St George Girls High School

Trial Higher School Certificate Examination

2006

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen
- Write your Student Number on every page
- All questions may be attempted.
- Begin each question in a new booklet.
- All necessary working must be shown.
- Marks may be deducted for careless or poorly presented work.
- Board-approved calculators may be used.
- A list of standard integrals is included at the end of this paper.
- The mark allocated for each question is listed at the side of the question.

Students are advised that this is a Trial Examination only and does not necessarily reflect the content or format of the Higher School Certificate Examination.

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Question 1 – (15 marks) – Start a new booklet

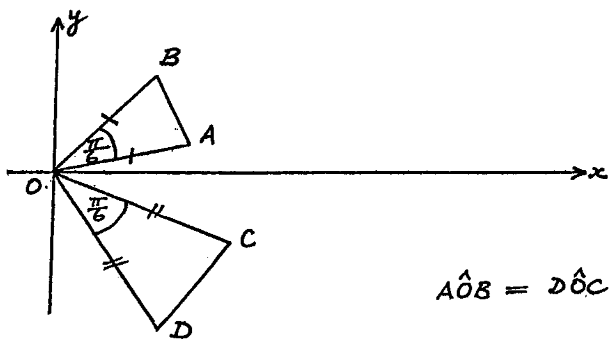
Marks

a) Prove that for complex numbers z, z_1 and z_2

(i) $z\bar{z} = |z|^2$ (ii) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$ 1, 1

b) Use the above two results to prove that $|z_1 z_2| = |z_1||z_2|$ 2

c)



$A\hat{O}B = D\hat{O}C = \frac{\pi}{6}$

The points A, B, C, D and O are points in the Argand plane such that $OD = OC$ and $OA = OB$. A corresponds to the complex number z and C corresponds to the complex number q. Let $w = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$.

(i) Explain why B corresponds to zw. 1

(ii) Find the complex number corresponding to D. 1

(iii) Prove, using complex numbers, that $BC = AD$. 1

d) (i) If α is a double zero of a polynomial $P(x)$ show that α is a single zero of $P'(x)$ 2

(ii) Find the integers 'a' and 'b' given that $(x+1)^2$ is a factor of $x^5 + 2x^2 + ax + b$ 3

e) (i) Show that $z = 1+i$ is a root of the equation $z^2 - (3-2i)z + (5-i) = 0$ 2

(ii) Find the other root of the equation.

Question 2 – (15 marks) – Start a new booklet

Marks

a) (i) Express $\frac{3x+7}{(x+1)(x+2)(x+3)}$ in partial fractions (show all working) 3

(ii) Hence evaluate $\int_0^1 \frac{3x+7}{(x+1)(x+2)(x+3)}\,dx$

b) (i) Simplify $\sin(A+B) + \sin(A-B)$ 3

(ii) Hence evaluate $\int_0^{\frac{\pi}{4}} \sin 5x \cos 3x \, dx$

c) Consider the ellipse $E: \frac{x^2}{9-\lambda} + \frac{y^2}{\lambda - 4} = 1$ where λ is a real number.

(i) Find the range of possible values of λ 1

(ii) For this range of values of λ, find the volume $V(\lambda)$ in simplest factored form, when the area enclosed by this ellipse is rotated about the x-axis. 3

(iii) Sketch the ellipse E when $\lambda = 5$ clearly showing intercepts, foci and directrices. 5

Question 3 – (15 marks) – Start a new booklet

Marks

a) (i) Sketch $y=\left[\sin^{-1}x\right]^{-1}$ clearly demonstrating its relationship to $y=\sin^{-1}x$ 2

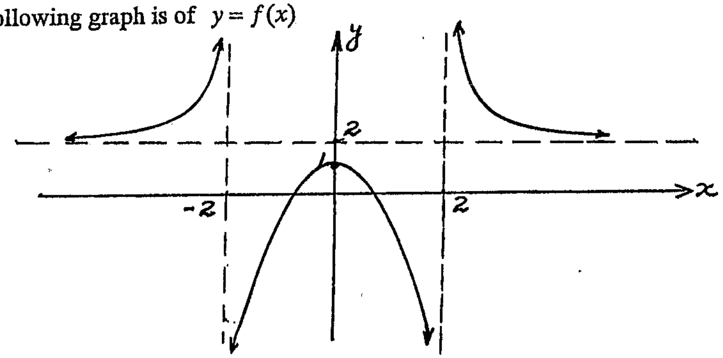
(ii) Sketch $y^2=\tan^{-1}x$ clearly showing its relationship to $y=\tan^{-1}x$ 2

b) $P(x)$ is an even monic polynomial of degree four with integer co-efficients. One zero is $3i$ and the product of the zeros is -18. Factor $P(x)$ fully over the real field. 3

c) The base of a solid is the circle $x^2+y^2=16$ 4

Find the volume of the solid if every cross-section perpendicular to the x-axis is an equilateral triangle with one side in the base of the solid.

d) The following graph is of $y=f(x)$



(i) Suggest a possible expression for $f(x)$ giving a clear explanation of your choice. 2

(ii) Sketch $y=f'(x)$ 2

Question 4 – (15 marks) – Start a new booklet

Marks

a) If α, β and γ are the roots of $x^3-2x^2+4x+2=0$ find the polynomial equation with roots 4

(i) $\alpha-1,\ \beta-1,\ \gamma-1$

(ii) $\alpha^2,\ \beta^2,\ \gamma^2$

b) Evaluate

(i) $\int_2^3 \frac{x^3}{x^2-1}\,dx$ 2

(ii) $\int_0^2 xe^x\,dx$ 2

c) Consider the curve $C: x^2+xy+y^2=3$

(i) Find $\frac{dy}{dx}$ 1

(ii) Find all stationary points and points where $\frac{dy}{dx}$ is not defined. 4

