St George Girls High School

Trial Higher School Certificate Examination

2013

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen.
- Write your student number on each booklet.
- Board-approved calculators may be used.
- A table of standard integrals is provided at the back of this paper.
- The mark allocated for each question is listed at the side of the question.
- Marks may be deducted for careless or poorly presented work.

Total Marks – 100

Section I – Pages 2 – 5
10 marks

- Attempt Questions 1 – 10.
- Allow about 15 minutes for this section.
- Answer on the sheet provided.

Section II – Pages 6 – 13
90 marks

- Attempt Questions 11 – 16
- Allow about 2 hours 45 minutes for this section.
- Begin each question in a new booklet.
- Show all necessary working in Questions 11 – 16.
- Templates for Q14(b) to be detached and placed in Q14 answer booklet.

Students are advised that this is a Trial Examination only and does not necessarily reflect the content or format of the Higher School Certificate Examination.

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Marks

Use the multiple-choice answer sheet for Questions 1–10.

1. Let $z = 1 + 2i$ and $w = -2 + i$. What is the value of $\bar{z}\,.\,w$

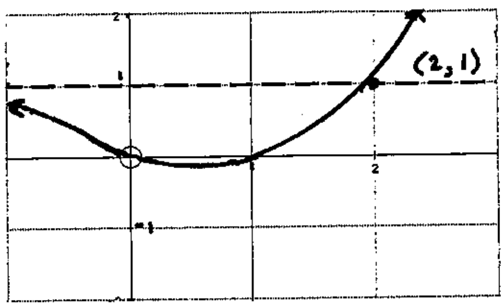
   (A) $5i$

   (B) $-4 + 5i$

   (C) $-3i$

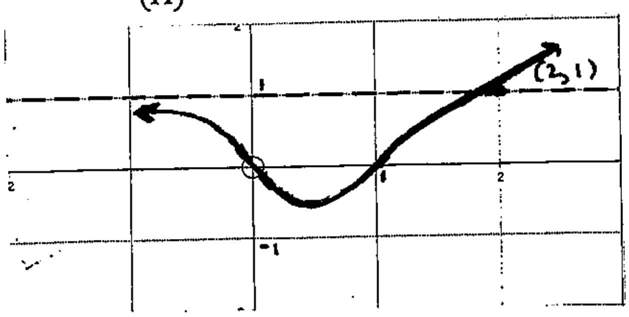
   (D) $-4 - 3i$

2. The graph of $y = f(x)$ is shown below

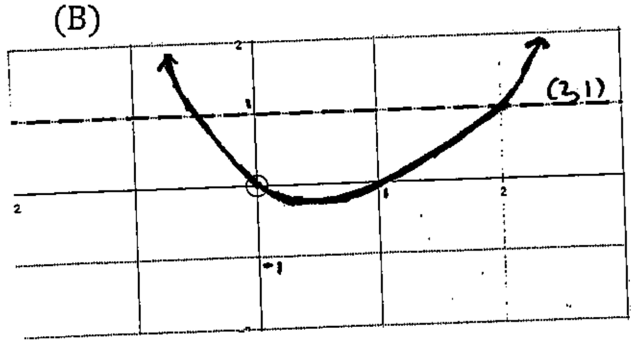


Which of the following is the graph of $y = \sqrt{f(x)}$?

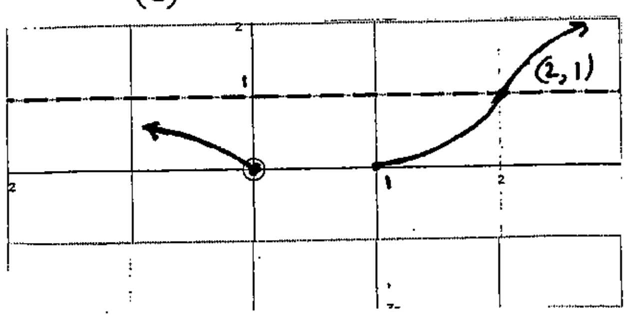
(A)



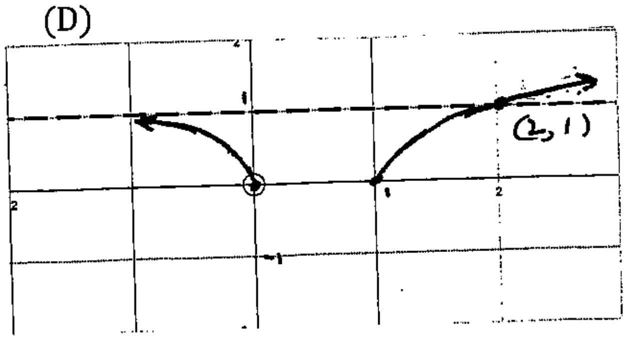
(B)



