St George Girls High School

Trial Higher School Certificate Examination

2009

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen
- Write your Student Number on every page
- Begin each question in a new booklet.
- All necessary working must be shown.
- Marks may be deducted for careless or poorly presented work.
- Board-approved calculators may be used.
- A list of standard integrals is included at the end of this paper.
- The mark allocated for each question is listed at the side of the question.

Total Marks -

- Attempt ALL questions.
- All questions are of equal value.

Students are advised that this is a Trial Examination only and does not necessarily reflect the content or format of the Higher School Certificate Examination.

Question 1 – (15 marks) – Start a new booklet Marks

a) Simplify i^{2009} 1

b) (i) Find real numbers x and y such that 2

$$x + iy = \sqrt{24 - 10i}$$

(ii) Solve the quadratic equation 2

$$z^2 + (1 - 3i)z - (8 - i) = 0$$

c) (i) Express $-\sqrt{3} + i$ in modulus-argument form. 2

(ii) Hence express $\left(-\sqrt{3} + i\right)^8$ in the form $a + bi$ where a and b are real numbers (in simplified form). 2

d) On an Argand diagram shade the region containing all points representing complex numbers, z, such that 3

$$2 \leq |z| \leq 3 \text{ and } \frac{-\pi}{3} < \arg z \leq \frac{2\pi}{3}$$

e) On separate diagrams draw a neat sketch of the locus specified by

(i) $\arg(z - 1 + i) = \frac{\pi}{4}$ 1

(ii) $\arg\left(\frac{z-1+i}{z-i}\right) = 0$ 2

Question 2 – (15 marks) – Start a new booklet Marks

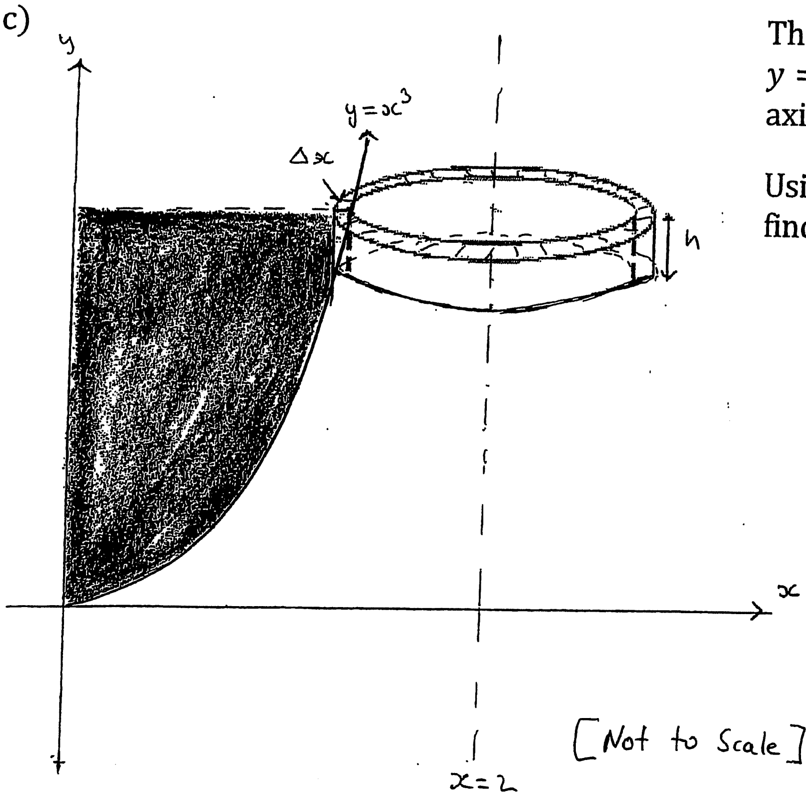
a) Using the substitution $u = \sqrt{x^3+1}$ or otherwise find 3

$$\int_0^2 \frac{x^5}{\sqrt{x^3+1}}\,dx$$

b) By completing the square find 2

$$\int \frac{dx}{\sqrt{7+6x-x^2}}$$

c)



The area enclosed by the curve $y = x^3$, $y = 1$ and the positive y-axis is rotated about the line $x = 2$. 3

Using the method of cylindrical shells find the volume of the solid generated.

d) (i) Show that $\sin(A+B)+\sin(A-B) = 2\sin A\cos B$ 1

(ii) Find all the solutions to the equation 3

$$\sin x + \sin 3x = \cos x$$

e) Use the substitution $t = \tan\dfrac{\theta}{2}$ to find 3

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{1+\cos\theta+3\sin\theta}$$

Question 3 - (15 marks) - Start a new booklet **Marks**

a) The remainder when $x^4 + ax + b$ is divided by $(x+3)(x-2)$ is $x - 3$. Find the values of a and b. 2

b) $z = 1 - i$ is a root of the equation $z^3 + mz^2 + nz + 6 = 0$ where m and n are real. 3

Find the values of m and n.

c) (i) Find the general solution of the equation $\cos 3\theta = \dfrac{1}{2}$ 1

(ii) Show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ 2

(iii) Using the substitution $x = \cos\theta$, and part (ii), express the equation in (i) as a polynomial in terms of x. 1

(iv) Hence, show that $\cos\dfrac{\pi}{9} + \cos\dfrac{5\pi}{9} + \cos\dfrac{7\pi}{9} = 0$ 2

(v) Find the polynomial of least degree that has zeros 2

$$\left(\sec\frac{\pi}{9}\right)^2, \left(\sec\frac{5\pi}{9}\right)^2, \left(\sec\frac{7\pi}{9}\right)^2$$

d) Find: 2

$$\int x.e^{2x}\,dx$$

Question 4 – (15 marks) – Start a new booklet

Marks

a) State whether the following is True or False. Give a brief reason. 1

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan^7\theta \, d\theta > 0$$

[Note: You are not required to find the primitive function]

b) The hyperbola H has equation

$$\frac{x^2}{9} - \frac{y^2}{16} = 1$$

(i) Find the eccentricity of H and hence write down the coordinates of the foci, S and S', and the equations of the directrices. 3

(ii) Write down the equations of the asymptotes of H. 1

(iii) Sketch H, clearly showing the foci, directrices and asymptotes. 2

(iv) $P(3\sec\theta,\ 4\tan\theta)$ is a point on H. Prove that the tangent at P has equation 2

$$\frac{x\sec\theta}{3} - \frac{y\tan\theta}{4} = 1$$

(v) This tangent cuts the asymptotes at A and B. Prove that

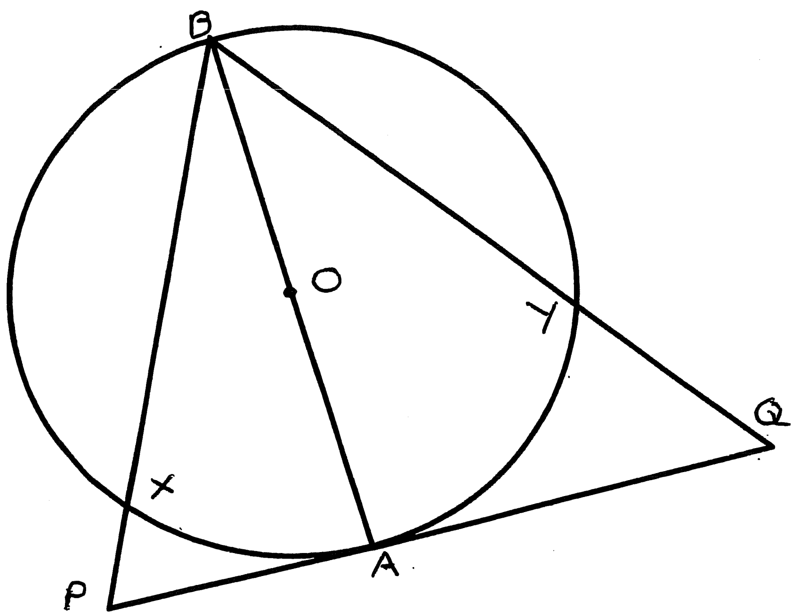
(α) $PA = PB$ and 3

(β) the area of ΔOAB is independent of the position of P on the hyperbola. 3

Question 5 – (15 marks) – Start a new booklet Marks

a) Find the equation of the tangent to the curve $x^3 - 2xy + y^2 = 4$ at the point $(-2,\ 2)$ 2

b)



PAQ is a tangent to the circle with centre O and AB is a diameter. 3

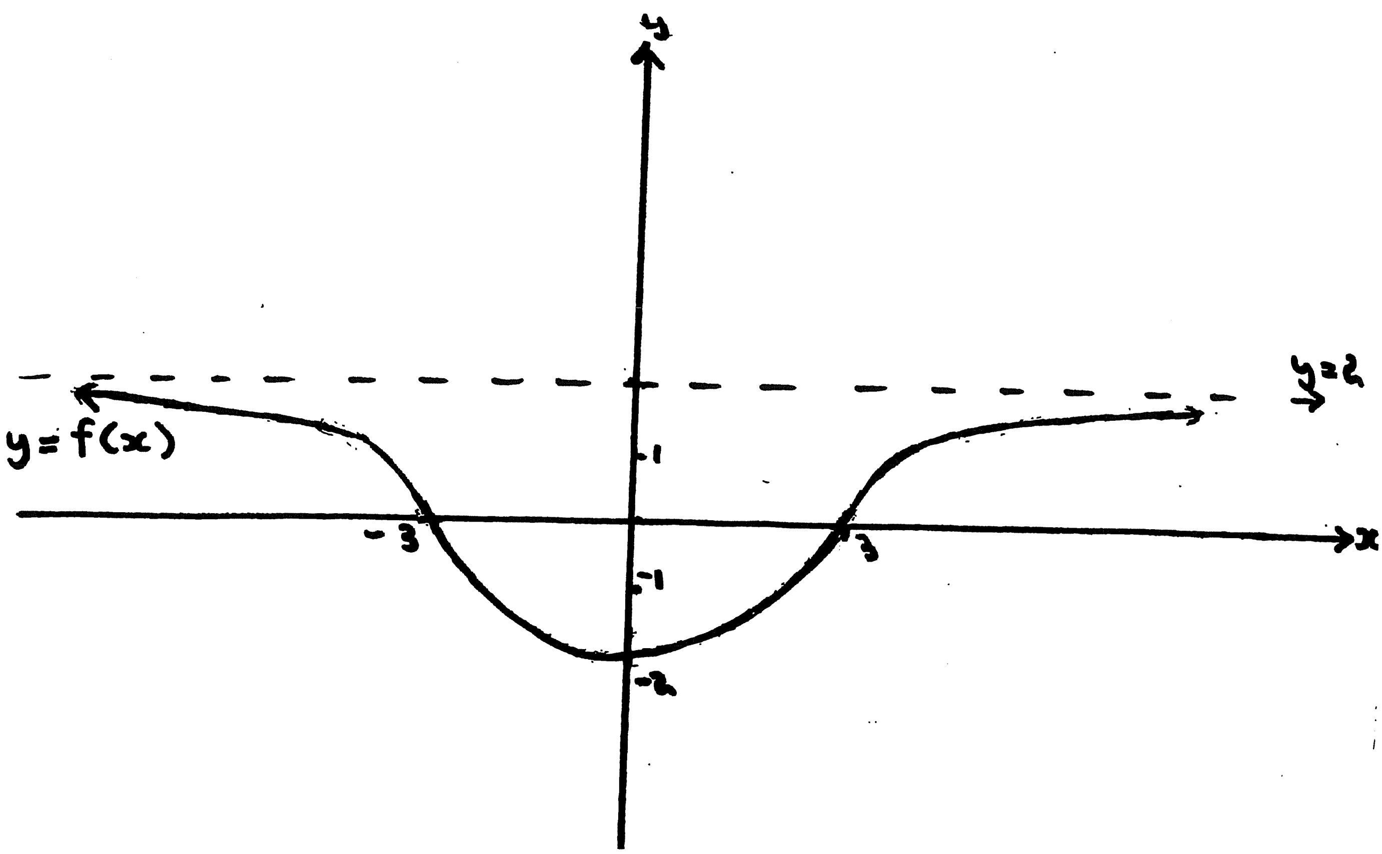
PB cuts the circle at X and QB cuts the circle at Y.

Prove that $PQYX$ is a cyclic quadrilateral.