(iii) Sketch C clearly showing the above features and intercepts on the x, y axes. 2

Marks

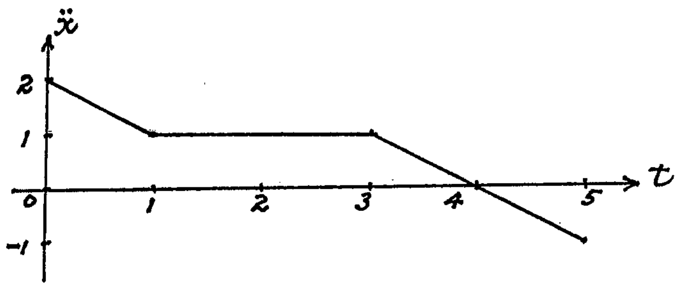
Question 5 – (15 marks) – Start a new booklet

a) (i) If $I_n = \int_0^{\frac{\pi}{2}} x^n \sin x \, dx$, $n \geq 0$ prove that for $n \geq 2$, $I_n + n(n-1)I_{n-2} = n\left(\frac{\pi}{2}\right)^{n-1}$ **5**

(ii) Calculate $\int_0^{\frac{\pi}{2}} x \sin x \, dx$

(iii) Deduce that $\int_0^{\frac{\pi}{2}} x^3 \sin x \, dx = \frac{3}{4}\pi^2 - 6$

b)



A particle starts from rest at the origin. The above sketch represents its acceleration $\ddot{x}$ as a function of time.

(i) Carefully describe the velocity of the particle for $1 \leq t \leq 3$. **2**

(ii) Find the velocity of the particle at $t = 5$ **2**

c) The region, in the first quadrant, bounded by the curve $y = \cos^{-1} x$ and the co-ordinate axes is rotated about the line $x = -1$. Use the method of cylindrical shells, clearly showing all working, to find the volume of the solid generated. **6**

Marks

Question 6 – (15 marks) – Start a new booklet

a) A solid is formed by rotating the ellipse $E: \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about the line $x = 2a$. **4**
Find the volume of this solid by taking slices perpendicular to the axis of rotation.

b) The point $P(x_1, y_1)$ lies on the hyperbola $H: \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.
The foci of H are S and S'.

The tangent to H at P cuts the x-axis at T.

(i) Prove that the equation of the tangent at P is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$ **2**

(ii) Find the coordinates of T. **1**

(iii) Prove that $\frac{PS}{PS^1} = \frac{TS}{TS^1}$ **3**

c) A particle of mass m kg is dropped from rest in a medium where the resistance force is mkv.

(i) Find the equation of motion and the terminal velocity v_T **2**

(ii) Find the time taken for the particle to reach a velocity of $\frac{v_T}{2}$. **3**

Marks

Question 7 – (15 marks) – Start a new booklet

a) A mass m kg is projected vertically upwards with velocity V m/s in a medium with resistance force mkv. Find the maximum height reached. 4

b) In any triangle ABC prove that $\tan A+\tan B+\tan C=\tan A\ \tan B\ \tan C$ 3

c) (i) If $t=\tan x$ prove that $\tan 4x=\dfrac{4t\left(1-t^2\right)}{t^4-6t^2+1}$ 2

(ii) If $\tan x\tan 4x=1$ deduce that $5t^4-10t^2+1=0$ 1

(iii) Use basic trigonometry to prove that both $x=18^\circ$ and $x=54^\circ$ satisfy the equation $\tan x\ \tan 4x=1$ 2

(iv) Deduce that $\tan 54^\circ=\sqrt{\dfrac{5+2\sqrt{5}}{5}}$ 3

Marks

Question 8 – (15 marks) – Start a new booklet

a) (i) Find the sum $1+10+10^2+\ldots+10^n$ 7

(ii) Use the method of mathematical induction to show that

$$1\times 9^2+11\times 9^2+111\times 9^2+\ldots+\underbrace{111\ldots1}_{n\ ones}\times 9^2$$

$$=10^{n+1}-9n-10 \text{ for all positive integers } n\geq 1$$

b) It is given that $z^5=1$ where $z\neq 1$

(i) Show that $z^2+z+1+z^{-1}+z^{-2}=0$ 1

(ii) Show that $z+z^{-1}=2\cos\dfrac{2k\pi}{5}$ $k=1,2,3,4$ 2

(iii) By letting $x=z+z^{-1}$ reduce the equation in (i) above to a quadratic equation in x. 3

(iv) Hence deduce that $\cos\dfrac{\pi}{5}\cdot\cos\dfrac{2\pi}{5}=\dfrac{1}{4}$ 2

End of Paper

Trial HSC
Ext. 2 Maths

(45)

Q1

a) let $z = x+iy$, $z_1 = x_1+iy_1$, $z_2 = x_2+iy_2$

(i) $z\bar{z} = (x+iy)(x-iy)$
$= x^2+y^2$
$= (\sqrt{x^2+y^2})^2$
$= |z|^2$ 1

(ii) $\overline{z_1 z_2} = \overline{(x_1+iy_1)(x_2+iy_2)}$

$= \overline{x_1x_2 - y_1y_2 + i(x_2y_1 + x_1y_2)}$

$= x_1x_2 - y_1y_2 - i(x_2y_1 + x_1y_2)$ $*$

and $\bar{z}_1\bar{z}_2 = (\overline{x_1+iy_1})(\overline{x_2+iy_2})$

$= (x_1 - iy_1)(x_2 - iy_2)$
$= x_1x_2 - ix_1y_2 - iy_1x_2 - y_1y_2$

$= x_1x_2 - y_1y_2 - i(x_2y_1 + x_1y_2)$ 1

$= \overline{z_1z_2}$ from $*$.