(C)

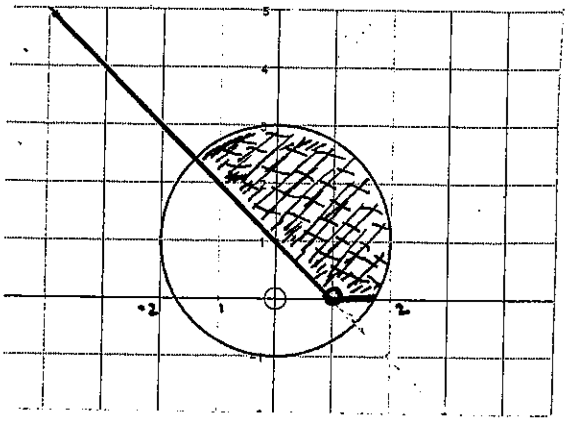


(D)



Section I (cont'd)

3. Consider the Argand diagram below.



Which inequality could define the shaded area?

(A) $|z - i| \leq 2$ and $0 \leq \arg(z-1) \leq \frac{3\pi}{4}$

(B) $|z + i| \leq 2$ and $0 \leq \arg(z-1) \leq \frac{3\pi}{4}$

(C) $|z - i| \leq 2$ and $0 \leq \arg(z-1) \leq \frac{\pi}{4}$

(D) $|z + i| \leq 2$ and $0 \leq \arg(z-1) \leq \frac{\pi}{4}$

4. The points $P(a\cos\theta, b\sin\theta)$ and $Q(a\cos\emptyset, b\sin\emptyset)$ lie on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2}$ and the chord PQ subtends a right angle at $(0, 0)$. Which of the following is the correct expression?

(A) $\tan\theta . \tan\emptyset = -\frac{b^2}{a^2}$

(B) $\tan\theta . \tan\emptyset = -\frac{a^2}{b^2}$

(C) $\tan\theta . \tan\emptyset = \frac{b^2}{a^2}$

(D) $\tan\theta . \tan\emptyset = \frac{a^2}{b^2}$

Section I (cont'd)

5. Which of the following is an expression for $\int \frac{\sin x . \cos x}{4 + \sin x}\, dx$

Use the substitution $u = 4 + \sin x$

(A) $-4 \ln |4 + \sin x| + C$
(B) $4 \ln |4 + \sin x| + C$
(C) $-\sin x - 4 \ln |4 + \sin x| + C$
(D) $4 + \sin x - 4 \ln |4 + \sin x| + C$

6. The polynomial $P(x) = x^5 - 3x^4 + 4x^3 - 4x^2 + 3x - 1$ has $x = 1$ as a root of multiplicity 3 and $x = i$ is a root. Which of the following expressions is a factorised form of $P(x)$ over the complex numbers?

(A) $P(x) = (x + 1)^3 (x - 1)(x + 1)$
(B) $P(x) = (x - 1)^3 (x - 1)(x + 1)$
(C) $P(x) = (x + 1)^3 (x - i)(x + i)$
(D) $P(x) = (x - 1)^3 (x - i)(x + i)$

7. What is the eccentricity of the hyperbola

$$\frac{(x-1)^2}{10} - \frac{(y+1)^2}{4} = 1$$

(A) $\frac{\sqrt{6}}{2}$ B. $\sqrt{\frac{7}{5}}$ C. $\frac{2}{\sqrt{6}}$ D. $\frac{\sqrt{14}}{2}$

8. The polynomial equation $x^3 - 3x^2 - x + 2 = 0$ has roots α, β and γ. Which of the following polynomial equations have roots $\frac{1}{\alpha}, \frac{1}{\beta}$ and $\frac{1}{\gamma}$

(A) $2x^3 - 2x^2 - 3x + 1 = 0$
(B) $2x^3 - x^2 - 3x + 1 = 0$
(C) $x^3 - 2x^2 - 3x + 1 = 0$
(D) $x^3 - x^2 - 3x + 1 = 0$

Section I (cont'd)

9. A particle of mass m is moving in a straight line under the action of a force

$$F = \frac{m}{x^3}(6 - 10x)$$

What of the following is an expression for its velocity in any position, if the particle starts from rest at $x = 1$?

