}^n x dx$ for $n \geq 0$, then 1
$$I_n = \frac{(\sqrt{2})^{n-2}}{n-1} + \frac{n-2}{n-1} I_{n-2} \text{ for } n \geq 2.$$

(iii) Deduce that $I_6 = \frac{28}{15}$ 2

Question 6

- (a) The equation $x^3 + 2x + 1 = 0$ has roots α, β and γ .
(i) Find the monic cubic equation with roots $\frac{1}{\alpha}, \frac{1}{\beta}$ and $\frac{1}{\gamma}$. 2
(ii) Find the monic cubic equation with roots $\frac{\beta+\gamma}{\alpha^2}, \frac{\gamma+\alpha}{\beta^2}$ and $\frac{\alpha+\beta}{\gamma^2}$. 3
- b) (i) If $4 - \tan\theta = 5\sin\theta\cos\theta$, show that $x = \tan\theta$ is a root of the equation $x^3 - 4x^2 + 6x - 4 = 0$ 2
(ii) Solve the equation $4 - \tan\theta = 5\sin\theta\cos\theta$ for $0^\circ \leq \theta \leq 360^\circ$ giving answers correct to the nearest degree. 3

c)



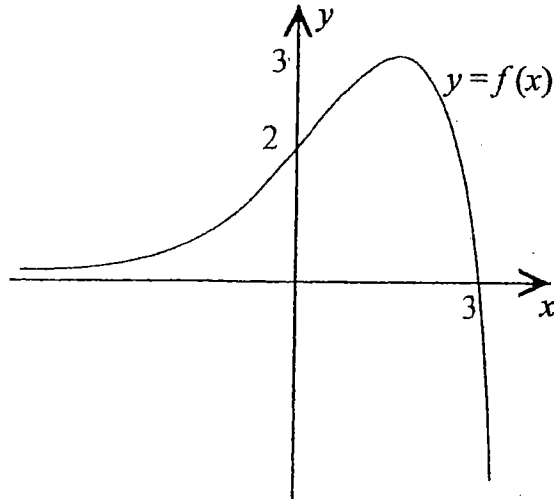
BAC, BAD are two circles such that tangents at C and D meet at T on AB produced. If CBD is a straight line prove that:

- | | | |
|-------|--------------------------------|---|
| (i) | TC=TD | 1 |
| (ii) | $\angle TAC = \angle TAD$ | 2 |
| (iii) | TCAD is a cyclic quadrilateral | 2 |

Question 7

- a) P represents the complex number z , where z satisfies $|z - 2| = 2$ and $0 < \arg z < \frac{\pi}{2}$
- | | | |
|------|---------------------------------------------------------------------------|---|
| (i) | Show that $ z^2 - 2z = 2 z $ | 2 |
| (ii) | Find the value of k (a real number) if $\arg(z - 2) = k \arg(z^2 - 2z)$ | 3 |
- b) Two sides of a triangle are in the ratio of 3:1 and the angles opposite these sides differ by $\frac{\pi}{6}$. Show that the smaller of the two angles is $\tan^{-1}\left(\frac{1}{6-\sqrt{3}}\right)$.

c)



Consider the above sketch of $y = f(x)$. On separate diagrams draw sketches of

- | | | |
|-------|----------------|---|
| (i) | $y = f(x) - 1$ | 1 |
| (ii) | $y = [f(x)]^3$ | 2 |
| (iii) | $y = f(x)$ | 2 |
| (iv) | $y = e^{f(x)}$ | 2 |

Question 8

- a) Prove that the y axis is a tangent to the curve $\sqrt{\frac{x}{u}} + \sqrt{\frac{y}{v}} = 1$ 2
 (u and v are positive constants).
- b) Gas is escaping from a spherical balloon. Find the radius of the balloon when the 2
 rate of decrease in the volume and the rate of decrease in the surface area are
 numerically equal.

Marks

- c) A particle of mass m kg is dropped from rest in a medium in which the resistance to motion has magnitude $\frac{1}{10}mv^2$ when the velocity of the particle is v ms^{-1} . After t seconds the particle has fallen x metres and has velocity v ms^{-1} and acceleration a ms^{-2} . Take the acceleration due to gravity as 10 ms^{-2} .
- (i) Draw a diagram showing forces acting on the particle. Hence show that $a = \frac{100-v^2}{10}$. 1
- (ii) Show that $t = \frac{1}{2} \ln \left[\frac{10+v}{10-v} \right]$ 3
- (iii) Find expressions in terms of t for v and x . 4
- (iv) Show that the terminal velocity is 10 ms^{-1} . 1
- (v) Find the exact time taken and the exact distance fallen by the particle in reaching a speed equal to 80% of its terminal velocity. 2

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln \left(x + \sqrt{x^2 - a^2} \right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln \left(x + \sqrt{x^2 + a^2} \right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$