b) $z_1z_2\overline{z_1z_2} = |z_1z_2|^2$ from a)(i)

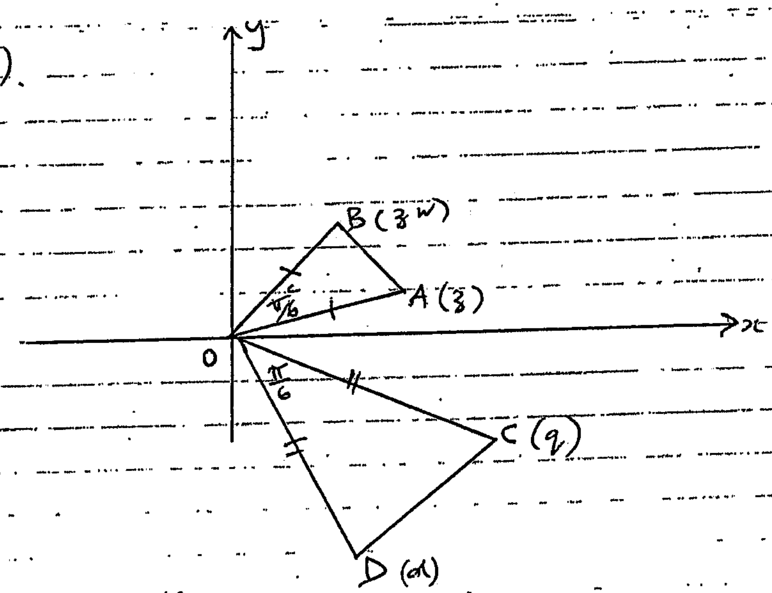
Now, $z_1z_2\overline{z_1z_2} = z_1z_2\bar{z}_1\bar{z}_2$ from a)(ii)

$= z_1\bar{z}_1 z_2\bar{z}_2$ (commutative law of $\times$ of complex n

$= |z_1|^2|z_2|^2$

$\therefore |z_1z_2|^2 = |z_1|^2|z_2|^2$
$|z_1z_2| = |z_1||z_2|$ on taking sq. rts.

c)(i).



(i) B represents $z \times 1\,\text{cis}\frac{\pi}{6}$ since to obtain the pos'n of B we have rotated A through $\frac{\pi}{6}^c$ 1
[$r\,\text{cis}\,\theta \times \text{cis}\,\phi = r\,\text{cis}(\theta+\phi)$]
$\therefore$ B corresponds to zw since $w = \text{cis}\frac{\pi}{6}$

(ii) Let d be the complex number corresponding to point D.
$\therefore d \times \text{cis}\frac{\pi}{6} = q$ 1

$\therefore d = \dfrac{q}{\text{cis}\frac{\pi}{6}} = \dfrac{q}{w}$

(iii) $\vec{BC} + \vec{OB} = \vec{OC}$
$\therefore \vec{BC} = \vec{OC} - \vec{OB}$
$= q - zw$
$\therefore BC = |q - zw|$

$\vec{OA} + \vec{AD} = \vec{OD}$
$\therefore \vec{AD} = \vec{OD} - \vec{OA}$
$= d - z$
$= \dfrac{q}{w} - z$
$= \dfrac{q - zw}{w}$

$AD = |q - zw|$

$= |q - zw|$ since $|w| = |cis\frac{\pi}{6}|$

$= \sqrt{\cos^2\frac{\pi}{6} + \sin^2\frac{\pi}{6}}$

$= \sqrt{1}$

$\therefore BC = AD.$ $= 1.$

d) (i) $P(x) = (x-\alpha)^2 Q(x)$, where $Q(\alpha) \neq 0$.

$\therefore P'(x) = 2(x-\alpha)Q(x) + (x-\alpha)^2 Q'(x)$

$= (x-\alpha)[2Q(x) + (x-\alpha)Q'(x)]$

Now, $Q(\alpha) \neq 0$ $\therefore$ there is no 2 factor of $x-\alpha$ in $2Q(x) + (x-\alpha)Q'(x)$.

$\therefore$ $P'(x)$ only has a single zero of $x=\alpha$

(ii) $(x+1)^2$ is a factor of $P(x) = x^5 + 2x^2 + ax + b$

$\therefore x+1$ is a factor of $P'(x)$ from (i)

ie $P'(-1) = 0$ (factor theorem).

Now, $P'(x) = 5x^4 + 4x + a$

$\therefore P'(-1) = 5 - 4 + a = 0$

$\therefore a = -1$

$\therefore P(x) = x^5 + 2x^2 - x + b = (x+1)^2 Q(x)$

$P(-1) = 0$

$\therefore (-1)^5 + 2(-1)^2 - (-1) + b = 0$

$-1 + 2 + 1 + b = 0$

$\therefore b = -2$.

e) (i) Let $P(z) = z^2 - (3-2i)z + (5-i) = 0$.