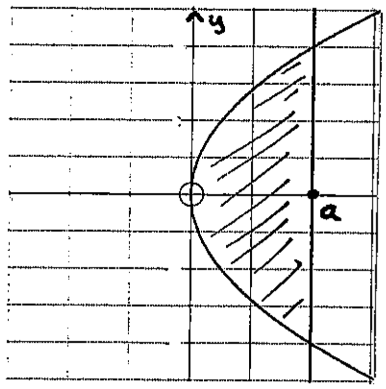
(A) $v = \pm\frac{1}{x}\sqrt{(-3 + 10x - 7x^2)}$

(B) $v = \pm x\sqrt{(-3 + 10x - 7x^2)}$

(C) $v = \pm\frac{1}{x}\sqrt{2(-3 + 10x - 7x^2)}$

(D) $v = \pm x\sqrt{2(-3 + 10x - 7x^2)}$

10. A solid is formed by rotating the region enclosed by the parabola $y^2 = 4ax$, its vertex $(0, 0)$ and the line $x = a$, about the y-axis.



What is the volume of this solid using the method of slicing.

(A) $8\pi a^3$

(B) $\frac{16\pi a^3}{5}$

(C) $\frac{8\pi a^3}{5}$

(D) $4\pi a^3$

End of Section I

Section II

90 marks
Attempt Questions 11 – 16
Allow about 2 hours 45 minutes for this section

Question 11 – Start A New Booklet – (15 marks) **Marks**

a) Evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{3+5\cos x}\,dx$ using the substitution $\tan\left(\frac{x}{2}\right) = t$ 3

b) Let $z = 3 - 2i$ and $w = 1 + \sqrt{2}i$

(i) Find $|z|$ 1

(ii) Express $\frac{w}{z}$ in the form $a + bi$ where a and b are real numbers. 2

c) Find $\int \frac{dx}{20 - 4x + x^2}$ 2

d) (i) Write $z = 1 + i$ in modulus-argument form. 2

(ii) Hence express z^{-4} in the form $x + yi$, where x and y are real. 2

e) The area bounded by the curve $y = \frac{1}{x+1}$, the x-axis, the line $x = 2$ and the line $x = 8$, is rotated about the y-axis.

Find the volume of the solid generated using the method of cylindrical shells. 3

Question 12 – Start A New Booklet – (15 marks)

Marks

a) Find $\int \frac{x}{(1-x)(1+x^2)}\,dx$ 3

b) Let $f(x) = \frac{1-x}{x}$

On <u>separate</u> diagrams sketch the graph of the following functions. For each graph label any asymptotes and critical points.

(i) $y = |f(x)|$ 2

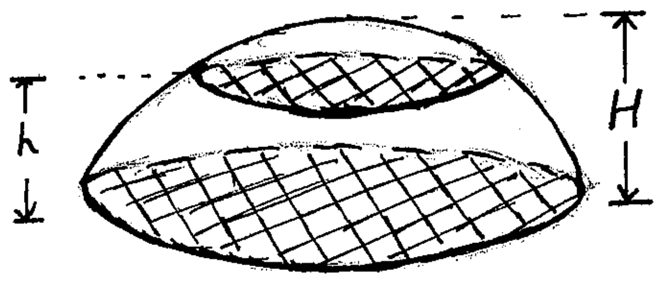
(ii) $y = e^{f(x)}$ 2

(iii) $y^2 = f(x)$ 2

c) (i) Verify that $\int_0^a \sqrt{a^2 - x^2}\,dx = \frac{1}{4}\pi a^2$ 1

(ii) Deduce that the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is πab 2

(iii)



The diagram shows a mound of height H. At height h above the horizontal base, the horizontal cross-section of the mound is an ellipse with equation 3

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \lambda^2 \text{ where } \lambda = 1 - \frac{h^2}{H^2}$$

(x, y are appropriate co-ordinates in the plane of the cross-section).

Show that the volume of the mound is $\frac{8\pi abH}{15}$

Question 13 – Start A New Booklet – (15 marks) Marks

a) On a school camp, one of the girls of mass M kg jumps vertically (feet first) from a rock ledge into a river below.