Question 5 (cont'd)

Marks

c)



The graph of $y = f(x)$ is shown. On the answer sheets provided draw the graphs of the following:

(i) $y = (f(x))^2$ — 2

(ii) $y = |f(x)|$ — 2

(iii) $y^2 = f(x)$ — 2

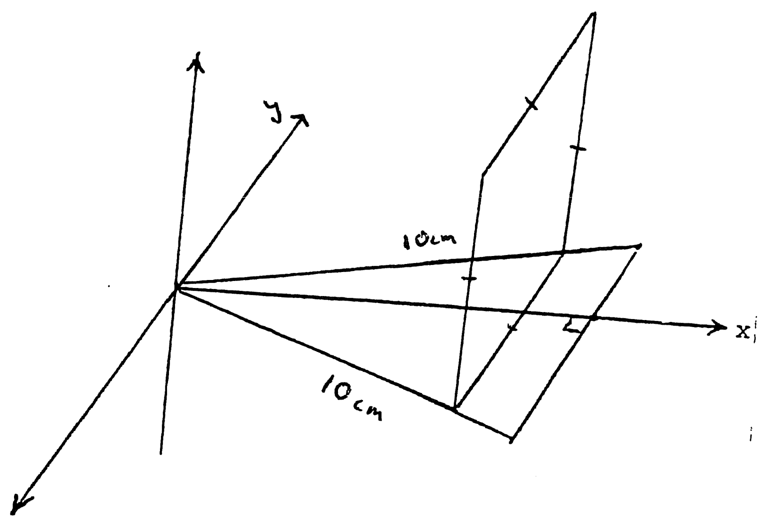
(iv) $y = \dfrac{1}{f(x)}$ — 2

(v) $y = f'(x)$ — 2

Question 6 – (15 marks) – Start a new booklet **Marks**

a) The base of a solid is an equilateral triangle of side length 10 cm, with one vertex at the origin and one side parallel to the y-axis as shown in the diagram.

Each cross-section perpendicular to the x-axis is a square with one side in the base of the solid.



(i) Show that the area of the cross-section x cm from the origin is **2**

$$A(x) = \frac{4x^2}{3}$$

(ii) Hence, find the volume of the solid. **3**

Question 6 (cont'd)

Marks

b) A particle of mass m is projected vertically upwards in a medium where it experiences a resistance of magnitude mkv^2 where k is a positive constant and v is the velocity of the particle.

During the downward motion the terminal velocity of the particle is V. Its initial velocity of projection is $\frac{1}{5}$ of this terminal velocity.

(i) By considering the forces on the particle during its downward motion, show that **2**

$$kV^2 = g$$

(where g is the acceleration due to gravity)

(ii) Show that during its upward motion the acceleration of the particle $\ddot{x}$ is given by **1**

$$\ddot{x} = -g\left(1 + \frac{v^2}{V^2}\right)$$

(iii) If the distance travelled by the particle in its upward motion is x when its velocity is v, show that the maximum height H reached is given by **3**

$$H = \frac{V^2}{2g} \text{In} \left(\frac{26}{25}\right)$$

(iv) If the velocity of the particle is v when it has fallen a distance of y from its maximum height, show that **2**

$$y = \frac{V^2}{2g} \text{In} \left[\frac{V^2}{V^2 - v^2}\right]$$

(v) The velocity of the particle is U when it returns to its point of projection. Show that **2**

$$\frac{V}{U} = \sqrt{26}$$

Question 7 – (15 marks) – Start a new booklet Marks

a) (i) Prove that 2

$$\int_0^a f(a-x)\,dx = \int_0^a f(x)\,dx$$

(ii) Hence evaluate 2

$$I = \int_0^1 \frac{x^{10}}{x^{10} + (1-x)^{10}}\,dx$$

b) If $P(cp, \dfrac{c}{p})$ and $Q(cq, \dfrac{c}{q})$ are two points on the rectangular hyperbola $xy = c^2$

(i) Show that the equation of the chord PQ is 2

$$x + pqy = c(p+q)$$

(ii) If the chord passes through the point $R(a,\ b)$ prove that the locus of the mid point of the chord is given by 3

$$2xy = ay + bx$$

c) (i) Use induction to prove that 3

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

for positive integers $n \geq 1$

(ii) Hence, or otherwise, find 3

$$2^2 + 5^2 + 8^2 + \cdots + (3n-1)^2$$

Question 8 – (15 marks) – Start a new booklet

Marks

a) ADB is a straight line with $AD = a$ and $DB = b$. A circle is drawn with AB as diameter. DC is drawn perpendicular to AB and meets the circle at C.

(i) By using similar triangles show that $DC = \sqrt{ab}$. **2**

(ii) Deduce geometrically that if a and b are positive real numbers then **1**

$$\sqrt{ab} \leq \frac{a+b}{2}$$

(iii) Using (ii), or otherwise, prove that if x, y, z are positive real numbers then **2**

$$(x+y)(y+z)(z+x) \geq 8xyz$$

b) For a certain series the nth term is given by

$$T_n = x^{n-1}(1 + x + x^2 + \cdots + x^{n-1})$$

(i) Show that S_n, the sum to n terms, of this series is given by **3**

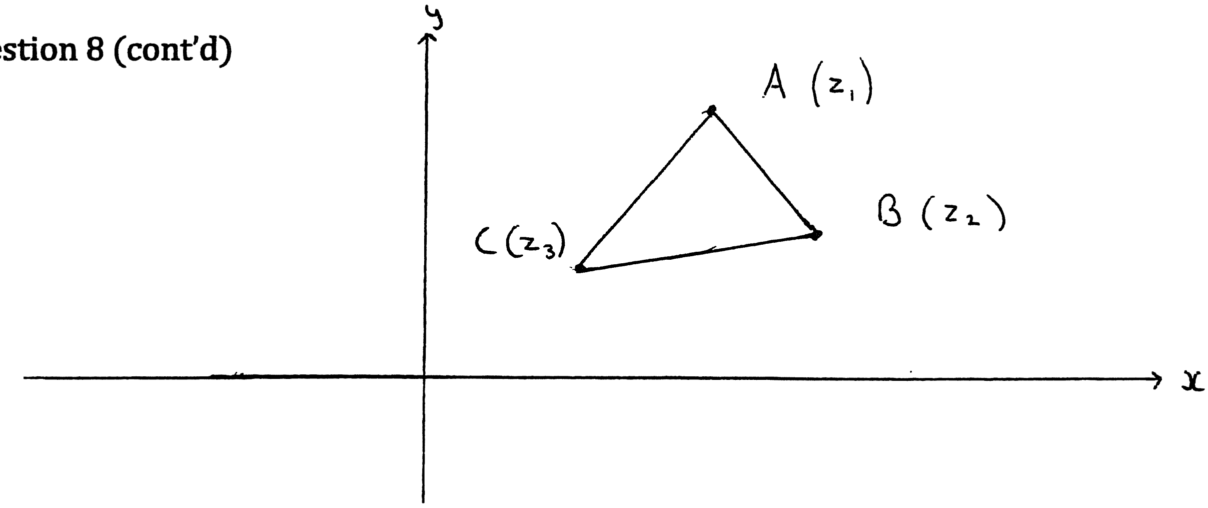
$$S_n = \frac{(1-x^n)(1-x^{n+1})}{(1-x)(1-x^2)} \qquad \text{provided } x^2 \neq 1$$

(ii) Deduce that **2**

$$\lim_{x \to 1} \frac{(1-x^n)(1-x^{n+1})}{(1-x)(1-x^2)} = \frac{1}{2}n(n+1)$$

Question 8 (cont'd) **Marks**

c) 5



A, B and C are the points that represent the complex numbers $z_1,\ z_2,\ z_3$ on the Argand diagram

Prove that if

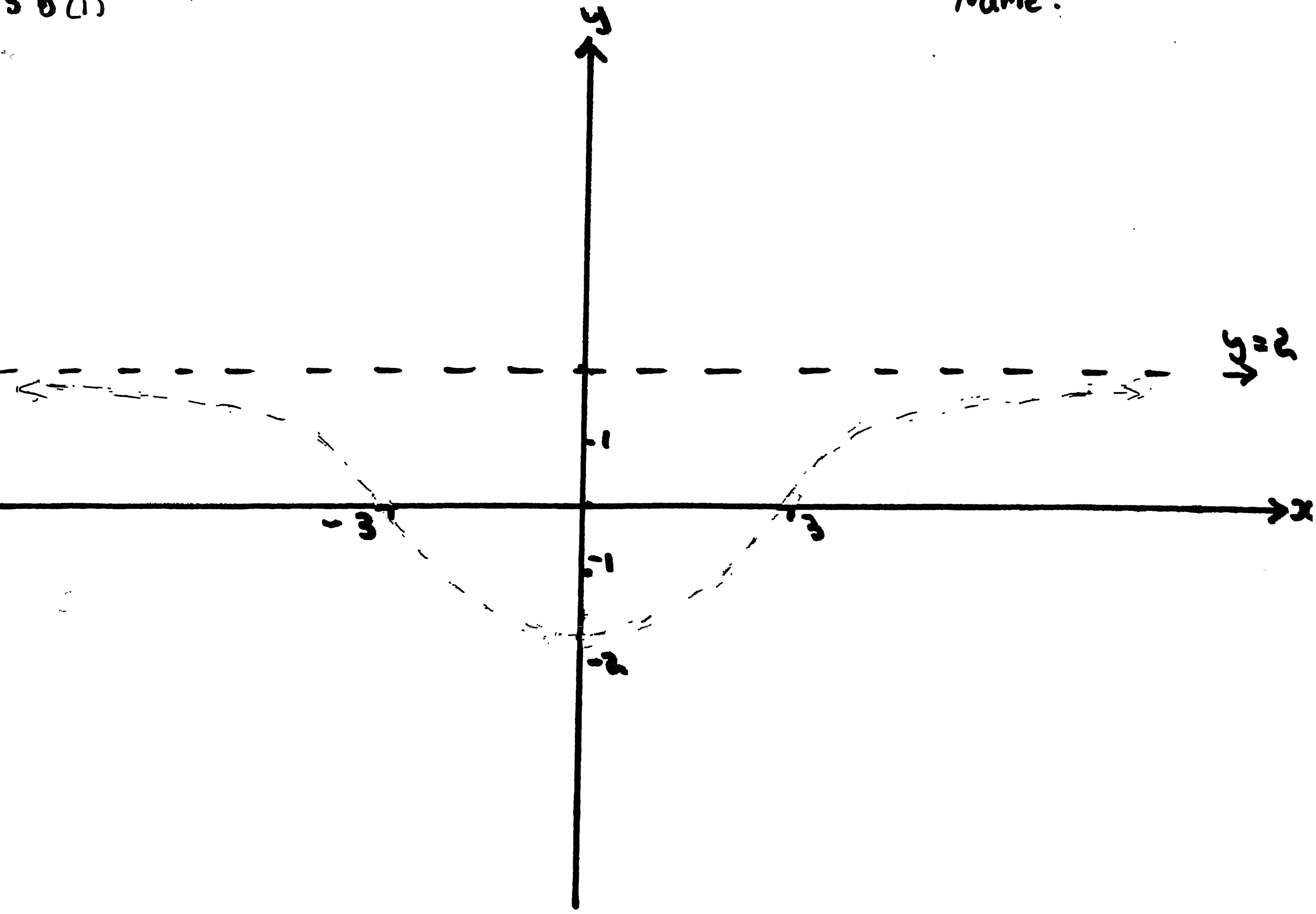
$$\frac{z_2 - z_3}{z_1 - z_3} = \frac{z_1 - z_3}{z_1 - z_2}$$

then ΔABC is equilateral.