$\therefore P(1+i) = (1+i)^2 - (3-2i)(1+i) + 5 - i$

$= 1 + 2i - 1 - 3 - 3i + 2i - 2 + 5 \cdot$

$= 0$

$\therefore z = 1+i$ is a root of $z^2 - (3-2i)z + 5$

(ii) Let α be the other root

$\therefore \alpha + 1 + i = 3 - 2i \quad \left(= \text{"}-\frac{b}{a}\text{"}\right)$

$\therefore \alpha = 2 - 3i$

(OR divide $P(z)$ by $z - 1 - i$)

Q2

a) (i) $\frac{3x+7}{(x+1)(x+2)(x+3)} \equiv \frac{a}{x+1} + \frac{b}{x+2} + \frac{c}{x+3}$

$\therefore 3x + 7 \equiv a(x+2)(x+3) + b(x+1)(x+3) + c(x+1)(x+2)$

Let $x = -2$

$\therefore 1 = b(-1)(1) \quad \therefore b = -1$

$x = -1$

$4 = a(1)(2) \quad \therefore a = 2$

$x = -3$

$1\frac{1}{2}$ $\quad -2 = c(-2)(-1) \quad \therefore c = -1$

$\frac{3x+7}{(x+1)(x+2)(x+3)} \equiv \frac{2}{x+1} - \frac{1}{x+2} - \frac{1}{x+3}$

(ii) $\therefore \int_0^1 \frac{3x+7}{(x+1)(x+2)(x+3)}\,dx = \int_0^1 \frac{2}{x+1} - \frac{1}{x+2} - \frac{1}{x+3}$

$= [2\ln(x+1) - \ln(x+2) - \ln($

$1\frac{1}{2}$ $\quad = 2\ln 2 - \cancel{\ln 3} - \ln 4 - 2\cancel{\ln 1} + \ln 2 +$

$= 3\ln 2 - 2\ln 2$

$= \ln 2$

b) (i) $\sin(A+B) + \sin(A-B)$

$= \sin A\cos B + \cos A\sin B + \sin A\cos B - \cos A\sin B$

1 $\quad = 2\sin A\cos B$

(ii) $\sin A\cos B = \frac{1}{2}\left(\sin(A+B) + \sin(A-B)\right)$

$\int_0^{\frac{\pi}{4}} \sin 5x\cos 3x\,dx = \frac{1}{2}\int_0^{\frac{\pi}{4}} \sin 8x + \sin 2x$

$(A = 5x,\ B = 3x)$

$= \frac{1}{2}\left[-\frac{1}{8}\cos 8x - \frac{1}{2}\cos 2x\right]_0^{\pi/4}$

2 $\quad = \frac{1}{2}\left(-\frac{1}{8}\cos 2\pi - \frac{1}{2}\cos\frac{\pi}{2} + \frac{1}{8} +\right.$

$= \frac{1}{2}\left(-\frac{1}{8} - 0 + \frac{1}{8} + \frac{1}{2}\right)$

$= \frac{1}{4}$

c) (i) We require $9 - \lambda > 0 \ \cap \ \lambda - 4 > 0$

ie $\lambda < 9 \ \cap \ \lambda > 4$

1 $\quad$ ie $4 < \lambda < 9$

$\frac{x^2}{9-\lambda} + \frac{y^2}{\lambda-4} =$

(ii) $V = 2\pi\int_0^{\sqrt{9-\lambda}} y^2\,dx$

$= 2\pi\int_0^{\sqrt{9-\lambda}} (\lambda-4)\left[1 - \frac{x^2}{9-\lambda}\right]dx$

$= 2\pi\left[(\lambda-4)x - \frac{\lambda-4}{9-\lambda}\frac{x^3}{3}\right]_0^{\sqrt{9-\lambda}}$

3 $\quad = 2\pi\left((\lambda-4)\sqrt{9-\lambda} - \frac{\lambda-4}{\cancel{9-\lambda}}\frac{(\cancel{9-\lambda})\sqrt{9-\lambda}}{3} - \right.$

$= 2\pi(\lambda-4)\left[\frac{2}{3}\sqrt{9-\lambda}\right]$

$= \frac{4\pi}{3}(\lambda-4)\sqrt{9-\lambda}$ units^3

(iii) If $\lambda = 5$, E becomes

$$\frac{x^2}{4} + \frac{y^2}{1} = 1.$$

i.e. $a = 2$, $b = 1$

Now, $b^2 = a^2(1-e^2)$

$\therefore 1 = 4(1-e^2)$

$\frac{1}{4} = 1 - e^2$

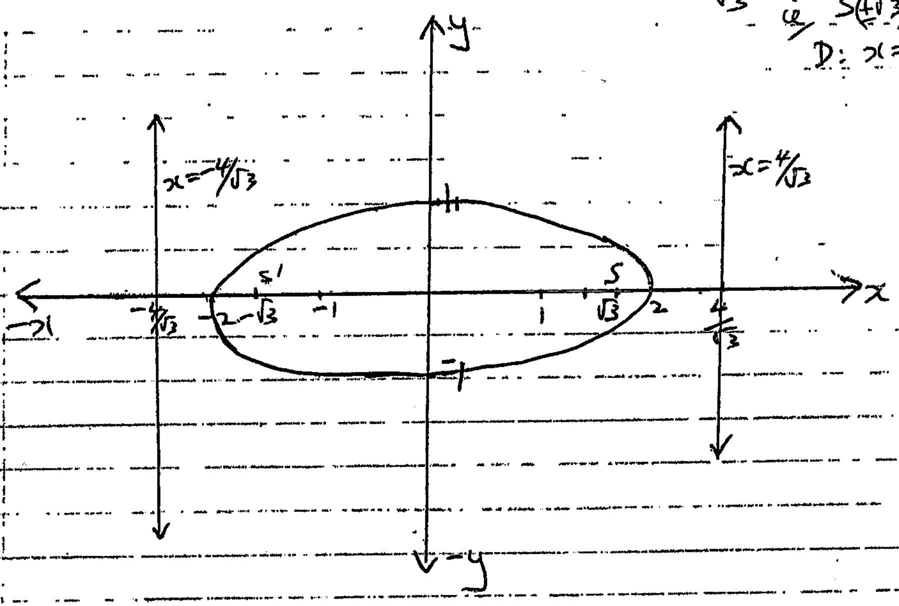
$e^2 = \frac{3}{4}$

$e = \frac{\sqrt{3}}{2}$

5.