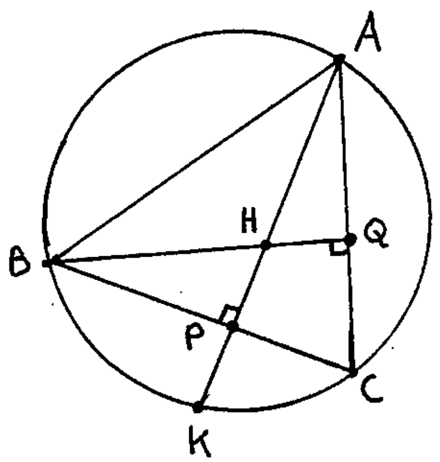
When she is falling at v m/s, she encounters air resistance equal to $\frac{Mv}{20}$ Newtons. She hits the water at a speed of V m/s. Let x be the displacement below the rock ledge at time t seconds after jumping.

(i) Show that $\ddot{x} = g - \frac{v}{20}$, where g is the acceleration due to gravity. 1

(ii) If it takes two seconds for her feet to hit the water, using $g = 10$ m/s^2 show that $V = 200\left(1 - e^{-\frac{1}{10}}\right)$ 3

(iii) Find the height of the rock ledge above the water (to the nearest 0.1 metre) 3

b)



The altitudes AP and BQ of the acute triangle ABC intersect at H. 3

AP produced cuts the circle at K.

Prove that $HP = PK$

Question 13 (cont'd) Marks

c) Show that the equation of the tangent to the ellipse 1

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \ \ (a>b>0)$$

at the point $P(a\cos\theta,\ b\sin\theta)$ is $\dfrac{x}{a}\cos\theta+\dfrac{y}{b}\sin\theta=1$

(i) The tangent at P meets the x-axis at S and the y-axis at T.

Find the area of ΔOST. 2

(ii) If A is the point $(a,0)$ and B is the point $(0,b)$, the area of ΔAPB is $\dfrac{1}{2}ab(\cos\theta+\sin\theta-1)$ [do not prove this]

Prove that, as θ varies in the interval $0<\theta<\dfrac{\pi}{2}$, the area of ΔAPB is a maximum when the tangent to the ellipse is parallel to AB. 2

Question 14 – Start A New Booklet – (15 marks) Marks

a) On an Argand diagram, sketch the locus of the points z such that 2

$$\arg\left\{\frac{(z-1)}{(z+1)}\right\} = \frac{\pi}{2}$$

b) $P\left(cp, \frac{c}{p}\right)$ and $Q\left(cq, \frac{c}{q}\right)$ are two points on the rectangular hyperbola $xy = c^2$.

Given that the equation of the tangent at P is

$$x + p^2 y = 2cp.$$

(i) The tangents at P and Q meet in T. Find the co-ordinates of T in terms of c, p and q. 2

(ii) If T lies on the hyperbola $xy = k^2$ for all positions of P and Q, prove that 1

$$\frac{pq}{(p+q)^2} = \frac{k^2}{4c^2}$$

c) (i) Given $z^5 + 1 = (z+1)(z^4 - z^3 + z^2 - z + 1)$ let w be a solution to $z^5 + 1 = 0$ where $w \neq -1$. Prove that $1 + w^2 + w^4 = w + w^3$ 1