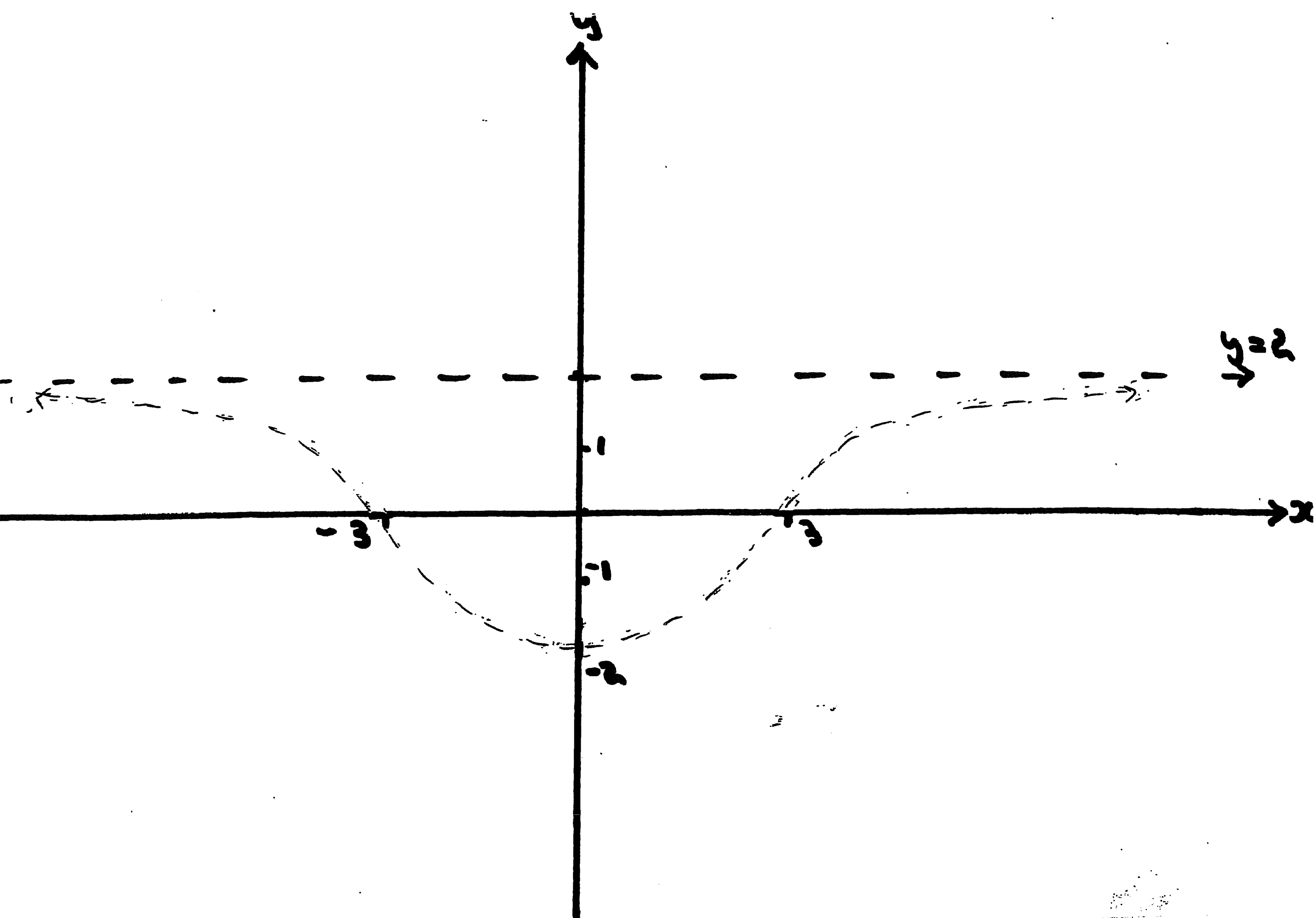
End of Paper

5 b (i)

Name :



5 b (ii)



5b (iii)

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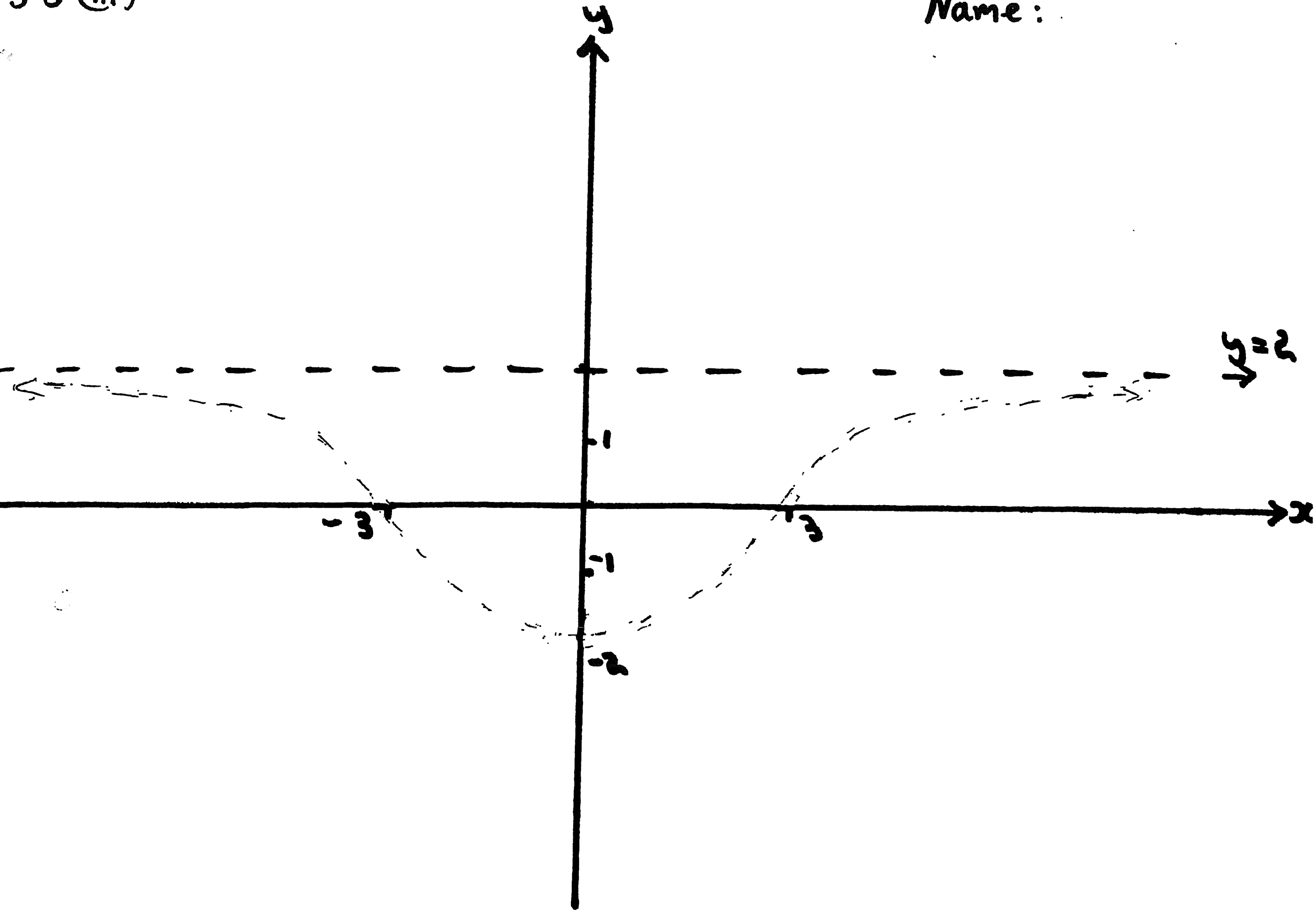
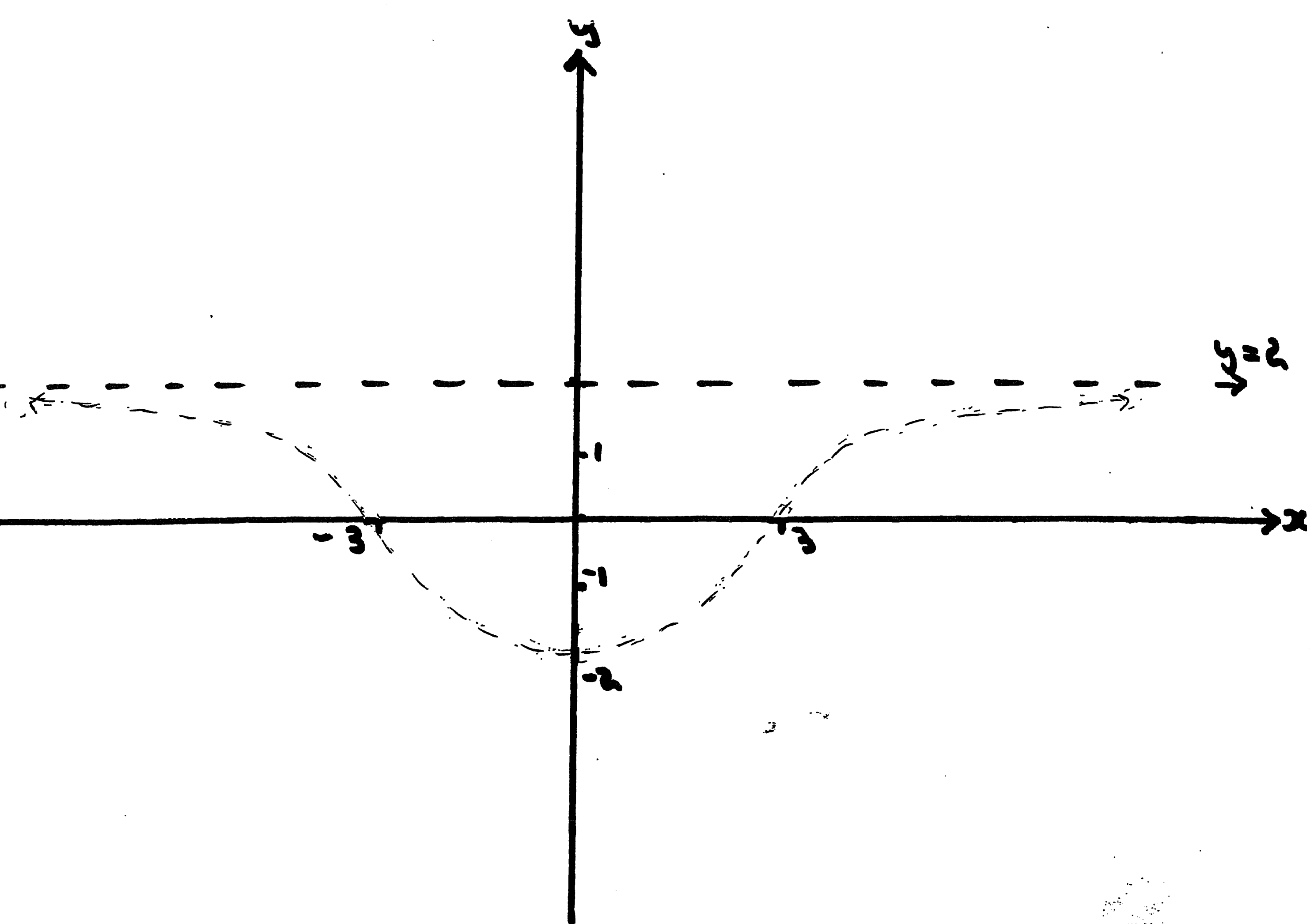


5b (iv)



b (v)