$\therefore ae = \sqrt{3}$, $\frac{a}{e} = \frac{4}{\sqrt{3}}$ i.e. $S(\pm\sqrt{3}, 0)$

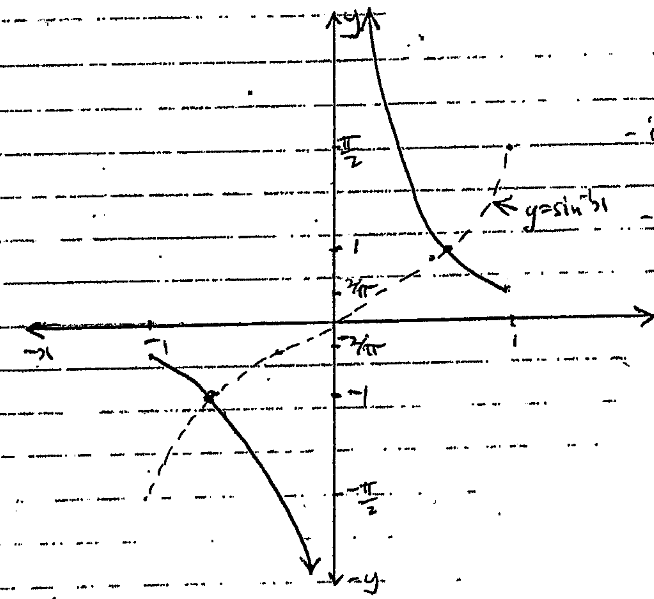
$D: x = \pm\frac{4}{\sqrt{3}}$



Q3

a) (i)

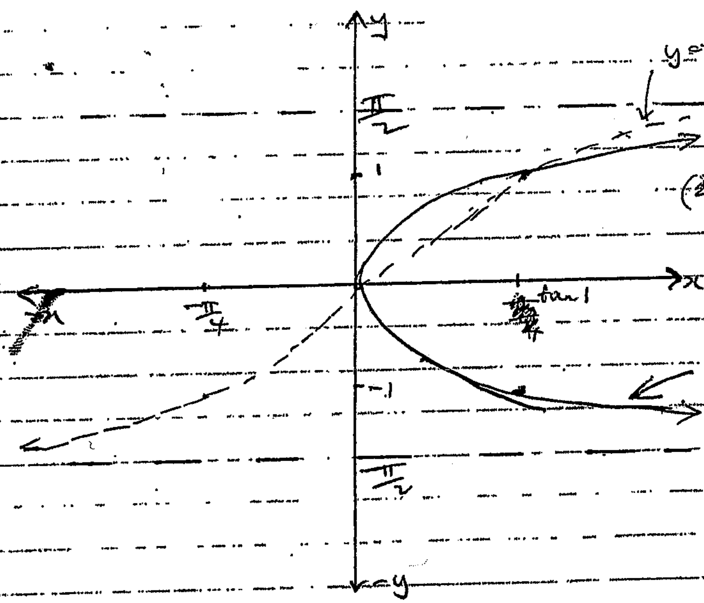
2



- intersect where $y = \pm 1$
$= (1, \frac{2}{\pi})$ and $(-1, -\frac{2}{\pi})$ on graph.
$x = 0$ is vert. asympt

(ii)

2



b) Let $P(x) = x^4 + ax^2 + b$ a, b integers.

If $x = 3i$ is a zero, so is $-3i$

(conjugate root theorem).

$(x+3i)(x-3i)$ is a factor

i.e. x^2+9 " " "

Let the other 2 roots be α, β

$\therefore 3i \times -3i \times \alpha\beta = b = -18$.

$\therefore 9\alpha\beta = -18$

$\alpha\beta = -2$ ①

Also, $\alpha+\beta+3i-3i = 0$.

$\therefore \alpha+\beta = 0$ ②

Solve ①, ② simultaneously

$\Rightarrow \alpha(-\alpha) = -2$

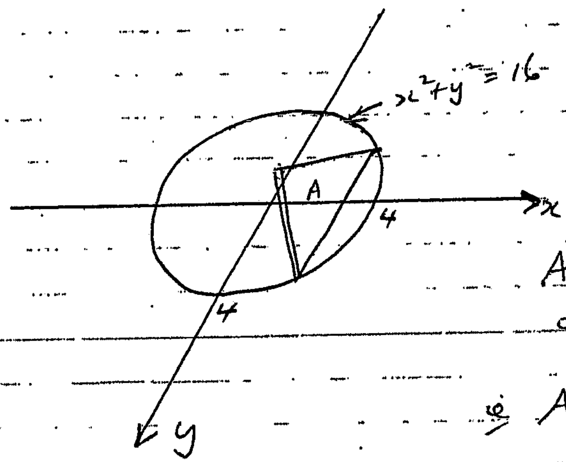
3

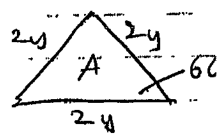
$\alpha^2 = 2$

i.e. $\alpha = \sqrt{2}, \ \beta = -\sqrt{2}$

$$\therefore P(x) = (x+3i)(x-3i)(x+\sqrt{2})(x-\sqrt{2})$$
$$= (x^2+9)(x^2-2)$$
$$= x^4 + 7x^2 - 18$$

c)