(ii) Hence show that 3

$$\cos\frac{\pi}{5} + \cos\frac{3\pi}{5} = \frac{1}{2}$$

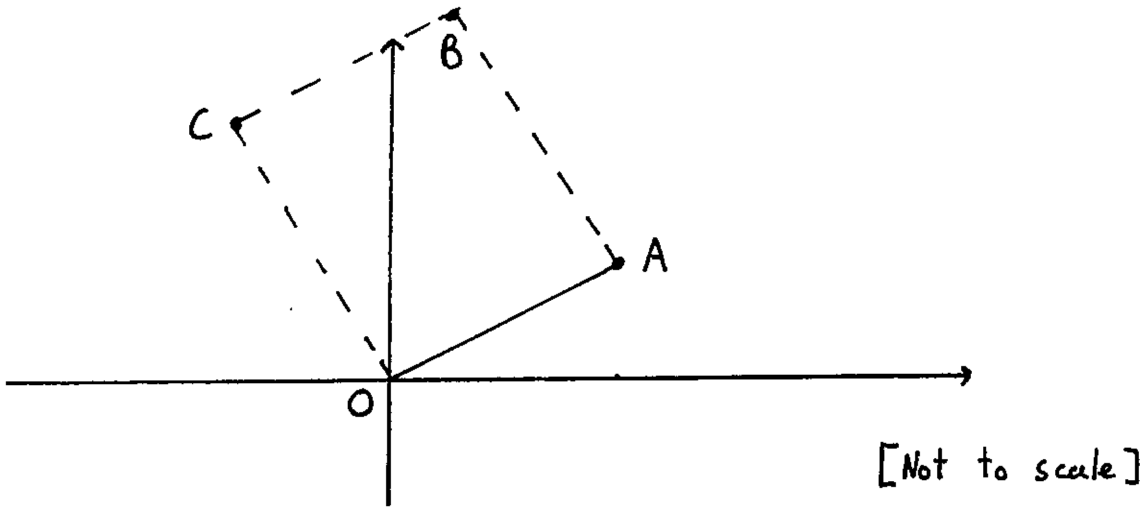
Question 14 (cont'd) Marks

d) (i) If $I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx$ where $n \geq 1$, use integration by parts to prove that 3

$$I_n = \frac{n-1}{n} I_{n-2}$$

(ii) Hence show that $I_5 = \frac{8}{15}$ 1

e) $OABC$ is a rectangle in an Argand diagram where O is the Origin and point A corresponds to the complex number $2 + i$



Given that the length of the rectangle is twice its breadth and OA is one of the shorter sides, find the complex number representing C. 2

Question 15 – Start A New Booklet – (15 marks) Marks

a) Suppose α, β and γ are the roots of the polynomial equation

$$x^3 + x + 12 = 0$$

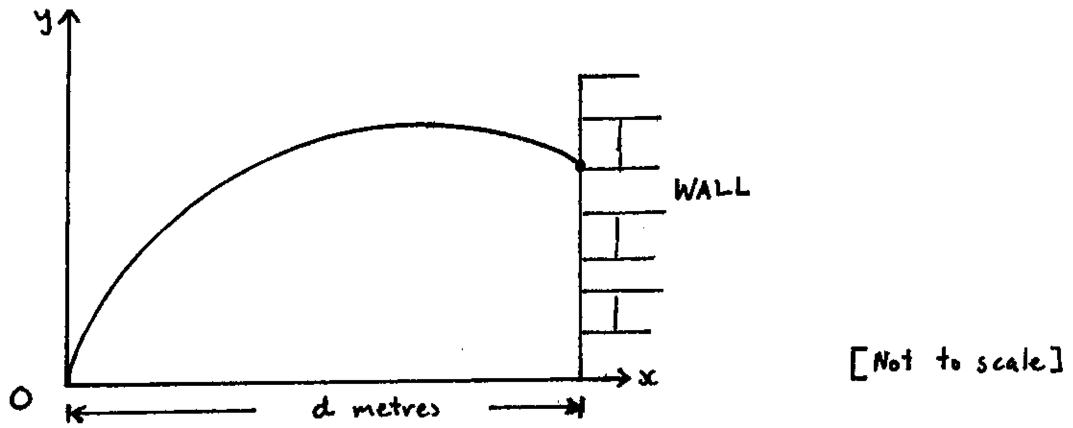
(i) Find $\alpha^2 + \beta^2 + \gamma^2$ 2

(ii) Hence explain why only one of the roots is real. 2

(iii) Let the real root be denoted by α. Prove that $-3 < \alpha < -2$ 1

(iv) Hence prove that the modulus of each of the other roots lies between 2 and $\sqrt{6}$ 3

b) 4



In the diagram above, the wall of a building stands on level ground, d metres from a fire hose located at O If water leaves the hose with velocity $V\text{ms}^{-1}$ at an angle θ to the ground:

Given $V > \sqrt{gd}$, (g is acceleration in the vertical plane due to gravity) show that the particle will strike the wall above ground level provided that $\beta < \theta < \frac{\pi}{2} - \beta$ where $\beta = \frac{1}{2}\sin^{-1}\left(\frac{gd}{V^2}\right)$

You may assume that the range on the horizontal plane from the point of projection is

$$\frac{V^2 \sin 2\theta}{g}$$

c) Find $\int \frac{dx}{x^{\frac{1}{2}} + x^{\frac{1}{3}}}$ 3

[Hint: • choose an appropriate substitution]
• $\frac{x^3}{x+1} = x^2 - x + 1 - \frac{1}{x+1}$]