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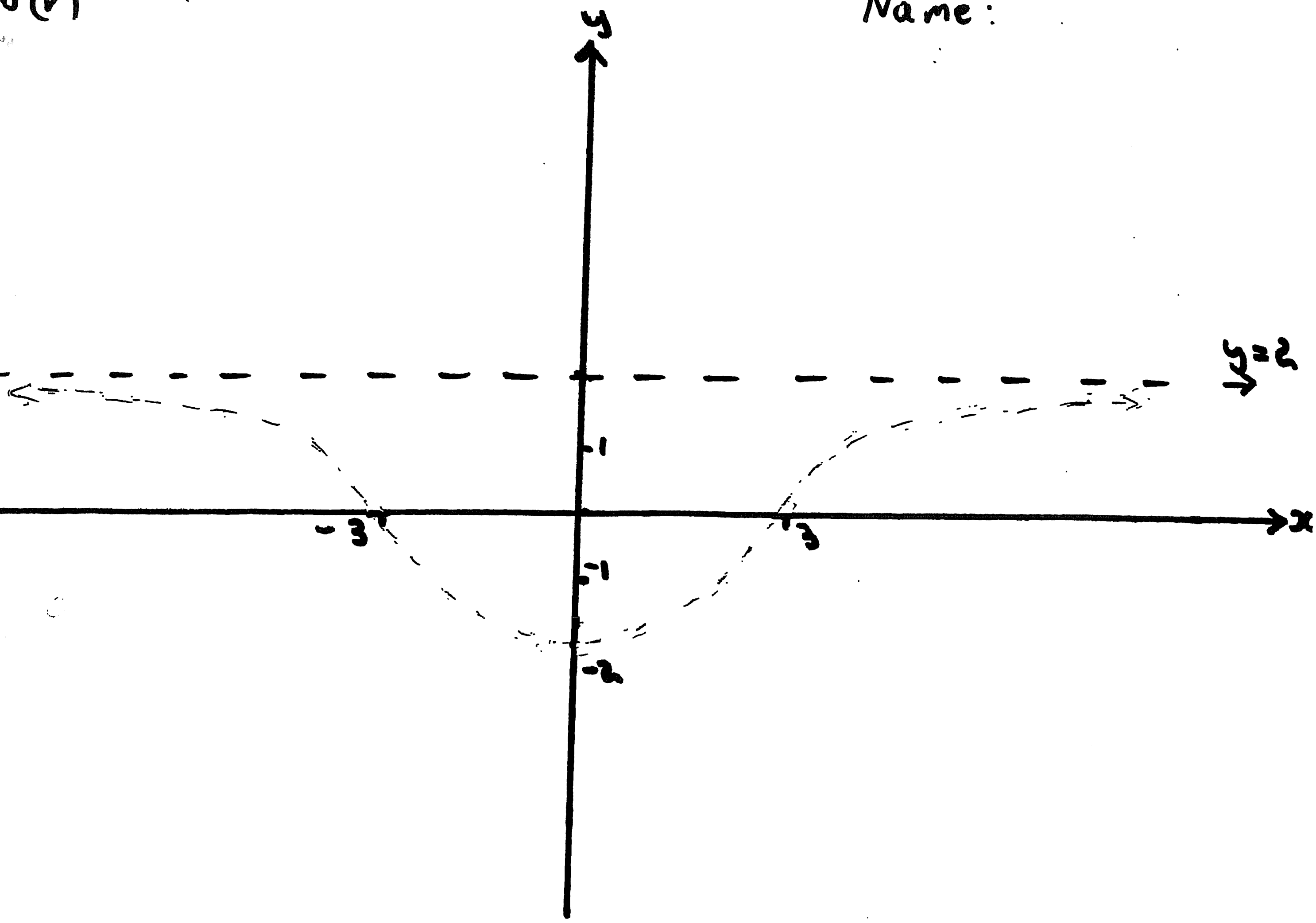


Trial HSC Mathematics Extension 2 2009 Solutions

Question 1

(a) $i^{2009} = (i^4)^{502} \times i$

$= 1^{502} \times i$

$= i$ (1)

(b)(i) $(x+iy)^2 = 24-10i$

$x^2 - y^2 = 24$ ①

$2xyi = -10i$

$xy = -5$

$y = -\frac{5}{x}$ ②

Subst ② in ①

$x^2 - \frac{25}{x^2} = 24$

$x^4 - 24x^2 - 25 = 0$ — (1)

$(x^2-25)(x^2+1) = 0$

$(x-5)(x+5)(x^2+1) = 0$

$x = 5, -5 \quad (x \in \mathbb{R})$

$y = -1, 1$

$\sqrt{24-10i} = \pm(5-i)$ — (1)

(ii) $z^2 + (1-3i)z - (8-i) = 0$

$\Delta = (1-3i)^2 - 4 \times 1 \times -(8-i)$

$= 1 - 6i + 9i^2 + 32 - 4i$

$= 24 - 10i$

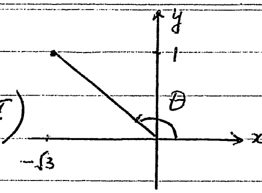
$z = \frac{-(1-3i) \pm \sqrt{24-10i}}{2}$ — (1)

$= \frac{-1+3i \pm (5-i)}{2}$

$= \frac{4+2i}{2}, \frac{-6+4i}{2} = 2+i, -3+2i$ — (1)

(c)(i) $-\sqrt{3} + i$

$= 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$



$|-\sqrt{3}+i|^2 = (\sqrt{3})^2 + 1^2$

$= 4$

$|-\sqrt{3}+i| = 2$ — (1)

$\arg(-\sqrt{3}+i) = \theta$

$\tan\theta = -\frac{1}{\sqrt{3}}$

$\therefore \theta = \frac{5\pi}{6}$ — (1)

(ii) $(-\sqrt{3}+i)^8 = 2^8\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)^8$

$= 256\left(\cos\frac{40\pi}{6} + i\sin\frac{40\pi}{6}\right)$

$\frac{40\pi}{6} = \frac{20\pi}{3}$

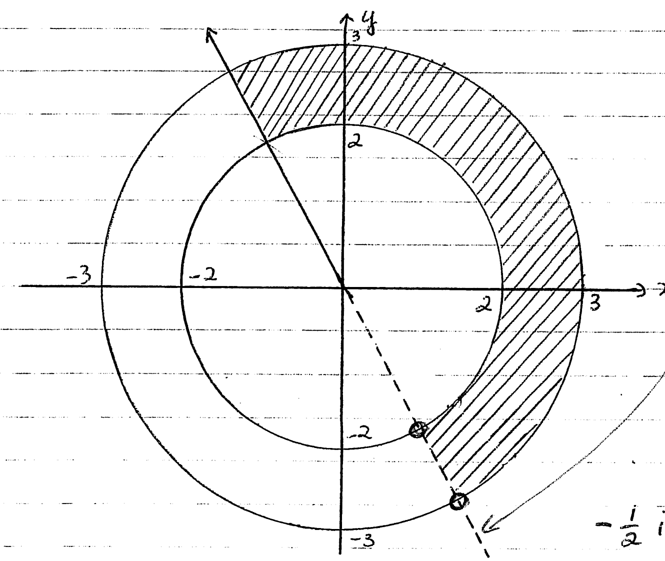
$= 6\pi + \frac{2\pi}{3}$

$= 256\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$ — (1)

$= 256\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

$= -128 + 128\sqrt{3}\,i$ — (1)

(d)



(3)

1 for line (dotted & solid)

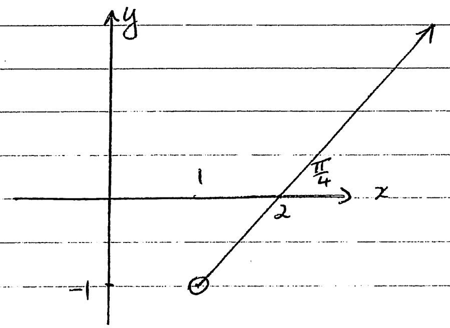
1 for annulus

1 for intersection

$-\frac{1}{2}$ if open circles missing

(e) (i) $\arg(z-1+i) = \frac{\pi}{4}$

$\arg(z-(1-i)) = \frac{\pi}{4}$



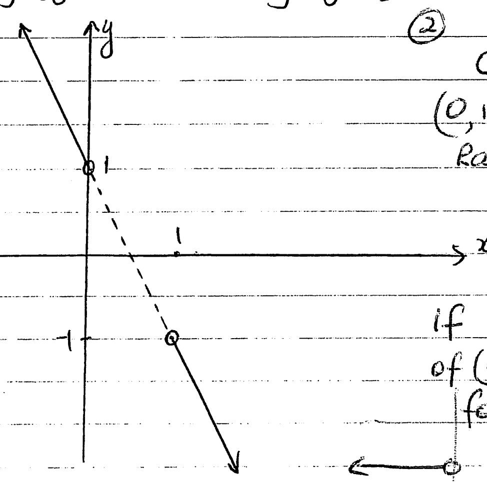
1

(ii) $\arg\left(\frac{z-1+i}{z-i}\right) = 0$

$\arg(z-(1-i)) - \arg(z-i) = 0$

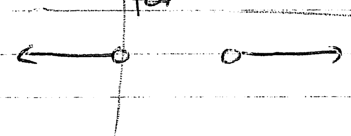
$\arg(z-(1-i)) = \arg(z-i)$

2



Open circles at $(0,1)$ and $(1,-1)$
Rays as shown

if used $(0,-1)$ instead of $(0,1)$ could get 1 for



Question 2

(a) $\int_0^2 \frac{x^5}{\sqrt{x^3+1}}\,dx$

$u = (x^3+1)^{1/2}$

$du = \frac{1}{2}(x^3+1)^{-1/2} \cdot 3x^2\,dx$

$\frac12$ $= \frac{3x^2}{2\sqrt{x^3+1}}\,dx$

When $x = 0$, $u = 1$

$x = 2$, $u = 3$ $\frac12$

$\frac12$ $= \frac{2}{3}\int_0^2 \frac{x^3 \cdot 3x^2}{2\sqrt{x^3+1}}\,dx$ $\frac12$

$= \frac{2}{3}\int_1^3 u^2 - 1\,du$ $\frac12$

$= \frac{2}{3}\left[\frac{u^3}{3} - u\right]_1^3$

$= \frac{2}{3}\left\{\left(\frac{27}{3} - 3\right) - \left(\frac{1}{3} - 1\right)\right\}$

$= \frac{40}{9}$ $\frac12$

$\boxed{2}$

OR $\int_0^2 \frac{x^5}{\sqrt{x^3+1}}\,dx$

$u = \sqrt{x^3+1}$

$u^2 = x^3+1$ $\frac12$

$2u\,du = 3x^2\,dx$

$x = 0$, $u = 1$

$x = 2$, $u = 3$ $\frac12$

$\frac12$ $= \frac{1}{3}\int_0^2 \frac{x^3 \cdot 3x^2\,dx}{\sqrt{x^3+1}}$ $\frac12$

$= \frac{1}{3}\int_1^3 \frac{(u^2-1) \cdot 2u\,du}{u}$

$\boxed{2}$

$= \frac{2}{3}\int_1^3 u^2 - 1\,du$ (then as above) $\frac12$ $\frac12$

(b) $7 + 6x - x^2 = 7 - (x^2 - 6x + 9 - 9)$

$= 16 - (x-3)^2$

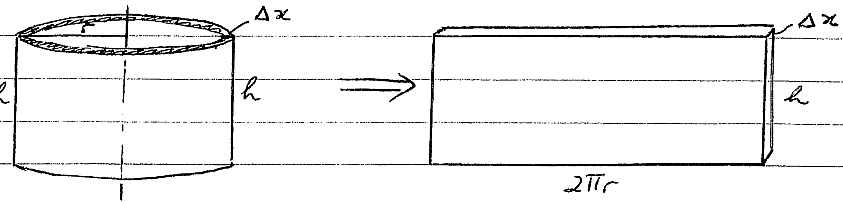
* Completing Square poorly done.

1

$\int \frac{dx}{\sqrt{16-(x-3)^2}} = \sin^{-1}\frac{(x-3)}{4} + C$

1

(c)



Volume of cylindrical shell $= \Delta V$

$$\Delta V \doteqdot 2\pi r h \,\Delta x \qquad h = 1 - y = 1 - x^3$$

$$= 2\pi(2-x)(1-x^3)\,\Delta x \qquad r = 2 - x$$

(½) (½) (½)

* Most missed part was $(1-y)$ for height.

$$V = \lim_{\Delta x \to 0} \sum_{x=0}^{1} \Delta V$$

$$= \lim_{\Delta x \to 0} \sum_{x=0}^{1} 2\pi(2 - x - 2x^3 + x^4)\,\Delta x \quad (½)$$

$$= 2\pi \int_0^1 2 - x - 2x^3 + x^4 \, dx$$

$$= 2\pi\left[2x - \frac{x^2}{2} - \frac{2x^4}{4} + \frac{x^5}{5}\right]_0^1 \quad (½)$$

$\boxed{3}$

$$= 2\pi\left\{\left(2 - \frac{1}{2} - \frac{1}{2} + \frac{1}{5}\right) - 0\right\}$$

$$= 2\pi \times \frac{6}{5}$$

$$\text{Volume} = \frac{12\pi}{5} \text{ units}^3 \quad (½)$$

(d) (i) $\sin(A+B) + \sin(A-B) = \sin A\cos B + \cos A \sin B + \sin A \cos B - \cos A \sin B$

$= 2\sin A \cos B$ $\boxed{1}$

(ii) $\sin x + \sin 3x$

$= 2\sin 2x \cos x$

Let $A + B = 3x$

$A - B = x$

$2A = 4x \quad A = 2x$

$2B = 2x \quad B = x$

(½) $A = 2x \quad B = x$

$\sin x + \sin 3x = \cos x$ * (−½) not general solns

$2\sin 2x \cos x - \cos x = 0$ (½) * Ugghh! Cannot divide by $\cos\theta$!!! Lost solution $\boxed{3}$