Area of triangular cross-section

$= \frac{1}{2} . 2y . 2y . \sin$

i.e. $A = \sqrt{3} y^2$

$\therefore \Delta V = A \, \Delta x$

$= \sqrt{3} y^2 \, \Delta x$

$= \sqrt{3}(16-x^2)\, \Delta x$

$\therefore V = \lim_{\Delta x \to 0} \sqrt{3} \sum_{x=-4}^{4} (16-x^2)\, \Delta x$

$= \sqrt{3} \int_{-4}^{4} 16 - x^2 \, dx$

$= 2\sqrt{3} \int_{0}^{4} 16 - x^2 \, dx$

$= 2\sqrt{3} \left[16x - \frac{x^3}{3} \right]_0^4$

4

$= 2\sqrt{3}\left(64 - \frac{64}{3} \right)$

$= 2\sqrt{3} \times \frac{2}{3} \times 64$

$= \frac{256\sqrt{3}}{3} \ u^3$

d) (i) Horizontal asymptote $y = 2$

i.e. $y \to 2$ as $x \to \pm\infty$

Vertical asymptotes $x = \pm 2$

i.e. denominator contains $x^2 - 4$.

Passes through $(0, 1)$.

$\therefore y = \frac{2x^2 + a}{x^2 - 4}$.

2. when $x = 0$, $y = 1$

$\therefore 1 = \frac{a}{-4}$

$\therefore a = -4$

$\therefore y = \frac{2x^2 - 4}{x^2 - 4}$.

(ii) $f(x)$ decreasing for $x>0$

" increasing " $x<0$

$\therefore f'(x) > 0$ for $x<0$

$f'(x) < 0$ for $x>0$

as $x \to \pm\infty$, $f(x) \to$ horizontal line

ie $f'(x) \to 0$.

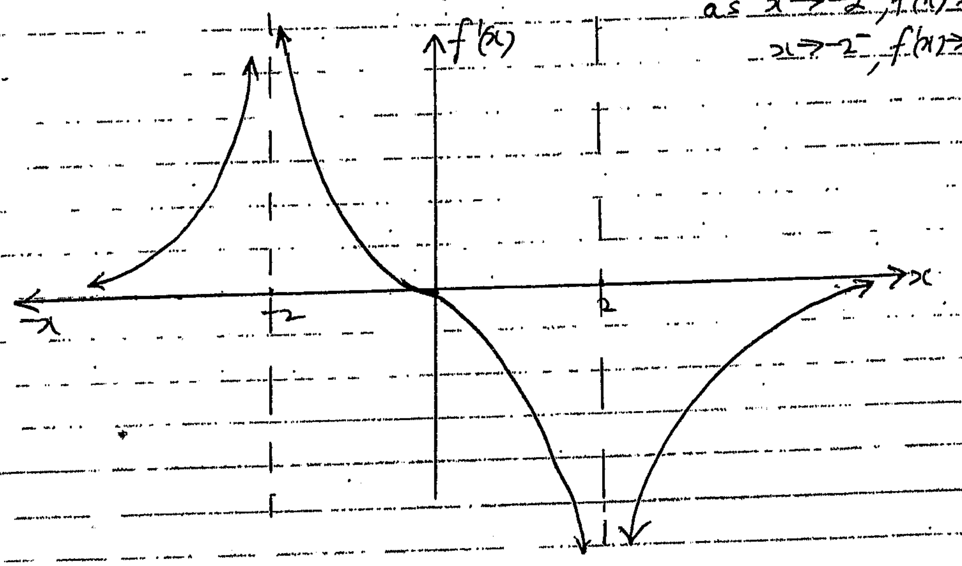
as $x \to 2^+$, $f'(x) \to -\infty$

$x \to 2^-$, $f'(x) \to -\infty$

as $x \to -2^+$, $f'(x) \to +\infty$

$x \to -2^-$, $f'(x) \to +\infty$

2 at $x=0$, $f'(x)=0$.



Q4

a) (i) $P(x) = x^3 - 2x^2 + 4x + 2 = 0$ has roots α, β, γ.

We want eqn in x where $x = \alpha - 1$.

Now, $P(\alpha) = 0$

$\therefore P(x+1) = 0$.

$P(x+1) = (x+1)^3 - 2(x+1)^2 + 4(x+1) + 2$

(2) $= x^3 + 3x^2 + 3x + 1 - 2x^2 - 4x - 2 + 4x + 6$

ie required eqn is

$x^3 + x^2 + 3x + 5 = 0$.

(ii) $x = \alpha^2$

$\therefore \alpha = \pm\sqrt{x}$.

$P(\alpha) = 0 \quad \therefore P(\pm\sqrt{x}) = 0$.

ie $(\pm\sqrt{x})^3 - 2(\pm\sqrt{x})^2 \pm 4\sqrt{x} + 2 = 0$.

ie $\pm\sqrt{x}(x+4) = 2x - 2$

$\therefore x(x+4)^2 = (2x-2)^2$

$x(x^2 + 8x + 16) = 4x^2 - 8x + 4$.