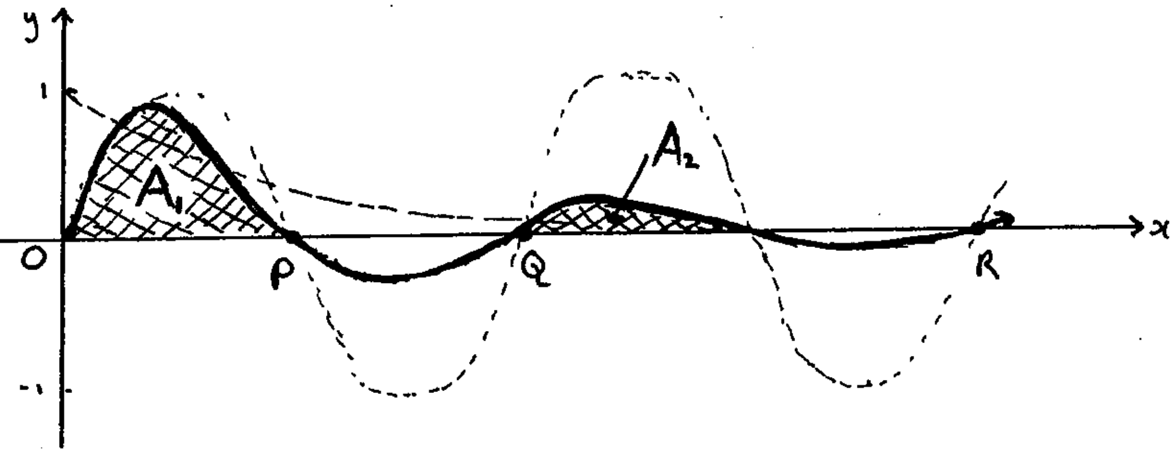
Question 16 – Start A New Booklet – (15 marks) Marks

a) A polynomial of degree n is given by $P(x) = x^n + ax - b$. It is given that the polynomial has a double root at $x = \alpha$.

(i) Find the derived polynomial $P'(x)$ and show that $\alpha^{n-1} = -\frac{a}{n}$ **2**

(ii) Show that $\left(\frac{a}{n}\right)^n + \left(\frac{b}{n-1}\right)^{n-1} = 0$ **4**

b)



The diagram shows a sketch of part of the curve $y = f(x)$ with equation

$$y = e^{-x}.\sin x, \qquad x \geq 0$$

(i) Find the coordinates of the points P, Q and R where $y = f(x)$ cuts the x-axis. **1**

(ii) Integrating by parts gives **3**

$$\int e^{-x}.\sin x\, dx = -\frac{1}{2}e^{-x}(\sin x + \cos x) + C$$

[you do <u>not</u> have to prove this]

If the terms $A_1, A_2, \ldots, A_n$ represent areas between $y = f(x)$ and the x-axis for successive portions of $y = f(x)$ where y is <u>positive</u>. (The areas represented by A_1 and A_2 are shown as the shaded regions in the diagram).

Show that $A_n = \frac{1}{2}\left(e^{(1-2n)\pi} + e^{(2-2n)\pi}\right)$

(iii) Show that $A_1 + A_2 + A_3 + \cdots$ is a geometric series and that **3**

$$S_\infty = \frac{e^{\pi}}{2(e^{\pi}-1)}$$

(iv) Given that $\int_0^\infty e^{-x}.\sin x\, dx = \frac{1}{2}$, find the exact value of $\int_0^\infty |e^{-x}.\sin x|\, dx$ **2**

End of Paper

Student Number: ______________________________ Teacher: ____________________

Year 12 Mathematics Extension 2 Trial HSC Examination 2013

Section I

Multiple-choice Answer Sheet – Questions 1 – 10

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample $2 + 4 =$ (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ☒ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate this by writing the word *correct* and drawing an arrow as follows:

correct

A ☒ B ☒ C ○ D ○

1.	A ○	B ○	C ○	D ○
2.	A ○	B ○	C ○	D ○
3.	A ○	B ○	C ○	D ○
4.	A ○	B ○	C ○	D ○
5.	A ○	B ○	C ○	D ○
6.	A ○	B ○	C ○	D ○
7.	A ○	B ○	C ○	D ○
8.	A ○	B ○	C ○	D ○
9.	A ○	B ○	C ○	D ○
10.	A ○	B ○	C ○	D ○