$\cos x(2\sin 2x - 1) = 0$

$\cos x = 0$ or $\sin 2x = \frac{1}{2}$ (½) for 2 solns

$x = (2k+1)\frac{\pi}{2}$ $\qquad 2x = \frac{\pi}{6} + 2k\pi$ or $\pi - \frac{\pi}{6} + 2k\pi$

$k \in \mathbb{Z}$ (½) $\qquad x = \frac{\pi}{12} + k\pi$ (½) or $\frac{5\pi}{12} + k\pi$ (½)

$$x = (2k+1)\frac{\pi}{2},\ \frac{\pi}{12} + k\pi,\ \frac{5\pi}{12} + k\pi \qquad k \in \mathbb{Z}$$

(e) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{d\theta}{1 + \cos\theta + 3\sin\theta}$

$t = \tan\frac{\theta}{2}$

$\theta = 2\tan^{-1} t$

$d\theta = \frac{2}{1+t^2}\, dt$ (½)

When $\theta = 0 \quad t = 0$

$\theta = \frac{\pi}{2} \quad t = 1$ (½)

$1 + \cos\theta + 3\sin\theta$

$= 1 + \frac{1-t^2}{1+t^2}$ (½) $+ \frac{3 \times 2t}{1+t^2}$ (½)

$$= \int_0^1 \frac{1}{1 + \frac{1-t^2}{1+t^2} + \frac{6t}{1+t^2}} \cdot \frac{2}{1+t^2}\, dt$$

$$= \int_0^1 \frac{2}{1 + t^2 + 1 - t^2 + 6t}\, dt$$

$$= \int_0^1 \frac{2}{2 + 6t}\, dt$$

$$= \int_0^1 \frac{1}{1+3t}\, dt \quad (½)$$

$$= \left[\frac{1}{3}\ln(1+3t)\right]_0^1$$

* (−1) if carried error made integral easier $\boxed{3}$

$$= \frac{1}{3}(\ln 4 - \ln 1)$$

$$= \frac{\ln 4}{3}$$

$$\left(= \frac{2\ln 2}{3}\right) \quad (½)$$

Question 3

(a) $x^4 + ax + b = (x+3)(x-2)Q(x) + (x-3)$

Subst $x = -3$ | Subst $x = 2$

$81 - 3a + b = -6$ | $16 + 2a + b = -1$ → 1

$3a - b = 87$ ① | $2a + b = -17$

$2a + b = -17$ ②

①+② $5a = 70$ ($\frac{1}{2}$ off for errors, minor)

$a = 14$

Subst in ① $42 - b = 87$ → 1

$b = -45$

(b) $z^3 + mz^2 + nz + 6 = 0$ has $z = 1 - i$ as a root

$(1-i)^2 = 1 - 2i + i^2 = -2i$

$(1-i)^3 = (1-i) \times -2i = -2 - 2i$

$\therefore -2 - 2i + m(-2i) + n(1-i) + 6 = 0$ → 1

$-2 + n + 6 + i(-2 - 2m - n) = 0$

Equating real and imaginary parts:

$n + 4 = 0$

$n = -4$ → 1

$-2 - 2m - n = 0$

$-2 - 2m + 4 = 0$

$2m = 2$ → 1

$m = 1$

OR Since the coefficients are real $1+i$ is also a root

Let the 3rd root be β

$(1-i)(1+i)\beta = -6$ → 1

$2\beta = -6$

$-m$ = sum of roots

$= 1 - i + 1 + i + -3$ → 1

$= -1$

$m = 1$

n = sum in pairs

$= (1-i)(1+i) + -3(1-i) + -3(1+i)$

$= 2 - 3(1 - i + 1 + i)$ → 1

$= 2 - 6$

$= -4$

(c) (i) $\cos 3\theta = \frac{1}{2}$

$3\theta = \frac{\pi}{3} + 2k\pi, \ -\frac{\pi}{3} + 2k\pi$ → 1

$\theta = \frac{2k\pi}{3} \pm \frac{\pi}{9}$ $\quad \frac{\pi}{9}(6k \pm 1)$

(ii) $\cos 3\theta = \cos(2\theta + \theta)$

$= \cos 2\theta \cos\theta - \sin 2\theta \sin\theta$

$= (2\cos^2\theta - 1)\cos\theta - 2\sin\theta\cos\theta . \sin\theta$ → 1

$= 2\cos^3\theta - \cos\theta - 2(1 - \cos^2\theta)\cos\theta$

$= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta$ → 1

$= 4\cos^3\theta - 3\cos\theta$

(iii) $4\cos^3\theta - 3\cos\theta = \frac{1}{2}$ $\quad x = \cos\theta$

$4x^3 - 3x = \frac{1}{2}$ → 1

$8x^3 - 6x - 1 = 0$

(iv) Roots of this cubic equation are

$x = \cos\theta$ where $\theta = \frac{\pi}{9}, \frac{7\pi}{9}, \frac{5\pi}{9}$

Note: $\theta = \frac{2k\pi}{3} + \frac{\pi}{9}$ gives $\frac{\pi}{9}, \frac{7\pi}{9}, \frac{13\pi}{9}, \ldots$ $k=0,1,2$

$\theta = \frac{2k\pi}{3} - \frac{\pi}{9}$ gives $-\frac{\pi}{9}, \frac{5\pi}{9}, \frac{11\pi}{9}, \ldots$ $k=0,1,2$

$\cos\left(\frac{\pi}{9}\right) = \cos\left(-\frac{\pi}{9}\right)$; $\cos\frac{7\pi}{9} = \cos\frac{11\pi}{9}$; $\cos\frac{13\pi}{9} = \cos\frac{5\pi}{9}$ → 1

$\therefore$ Roots are $\cos\frac{\pi}{9}, \cos\frac{5\pi}{9}, \cos\frac{7\pi}{9}$

Sum of roots $= \frac{-\text{coeff } x^2}{\text{coeff } x^3}$

$= 0$

$\therefore \cos\frac{\pi}{9} + \cos\frac{5\pi}{9} + \cos\frac{7\pi}{9} = 0$ → 1

(v) Let α, β, γ be the roots of $8x^3 - 6x - 1 = 0$
Require the polynomial with roots $\frac{1}{\alpha^2}, \frac{1}{\beta^2}, \frac{1}{\gamma^2}$

Let $P(x) = 8x^3 - 6x - 1$
Required equation is
$P\left(\frac{1}{\sqrt{x}}\right) = 0$ → 1

$8\left(\frac{1}{\sqrt{x}}\right)^3 - 6 \times \frac{1}{\sqrt{x}} - 1 = 0$

$\frac{8}{x\sqrt{x}} - \frac{6}{\sqrt{x}} - 1 = 0$

$8 - 6x - x\sqrt{x} = 0$

$8 - 6x = x\sqrt{x}$

$64 - 96x + 36x^2 = x^3$

$x^3 - 36x^2 + 96x - 64 = 0$ → 1

(d) $\int x e^{2x}\,dx = \int x \cdot \frac{d}{dx}\left(\frac{1}{2}e^{2x}\right) dx$

$= \frac{1}{2}xe^{2x} - \int 1 \cdot \frac{1}{2}e^{2x}\,dx$ → 1

$= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + c$ → 1

Question 4

(a) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan^7\theta\,d\theta > 0$ False $\frac{1}{2}$

$f(\theta) = (\tan\theta)^7$ is an odd function

$f(-\theta) = (\tan(-\theta))^7$

$= (-\tan\theta)^7$

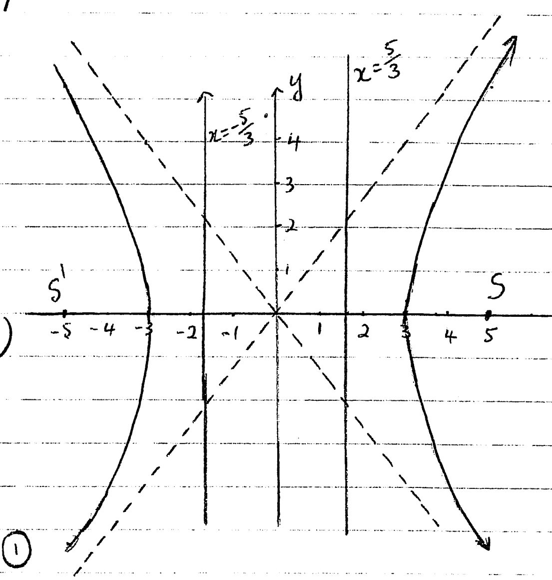
$= -(\tan\theta)^7$

$= -f(\theta)$ $\frac{1}{2}$

Hence $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan^7\theta\,d\theta = 0$

$\boxed{1}$

(b) $\frac{x^2}{9} - \frac{y^2}{16} = 1$



(i) $b^2 = a^2(e^2 - 1)$

$16 = 9(e^2 - 1)$

$e^2 = \frac{16}{9} + 1$

$= \frac{25}{9}$

1 $e = \frac{5}{3}$ $(e > 0)$

$ae = 3 \times \frac{5}{3} = 5$

$\frac{a}{e} = \frac{3}{5/3} = \frac{9}{5}$

$S(5, 0)$ $S'(-5, 0)$ 1

$\boxed{3}$

Directrices : $x = \pm\frac{5}{3}$ 1

(ii) $y = \pm\frac{4}{3}x$ 1

$\boxed{1}$

* Write equation of directrix & assymptote.

* Very poorly done. Learn basics - check difference between Ellipse + Hyperbola going ⊂⊃ ( ) + ⟩ ⟨

(iv) $P(3\sec\theta, 4\tan\theta)$

$$\frac{x^2}{9} - \frac{y^2}{16} = 1$$

$$\frac{2x}{9} - \frac{2y}{16}\frac{dy}{dx} = 0$$

$$\frac{2x}{9} = \frac{2y}{16}\frac{dy}{dx}$$

$$\frac{2x}{9} \times \frac{16}{2y} = \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{16x}{9y} \quad \frac{1}{2}$$

* If answer in $\sec\theta$ + $\tan\theta$ leave working in $\sec\theta$ + $\tan\theta$.

At P $$\frac{dy}{dx} = \frac{16 . 3\sec\theta}{9 . 4\tan\theta}$$

$$= \frac{4\sec\theta}{3\tan\theta} \quad \frac{1}{2}$$

* Practise such questions — marks thrown away. (and for part(ii))

Eq^n of tangent is

$$y - 4\tan\theta = \frac{4\sec\theta}{3\tan\theta}(x - 3\sec\theta) \quad \frac{1}{2} \quad \left(\times \frac{\tan\theta}{4}\right)$$

$$\frac{y\tan\theta}{4} - \tan^2\theta = \frac{x\sec\theta}{3} - \sec^2\theta$$

$\boxed{2}$

$$\sec^2\theta - \tan^2\theta = \frac{x\sec\theta}{3} - \frac{y\tan\theta}{4} \quad \frac{1}{2}$$

$$\frac{x\sec\theta}{3} - \frac{y\tan\theta}{4} = 1$$

(v) When $y = \frac{4}{3}x$: $$\frac{x\sec\theta}{3} - \frac{4x}{3}.\frac{\tan\theta}{4} = 1$$

$$\frac{x}{3}(\sec\theta - \tan\theta) = 1$$

$$x = \frac{3}{\sec\theta - \tan\theta} \quad \frac{1}{2}$$

$$y = \frac{4}{3} . \frac{3}{\sec\theta - \tan\theta}$$

$$= \frac{4}{\sec\theta - \tan\theta} \quad \frac{1}{2}$$

* Algebra poor.