(2) $x^3 + 8x^2 + 16x = 4x^2 - 8x + 4$

ie $x^3 + 4x^2 + 24x - 4 = 0$

b) (i) $\int_2^3 \frac{x^3}{x^2-1}\,dx = \int_2^3 \frac{x(x^2-1) + x}{x^2-1}\,dx$

$= \int_2^3 x + \frac{x}{x^2-1}\,dx$

$= \left[\frac{1}{2}x^2 + \frac{1}{2}\ln(x^2-1)\right]_2^3$

(2) $= \frac{9}{2} + \frac{1}{2}\ln 8 - 2 - \frac{1}{2}\ln 3$

$= 2\frac{1}{2} + \frac{1}{2}\ln \frac{8}{3}$

$= \frac{5 + \ln(8/3)}{2}$

Q.4 b)

(ii) $\int_0^2 x e^x \, dx$

$= \int_0^2 x \frac{d}{dx} e^x \, dx$

$= xe^x\Big]_0^2 - \int_0^2 e^x . 1 \, dx$

$= \Big[xe^x - e^x\Big]_0^2$

(2)

$= 2e^2 - e^2 + 1$

$= e^2 + 1.$

e) (i) $x^2 + xy + y^2 = 3$ *

$\therefore \frac{d}{dx}(x^2 + xy + y^2) = \frac{d}{dx} 3$

ie $2x + xy' + y + 2yy' = 0$

(1) $y'(x+2y) = -2x - y$

$\therefore y' = \frac{-2x-y}{x+2y}$

(ii) $y' = 0 \Rightarrow -2x - y = 0$

ie $y = -2x$ (+)

Sub. $y = -2x$ in *

$\Rightarrow x^2 + x(-2x) + 4x^2 = 3$

$3x^2 = 3$

$x = \pm 1$

(2) when $x = 1$, $y = -2$

$x = -1$, $y = 2$

ie $(1,-2)$ & $(-1,2)$ are st. pts.

Also, $\frac{dy}{dx}$ undefined where $x + 2y = 0$

ie $x = -2y$

Sub. into *

$\Rightarrow 4y^2 - 2y^2 + y^2 = 3$

$3y^2 = 3$

$y = \pm 1$

when $y = 1$, $x = -2$

(2) $y = -1$, $x = 2$

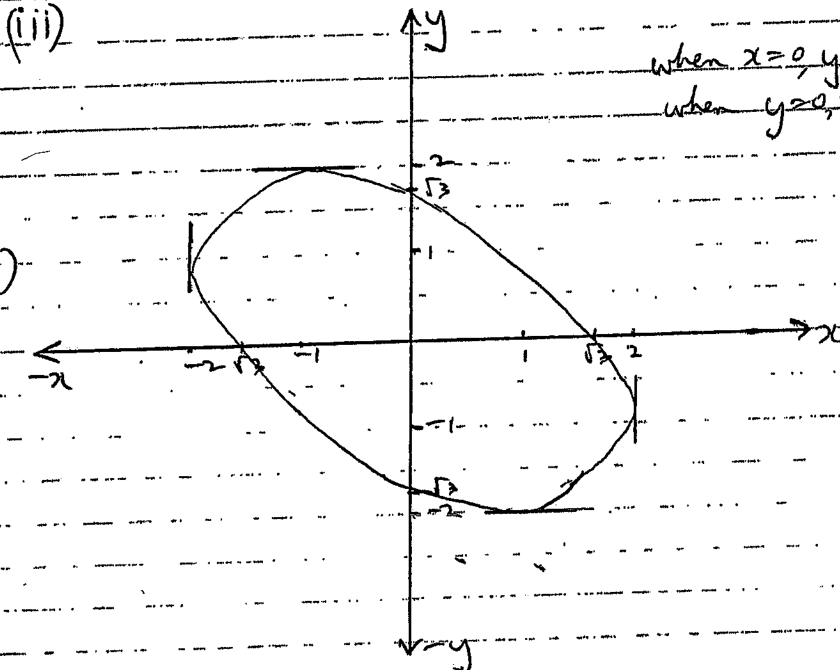
$\therefore y'$ undefined at $(-2,1)$, $(2,-1)$

(iii)

when $x = 0$, $y = \pm\sqrt{3}$

when $y = 0$, $x = \pm$

(2)



Q5

a) (i) $I_n = \int_0^{\frac{\pi}{2}} x^n \sin x \, dx, \quad n \geq 0$

$= \int_0^{\frac{\pi}{2}} x^n \frac{d}{dx}(-\cos x) \, dx$

$= -x^n \cos x \Big]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos x \cdot n x^{n-1} \, dx$

$= n \int_0^{\frac{\pi}{2}} \cos x \; x^{n-1} \, dx$

$= n \int_0^{\frac{\pi}{2}} \frac{d}{dx}(\sin x) \; x^{n-1} \, dx$

$= n \left\{ x^{n-1} \sin x \Big]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x \, (n-1) x^{n-2} \, dx \right\}$

ie $I_n = n\left(\frac{\pi}{2}\right)^{n-1} \cdot 1 - 0 - n(n-1)\int_0^{\frac{\pi}{2}} x^{n-2} \sin x \, dx$ (6)

ie $I_n + n(n-1) I_{n-2} = n\left(\frac{\pi}{2}\right)^{n-1}$. #

(5) (ii) $I_1 = \int_0^{\frac{\pi}{2}} x \sin x \, dx$

$= \int_0^{\frac{\pi}{2}} x \cdot \frac{d}{dx}(-\cos x) \, dx$

$= -x \cos x \Big]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos x \cdot 1 \, dx$

$= 0 + \left[\sin x\right]_0^{\frac{\pi}{2}}$

$= 0 + 1 - 0$

$= 1$

(iii) Let $n = 3$ in #

ie $I_3 + 3 \cdot 2 I_1 = 3\left(\frac{\pi}{2}\right)^2$

$\therefore \int_0^{\frac{\pi}{2}} x^3 \sin x \, dx = I_3 = \frac{3\pi^2}{4} - 3 \cdot 2 \cdot 1$