A has coords $\left(\frac{3}{\sec\theta - \tan\theta}, \frac{4}{\sec\theta - \tan\theta}\right)$

When $y = -\frac{4}{3}x$ $$\frac{x\sec\theta}{3} + \frac{4x}{3}.\frac{\tan\theta}{4} = 1$$

$$\frac{x}{3}(\sec\theta + \tan\theta) = 1$$

$$x = \frac{3}{\sec\theta + \tan\theta} \quad \frac{1}{2}$$

$$y = \frac{-4}{\sec\theta + \tan\theta} \quad \frac{1}{2}$$

B has coords $\left(\frac{3}{\sec\theta + \tan\theta}, \frac{-4}{\sec\theta + \tan\theta}\right)$

(a) Midpt of AB is :

$$x = \frac{1}{2}\left(\frac{3}{\sec\theta + \tan\theta} + \frac{3}{\sec\theta - \tan\theta}\right)$$

$$= \frac{3(\sec\theta - \tan\theta) + 3(\sec\theta + \tan\theta)}{2(\sec^2\theta - \tan^2\theta)}$$

$$= \frac{6\sec\theta}{2} \quad \frac{1}{2}$$

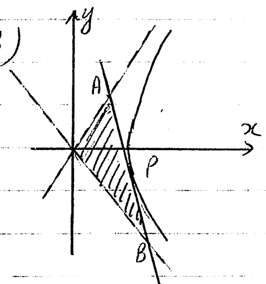
$$= 3\sec\theta$$

$$y = \frac{1}{2}\left(\frac{-4}{\sec\theta+\tan\theta} + \frac{4}{\sec\theta-\tan\theta}\right)$$
$$= \frac{1}{2}\left(\frac{-4\sec\theta+4\tan\theta+4\sec\theta+4\tan\theta}{\sec^2\theta-\tan^2\theta}\right)$$
$$= \frac{8\tan\theta}{2\times 1}$$
$$= 4\tan\theta \quad \frac{1}{2} \qquad \boxed{3}$$

$\therefore$ Midpt of $AB = (3\sec\theta, 4\tan\theta) = P$

ie P is the midpoint of AB ie $AP = BP$

(β)



Area $\Delta AOB = \frac{1}{2}\times OA\times OB\times \sin A\hat{O}B$

$A\hat{O}B = 2\times A\hat{O}x$

$= 2\theta \quad \frac{1}{2}$ where $\tan\theta = \frac{4}{3}$

$\sin A\hat{O}B = 2\sin\theta\cos\theta \qquad \sin\theta = \frac{4}{5}$

$= 2\times\frac{4}{5}\times\frac{3}{5} \qquad \cos\theta = \frac{3}{5}$

$= \frac{24}{25} \quad \frac{1}{2}$

* Also done by $A=\frac{1}{2}bh$ using perpendicular distance formula.

* Not necessary to calculate $2\tan^{-1}\left(\frac{4}{3}\right)=\theta$

$$OA^2 = \frac{9}{(\sec\theta-\tan\theta)^2} + \frac{16}{(\sec\theta-\tan\theta)^2} \qquad OB^2 = \frac{9+16}{(\sec\theta+\tan\theta)^2}$$
$$= \frac{25}{(\sec\theta-\tan\theta)^2} \qquad \boxed{3}$$
$$OA = \frac{5}{|\sec\theta-\tan\theta|} \quad \frac{1}{2} \qquad OB = \frac{5}{|\sec\theta+\tan\theta|} \quad \frac{1}{2}$$

$$\text{Area } \Delta AOB = \frac{1}{2}\times\frac{5}{|\sec\theta-\tan\theta|}\times\frac{5}{|\sec\theta+\tan\theta|}\times\frac{24}{25} \quad \frac{1}{2}$$
$$= \frac{12}{|\sec^2\theta-\tan^2\theta|} = \frac{12}{1} = 12 \quad \frac{1}{2}$$

Question 5

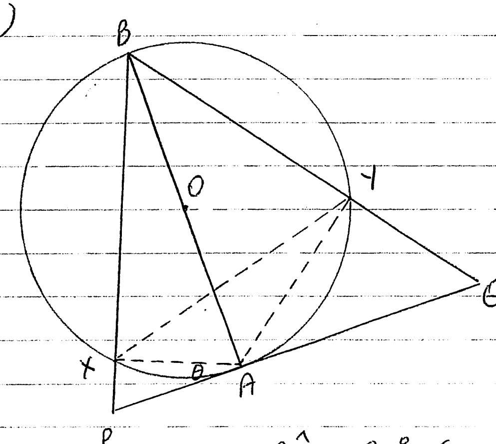
(a) $x^3 - 2xy + y^2 = 4$

$$3x^2 - \left(2y + 2x\frac{dy}{dx}\right) + 2y\frac{dy}{dx} = 0$$
$$3x^2 - 2y = (2x-2y)\frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{3x^2-2y}{2x-2y} \quad — 1$$

At $(-2,2)$ $\frac{dy}{dx} = \frac{3(-2)^2-2\times 2}{2(-2)-2\times 2}$
$$= \frac{8}{-8}$$
$$= -1$$

Equation of tangent is
$$y - 2 = -1(x+2)$$
$$y = -x \quad — 1$$

(b)



Join AX, AY, XY

Let $P\hat{A}X = \theta$

$\therefore A\hat{B}X = \theta$

(angle between chord & tangent = angle in alternate segment)

$A\hat{Y}X = A\hat{B}X$ (angles in same segment)

$= \theta \quad — 1$

$B\hat{X}A = 90°$ (angle in a semicircle)

$\therefore P\hat{X}A = 90°$ ($B\hat{X}P$ is a straight angle)

$\therefore X\hat{P}A = 90° - \theta$ (angle sum of $\Delta = 180°$) $\quad — 1$

Similarly $B\hat{Y}A = Q\hat{Y}A = 90°$

Name :

Hence $\quad Q\hat{Y}X = 90^\circ + \theta$

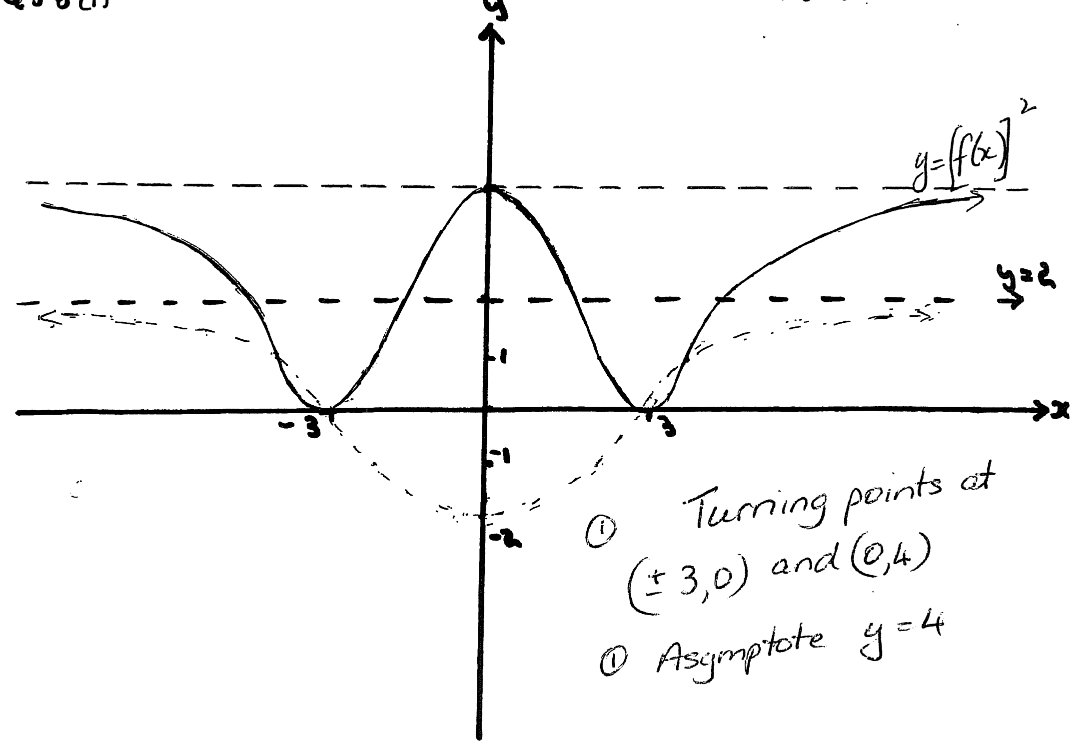
$\therefore \quad Q\hat{P}X + Q\hat{Y}X = 90^\circ - \theta + 90^\circ + \theta$

$\qquad\qquad\qquad\qquad = 180^\circ \qquad$ — 1

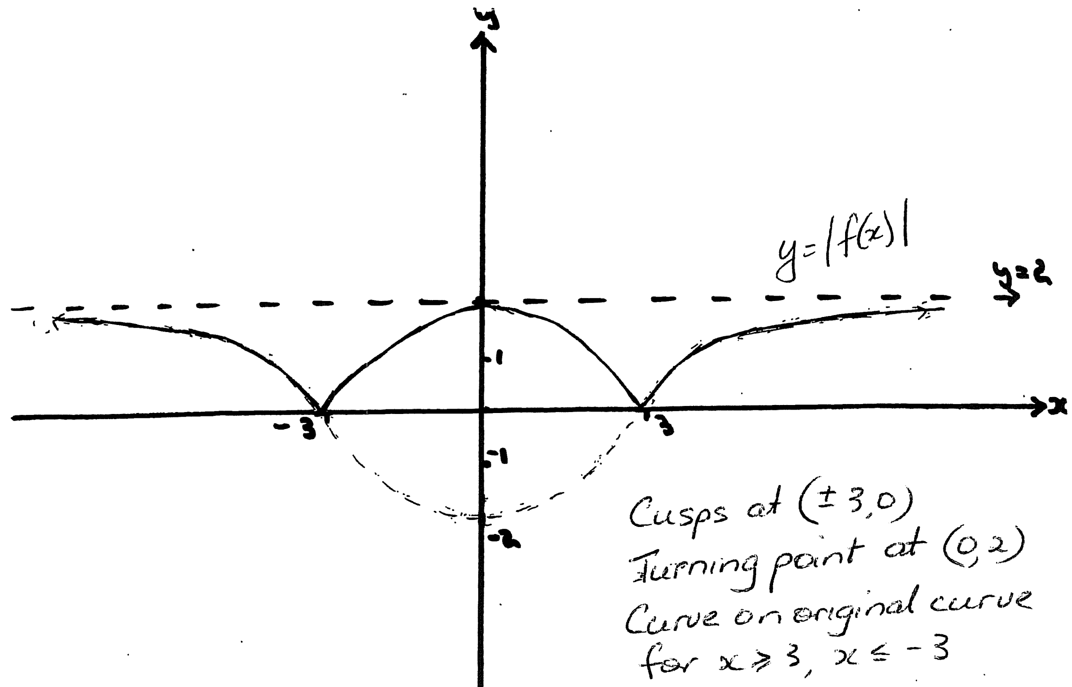
$\therefore$ PQYX is a cyclic quadrilateral since opposite angles are supplementary

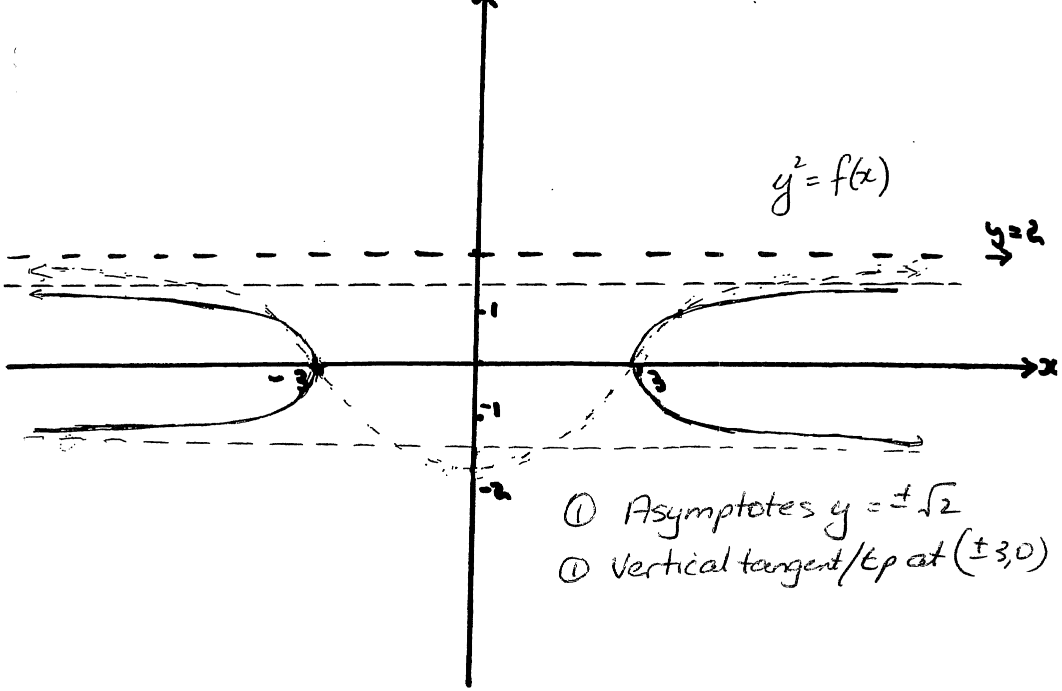
There are many methods to get to the result – each was marked according to the correct logic displayed

Q 5 b (i)



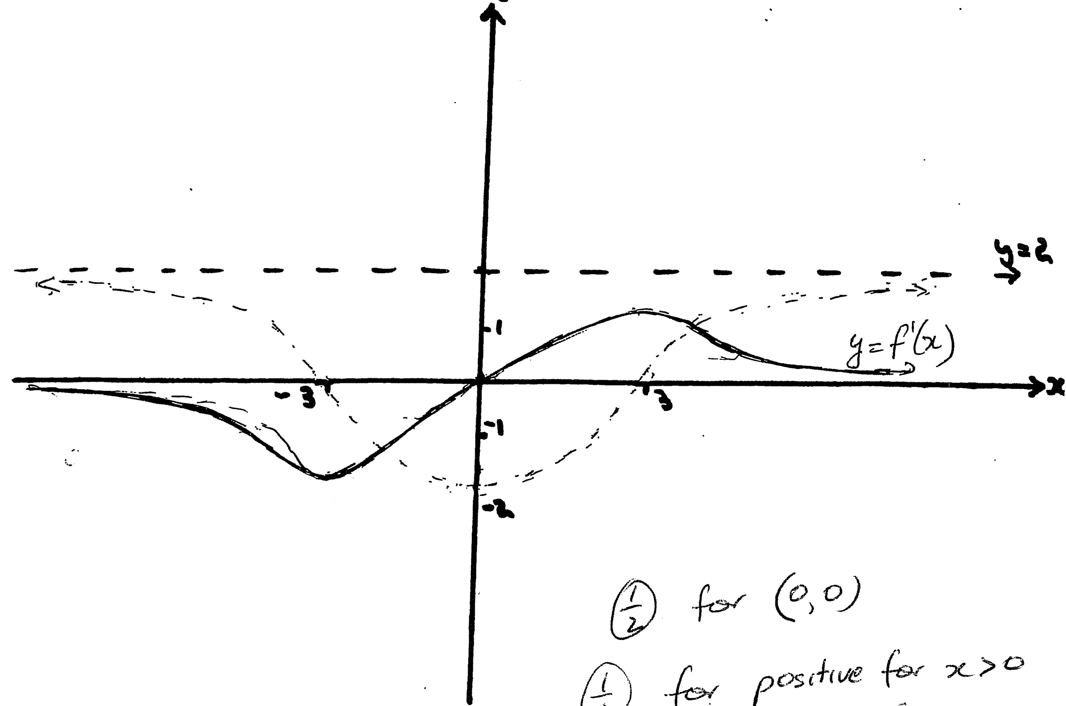
Q 5 b (ii)





① Asymptotes $y = \pm\sqrt{2}$

① Vertical tangent/tp at $(\pm 3, 0)$

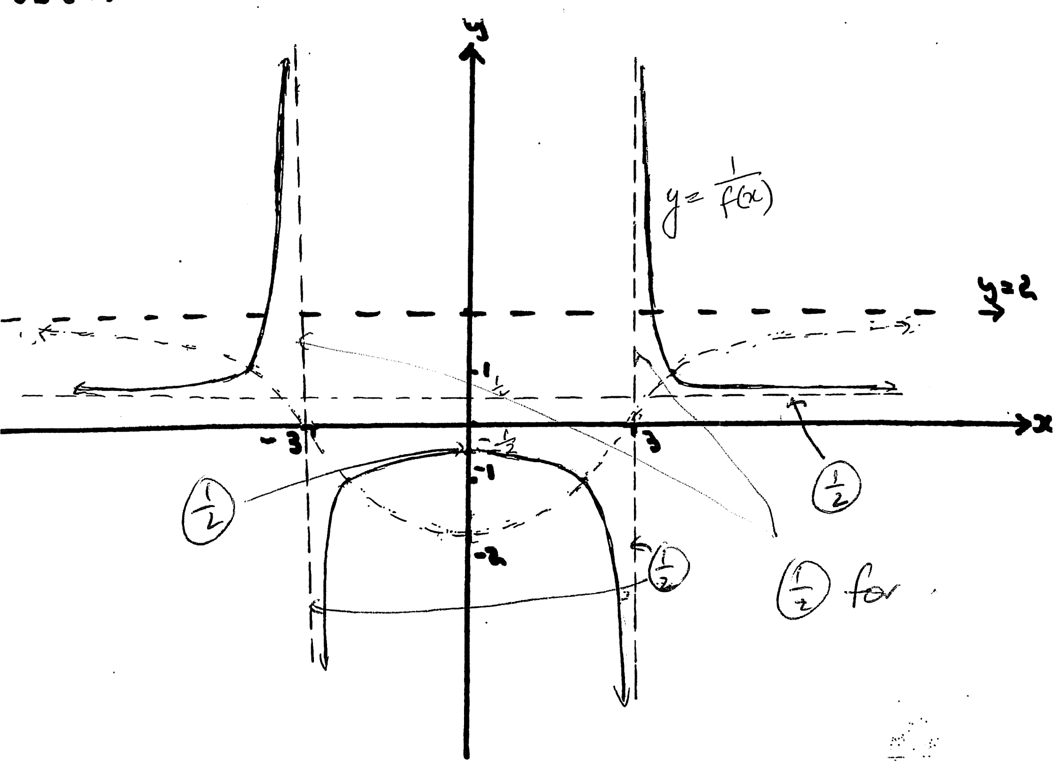


$\frac{1}{2}$ for $(0,0)$

$\frac{1}{2}$ for positive for $x > 0$
negative for $x < 0$

① for $\to 0$ as $x \to \pm\infty$

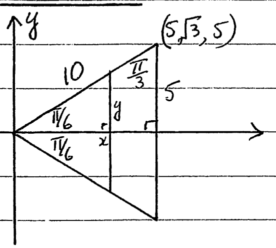
5b (iv)



$\frac{1}{2}$ for

Question 6

(a)

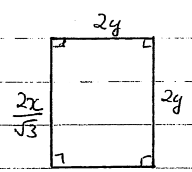


When $y = 5$ $\quad \frac{x}{5} = \tan\frac{\pi}{3}$

$x = 5\sqrt{3}$ ✓ → $\frac{1}{2}$

x ~~units~~ cm from origin

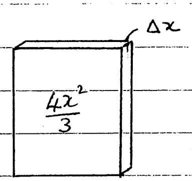
$$\frac{y}{x} = \tan\frac{\pi}{6}$$ ✓



$$\left.\begin{aligned} y &= \frac{x}{\sqrt{3}} \\ 2y &= \frac{2x}{\sqrt{3}} \end{aligned}\right\} \rightarrow 1$$

$$A(x) = (2y)^2 \quad ✓ \quad \rightarrow \frac{1}{2}$$
$$= \frac{4x^2}{3}$$

(ii)



$$\left.\begin{aligned} \Delta V &= \frac{4x^2}{3}\Delta x \\ V &= \lim_{\Delta x \to 0} \sum_{x=0}^{5\sqrt{3}} \frac{4x^2}{3}\Delta x \end{aligned}\right\} \rightarrow 1$$

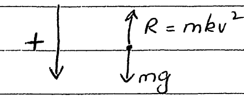
$$\left.\begin{aligned} &= \int_0^{5\sqrt{3}} \frac{4x^2}{3}\,dx \\ &= \frac{4}{3}\left[\frac{x^3}{3}\right]_0^{5\sqrt{3}} \end{aligned}\right\} ✓ \rightarrow 1$$

$$= \frac{4}{9}\left((5\sqrt{3})^3 - 0\right)$$

$$= \frac{4 \times 125 \times 3\sqrt{3}}{9} \quad ✓ \quad \rightarrow 1$$

$$\text{Volume} = \frac{500\sqrt{3}}{3}\ \text{cm}^3$$

(b) (i) Downwards motion

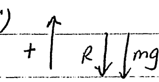


$$\left.\begin{aligned} m\ddot{x} &= mg - mkv^2 \\ \ddot{x} &= g - kv^2 \end{aligned}\right\} \rightarrow 1$$

For terminal velocity $\ddot{x} \rightarrow 0$

$$\left.\begin{aligned} 0 &= g - kV^2 \\ kV^2 &= g \end{aligned}\right\} - 1$$

(ii) Upwards motion



$$m\ddot{x} = -mg - mkv^2$$

$$\ddot{x} = -(g + kv^2) \rightarrow \frac{1}{2}$$

$$\left.\begin{aligned} &= -g\left(1 + \frac{k}{g}v^2\right) \\ &= -g\left(1 + \frac{1}{V^2}\cdot v^2\right) \\ &= -g\left(1 + \frac{v^2}{V^2}\right) \end{aligned}\right\} \rightarrow \frac{1}{2}$$

$V^2 = \frac{g}{k}$

(iii)

$$\left.\begin{aligned} v\frac{dv}{dx} &= -g\left(1 + \frac{v^2}{V^2}\right) \\ \frac{dv}{dx} &= -g\left(\frac{V^2 + v^2}{V^2 v}\right) \\ \frac{dx}{dv} &= -\frac{V^2 v}{g(V^2 + v^2)} \end{aligned}\right\} \rightarrow 1$$

$$x = -\frac{V^2}{2g}\ln(V^2 + v^2) + c$$

When $x = 0$ $\quad v = \frac{V}{5}$

$$\left.\begin{aligned} 0 &= \frac{-V^2}{2g}\ln\left(V^2 + \frac{V^2}{25}\right) + c \\ c &= \frac{V^2}{2g}\ln\left(\frac{26V^2}{25}\right) \end{aligned}\right\} \rightarrow 1$$

$$x = \frac{V^2}{2g}\ln\left(\frac{26V^2}{25}\right) - \frac{V^2}{2g}\ln\left(V^2+v^2\right))$$

When $v = 0$ $\quad x = H$ (max height reached)

$$H = \frac{V^2}{2g}\ln\left(\frac{26V^2}{25}\right) - \frac{V^2}{2g}\ln V^2$$

$$= \frac{V^2}{2g}\ln\left(\frac{26}{25}V^2 \div V^2\right)$$

$$= \frac{V^2}{2g}\ln\left(\frac{26}{25}\right)$$

$\rightarrow 1$

($\frac{1}{2}$ off minor errors)

OR $\quad H = \int_{\frac{V}{5}}^{0} \frac{-V^2 v}{g(V^2+v^2)}\,dv$

$$= \left[\frac{-V^2}{2g}\ln(V^2+v^2)\right]_{\frac{V}{5}}^{0} \rightarrow 2$$

$$= -\frac{V^2}{2g}\ln V^2 + \frac{V^2}{2g}\ln\left(V^2+\frac{V^2}{25}\right)$$

$$= \frac{V^2}{2g}\left(\ln\frac{26V^2}{25} - \ln V^2\right)$$

$$= \frac{V^2}{2g}\ln\left(\frac{26}{25}\right)$$

(iv) Downwards motion

$+\downarrow$

$$\ddot{x} = g - kv^2$$

$$v\frac{dv}{dx} = g - kv^2$$

$$\frac{dv}{dx} = \frac{g-kv^2}{v}$$

$$\frac{dx}{dv} = \frac{v}{g-kv^2}$$

$$\frac{dx}{dv} = \frac{-1}{2k}\cdot\frac{-2kv}{g-kv^2}$$

$$=$$

$$x = -\frac{1}{2k}\ln(g-kv^2) + c$$

$\rightarrow 1$

When $x = 0$ $\quad v = 0$

$$0 = -\frac{1}{2k}\ln g + c$$

$$c = \frac{1}{2k}\ln g$$

$$x = \frac{1}{2k}\ln g - \frac{1}{2k}\ln(g-kv^2)$$

$$= \frac{1}{2k}\ln\left(\frac{g}{g-kv^2}\right) \rightarrow 1$$

$$= \frac{1}{2k}\ln\left(\frac{1}{1-\frac{k}{g}v^2}\right)$$

When $x = y$ velocity is v ; $\quad \frac{1}{k} = \frac{V^2}{g}$, $\quad \frac{k}{g} = \frac{1}{V^2}$

$$y = \frac{V^2}{2g}\ln\left(\frac{1}{1-\frac{v^2}{V^2}}\right) \rightarrow 1$$

$$= \frac{V^2}{2g}\ln\left(\frac{V^2}{V^2-v^2}\right)$$

(v) When $y = H = \frac{V^2}{2g}\ln\left(\frac{26}{25}\right)$, $\quad v = U$ $\quad \rightarrow 1$

$$\frac{V^2}{2g}\ln\frac{26}{25} = \frac{V^2}{2g}\ln\left(\frac{V^2}{V^2-U^2}\right)$$

$$\frac{V^2}{V^2-U^2} = \frac{26}{25}$$

$$\frac{1}{1-\left(\frac{U}{V}\right)^2} = \frac{26}{25}$$

$$25 = 26 - 26\left(\frac{U}{V}\right)^2$$

$$26\left(\frac{U}{V}\right)^2 = 1 \quad \rightarrow 1$$

$$\left(\frac{V}{U}\right)^2 = 26$$

$$\frac{V}{U} = \sqrt{26}$$

Question 7

(a)(i) $\int_0^a f(a-x)\,dx$ Let $u = a - x$, $du = -dx$ $\frac{1}{2}$

When $x = 0$, $u = a$; $x = a$, $u = 0$ $\frac{1}{2}$

$$= \int_a^0 f(u)\cdot(-1)\,du$$

$$= \int_0^a f(u)\,du \quad 1$$

$$= \int_0^a f(x)\,dx \quad \boxed{2}$$

(ii)
$$I = \int_0^1 \frac{x^{10}}{x^{10}+(1-x)^{10}}\,dx$$

$1-(1-x) = x$

$$= \int_0^1 \frac{(1-x)^{10}}{(1-x)^{10}+x^{10}} \quad \text{(from (i))}$$

$$\therefore 2I = \int_0^1 \frac{x^{10}}{x^{10}+(1-x)^{10}}\,dx + \int_0^1 \frac{(1-x)^{10}}{(1-x)^{10}+x^{10}}\,dx$$

$$= \int_0^1 \frac{x^{10}+(1-x)^{10}}{x^{10}+(1-x)^{10}}\,dx \quad \boxed{2}$$

$$= \int_0^1 1\cdot dx$$

$$= \left[x\right]_0^1$$

$$= 1 - 0$$

$$\therefore I = \frac{1}{2}$$

*Wrong number for a 1 only

(b) (i) $P\left(cp, \frac{c}{p}\right)$ $Q\left(cq, \frac{c}{q}\right)$

$$\text{Grad } PQ = \frac{\frac{c}{p} - \frac{c}{q}}{cp - cq} = \frac{c \cdot \frac{q-p}{pq}}{c(p-q)} = -\frac{1}{pq} \quad 1$$

$\therefore$ Eq^n of PQ is

$$y - \frac{c}{p} = -\frac{1}{pq}(x - cp)$$

$$pqy - cq = -x + cp \quad 1$$

$$x + pqy = c(p+q)$$

$\boxed{2}$

(ii) $R(a, b)$ lies on PQ

$$a + pqb = c(p+q) \quad \text{— } 1 \quad \tfrac{1}{2}$$

Let Midpt of PQ be (x, y)

$$x = \frac{cp+cq}{2} = \frac{c(p+q)}{2} \quad \tfrac{1}{2} \qquad y = \frac{1}{2}\left(\frac{c}{p} + \frac{c}{q}\right) = \frac{c(p+q)}{2pq} \quad \tfrac{1}{2}$$

$$2x = c(p+q)$$

From 1 $\quad pq = \frac{c(p+q) - a}{b} = \frac{2x - a}{b} \quad \tfrac{1}{2}$

$\boxed{3}$

$$\therefore \tfrac{1}{2} \; y = \frac{\cancel{2}x}{\cancel{2}\left[\frac{2x-a}{b}\right]}$$

$$2xy - ay = bx$$

$$2xy = ay + bx \quad \tfrac{1}{2}$$

(c) Aim to show $1^2+2^2+3^2+\ldots+n^2 = \frac{n(n+1)(2n+1)}{6}$

When $n=1$ $\quad$ LHS $= 1^2 = 1$

RHS $= \frac{1(1+1)(2\times1+1)}{6} = 1 = $ LHS $\quad -\tfrac{1}{2}$ not showing

$\therefore$ Proposition is true for $n=1$ $\quad 1$

Let k be a positive integer for which proposition is true

ie $\quad 1^2+2^2+3^2+\ldots+k^2 = \frac{k(k+1)(2k+1)}{6}$

Aim to show ~~po~~proposition is then true for $n=k+1$

ie $\quad 1^2+2^2+3^2+\ldots+k^2+(k+1)^2 = \frac{(k+1)(k+1+1)(2(k+1)+1)}{6}$

$\tfrac{1}{2}$ for here or at end showing $\frac{(k+1)(\overline{k+1}+1)(2\overline{k+1}+1)}{6}$

$$\text{LHS} = 1^2+2^2+\ldots+k^2+(k+1)^2$$

$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \quad \tfrac{1}{2}$$

$$= \frac{(k+1)}{6}\left[k(2k+1) + 6(k+1)\right] \quad \tfrac{1}{2}$$

$$= \frac{(k+1)}{6}(2k^2+7k+6)$$

$$= \frac{(k+1)(k+2)(2k+3)}{6} \quad \tfrac{1}{2}$$

$$= \text{RHS}$$

$\boxed{3}$

$\therefore$ Proposition is true for $n=k+1$ if true for $n=k$

etc

(ii) $2^2 + 5^2 + 8^2 + \dots + (3n-1)^2$

$= \sum_{k=1}^{n} (3k-1)^2$ $\left(\frac{1}{2}\right)$

$= \sum_{k=1}^{n} (9k^2 - 6k + 1)$

$= 9\sum_{k=1}^{n} k^2 - 6\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1$ $\left(\frac{1}{2}\right)$

$= 9\frac{n(n+1)(2n+1)}{6} - 6\frac{n(n+1)}{2} + n$ $\left(\frac{1}{2}\right)$ $\left(\frac{1}{2}\right)$ $\left(\frac{1}{2}\right)$

$= \frac{3n(n+1)(2n+1) - 6n(n+1) + 2n}{2}$

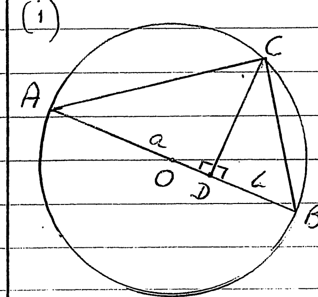
$= \frac{n\left[3(n+1)(2n+1) - 6(n+1) + 2\right]}{2}$

$= \frac{n}{2}\left[6n^2 + 9n + 3 - 6n - 6 + 2\right]$ $\boxed{3}$

$= \frac{n}{2}(6n^2 + 3n - 1)$ $\left(\frac{1}{2}\right)$

Question 8

(i)



$A\hat{C}B = 90^\circ$ (angle in a semicircle)

Let $C\hat{A}D = \theta$ $\therefore BCD$

$\therefore A\hat{C}D = 90^\circ - \theta$ (angle sum of ΔACD is 180°)

$C\hat{B}A = 90^\circ - \theta$ (angle sum of ΔABC is 180°)

$B\hat{C}D = \theta$ (complement of $A\hat{C}D$)

$\Delta ACD \ ||| \ \Delta CBD$ (equiangular)

$\frac{CD}{BD} = \frac{AD}{CD}$ (corresponding sides in same ratio)

$CD^2 = AD.BD$

$= ab$

$CD = \sqrt{ab}$ $(CD > 0)$

(ii) $CD \leq$ radius of circle — (1)

$\sqrt{ab} \leq \frac{a+b}{2}$

(iii) $\therefore a+b \geq 2\sqrt{ab}$ for positive real numbers

$\therefore$ If x, y, z are positive real numbers

$$\left.\begin{aligned} x+y &\geq 2\sqrt{xy} \\ y+z &\geq 2\sqrt{yz} \\ z+x &\geq 2\sqrt{zx} \end{aligned}\right\} \leftarrow (1)$$

$\therefore (x+y)(y+z)(z+x) \geq 8\sqrt{xy.yz.zx}$

$= 8\sqrt{x^2y^2z^2}$ — (1)

$= 8xyz$

(a) $T_n = x^{n-1}(1+x+x^2+\dots+x^{n-1})$

(i) $T_n = x^{n-1} \cdot \dfrac{(1-x^n)}{1-x}$

$= \dfrac{x^{n-1}-x^{2n-1}}{1-x}$ — (1)

$S_n = T_1 + T_2 + T_3 + \dots + T_n$

$= \dfrac{1}{1-x}\left[1+x+x^2+\dots+x^{n-1} - (x+x^3+x^5+\dots+x^{2n-1})\right]$

$= \dfrac{1}{1-x}\left[\dfrac{1(1-x^n)}{1-x} - \dfrac{x(1-(x^2)^n)}{1-x^2}\right]$ — (1) $x \neq 1$, $x^2 \neq 1$

$= \dfrac{1}{(1-x)} \cdot \dfrac{(1-x^n)(1+x) - x(1-x^{2n})}{(1-x^2)}$ $(1-x^n)(1+x^n)$

$= \dfrac{(1-x^n)\left[1+x-x(1+x^n)\right]}{(1-x)(1-x^2)}$

$= \dfrac{(1-x^n)(1-x^{n+1})}{(1-x)(1-x^2)}$ — (1) $(x^2 \neq 1)$

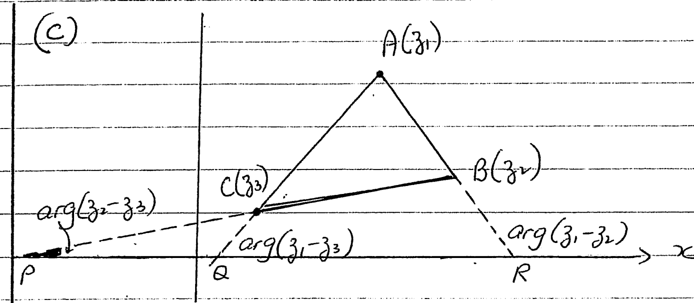
(ii) $\lim\limits_{x\to 1} \dfrac{(1-x^n)(1-x^{n+1})}{(1-x)(1-x^2)} = \lim\limits_{x\to 1}(T_1+T_2+T_3+\dots+T_n)$

$= 1+2+3+\dots+n$ — (1)

$= \dfrac{n}{2}(1+n)$

$= \dfrac{1}{2}n(n+1)$ — (1)

(c)



(1) for diagram

$\dfrac{z_2-z_3}{z_1-z_3} = \dfrac{z_1-z_3}{z_1-z_2}$ — (1)

Let AC meet x axis at Q $\quad C\hat{Q}R = \arg(z_1-z_3)$

BC " " " " P $\quad C\hat{P}Q = \arg(z_2-z_3)$

AB " " " " R $\quad B\hat{R}x = \arg(z_1-z_2)$

From (1) $\arg(z_2-z_3) - \arg(z_1-z_3) = \arg(z_1-z_3) - \arg(z_1-z_2)$

$\arg(z_1-z_2) - \arg(z_1-z_3) = \arg(z_1-z_3) - \arg(z_2-z_3)$

LHS $= C\hat{A}B$ $= P\hat{C}Q$ $=$ RHS — (1)

(exterior ∠ of ΔAQR = sum of interior opp angles) (exterior ∠ of ΔPCQ = sum of interior opp ∠s)

$P\hat{C}Q = A\hat{C}B$ (vertically opp ∠s)

$\therefore A\hat{C}B = C\hat{A}B \; (= P\hat{C}Q)$ — (1)

Hence $AB = BC$ (equal sides opposite equal angles in a Δ)

$|z_1-z_2| = |z_2-z_3|$ — (2) — (1)

From (1) $\dfrac{|z_2-z_3|}{|z_1-z_3|} = \dfrac{|z_1-z_3|}{|z_1-z_2|}$

$$|z_1 - z_3|^2 = |z_2 - z_3||z_1 - z_2|$$

$$= |z_1 - z_2||z_1 - z_2| \quad \text{from } (2)$$

$$\therefore |z_1 - z_3| = |z_1 - z_2| \qquad \text{— } (1)$$

Hence $|z_1 - z_3| = |z_1 - z_2| = |z_2 - z_3|$ from (2)

$\therefore \quad AC = AB = BC$

ie ΔABC is equilateral

There are other methods – each scored part marks for relevant facts that were established.