$= \frac{3\pi^2}{4} - 6$

$\int_0^1 \frac{x}{\sqrt{1-x^2}} \, dx$

let $u = 1 - x^2$ $\qquad x = 0, u = 1$

$\therefore \frac{du}{dx} = -2x$ $\qquad x = 1, u = 0$

$x \, dx = -\frac{1}{2} du$

$\therefore I = \int_1^0 -\frac{1}{2} u^{-\frac{1}{2}} \, du$

$= \left[-\frac{1}{2} \cdot 2u^{\frac{1}{2}}\right]_1^0$

$= \left[-u^{\frac{1}{2}}\right]_1^0$

$= 1$

$\therefore \int_0^1 \cos^{-1} x \, dx = 0 - 0 + 1$

$= 1$

$\therefore V = 2\pi \int_0^1 x \cos^{-1} x + \cos^{-1} x \, dx$

$= 2\pi\left(\frac{\pi}{8} + 1\right) \, v^3$

Q4

a) (i) ~~$1 + 10 + 10^2 + \ldots + 10^n$~~

~~G.P. $\quad a = 1, \quad r = 10 \quad N = n+1$~~

~~$S_N = \frac{a(r^N - 1)}{r-1}$~~

~~$= \frac{10^{n+1} - 1}{9}$~~

(ii) ~~$1 \times 9^2 + 11 \times 9^2 + 111 \times 9^2 + \ldots + (\underbrace{111\ldots1}_{n \text{ ones}}) \times 9^2$~~

$= 10^{n+1} - 9n -$

b) at $t=0$, $x=0$, $v=0$.

$\ddot{x}=1$ for $1\le t\le 3$

$\therefore \frac{dv}{dt}=1, \quad 1\le t\le 3$.

$v = t + C$. $*$

Now, $\int_0^1 \ddot{x}\,dt = \int_0^1 \frac{dv}{dt}\,dt$

$= v(t)\Big]_0^1$

$= v(1)-v(0)$

$= v(1)$ since $v(0)=0$

$\therefore$ vel. after 1 sec $= 3/2$ (area under graph)

$\therefore$ when $t=1$, $v=3/2$.

2

Sub into $*$

$\Rightarrow 3/2 = 1 + C$

$\therefore C = \frac{1}{2}$.

$v = t + \frac{1}{2}, \quad 1\le t\le 3$.

(ii) $\therefore$ when $t=3$, $v=3\frac{1}{2}$ from (1)

Now, for $3\le t\le 5$, $\ddot{x} = -t+4$

$\therefore v = -\frac{t^2}{2} + 4t + k$

2

when $t=3$, $v=3\frac{1}{2}$

$\therefore 3\frac{1}{2} = -\frac{9}{2} + 12 + k$

$\therefore k = 8-12 = -4$.

$\therefore v = -\frac{t^2}{2} + 4t - 4$

when $t=5$,

$* \int_0^5 \ddot{x}\,dt$

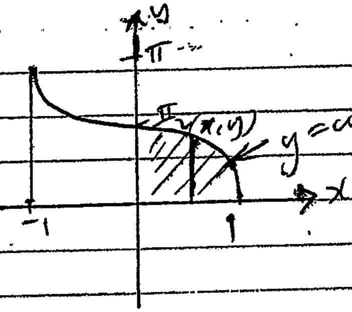
$= v(5)-v(0)$

$= v(5)$

$\therefore$ velocity after 5s $= 4-\frac{1}{2} = 3\frac{1}{2}$

OR $*$

c)



$\Delta V \doteq 2\pi(x+1).y.\Delta x$

$V = \lim_{\Delta x\to 0} \sum_{x=0}^{1} 2\pi(x+1)y\,\Delta x$

$= 2\pi \int_0^1 (x+1)\cos^{-1}x\,dx$.

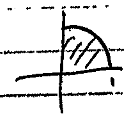
Now, $\int_0^1 x\cos^{-1}x\,dx = \int_0^1 \cos^{-1}x \cdot \frac{d}{dx}\left(\frac{1}{2}x^2\right)dx$

$= \frac{1}{2}x^2\cos^{-1}x\Big]_0^1 - \frac{1}{2}\int x^2 \cdot \frac{-1}{\sqrt{1-x^2}}$

$= \frac{1}{2}x^2\cos^{-1}x\Big]_0^1 + \frac{1}{2}\int_0^1 \frac{x^2}{\sqrt{1-x^2}}\,dx$

$\int_0^1 \frac{x^2}{\sqrt{1-x^2}}\,dx = -\int_0^1 \frac{1-x^2}{\sqrt{1-x^2}}\,dx + \int_0^1 \frac{dx}{\sqrt{1-x^2}}$

(6)



$A = \frac{1}{4}\pi \cdot 1^2$

$= -\int_0^1 \sqrt{1-x^2}\,dx + \left[\sin^{-1}x\right]_0^1$

$= -\frac{\pi}{4} + \sin^{-1}1 - \sin^{-1}0$.

$= -\frac{\pi}{4} + \frac{\pi}{2} = \frac{\pi}{4}$

$\therefore \int_0^1 x\cos^{-1}x\,dx = 0-0 + \frac{1}{2}\times\frac{\pi}{4}$

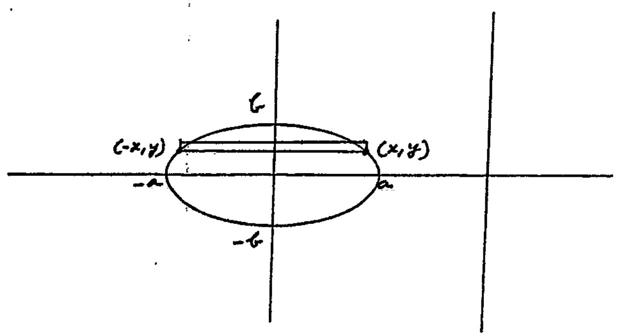
$= \frac{\pi}{8}$.

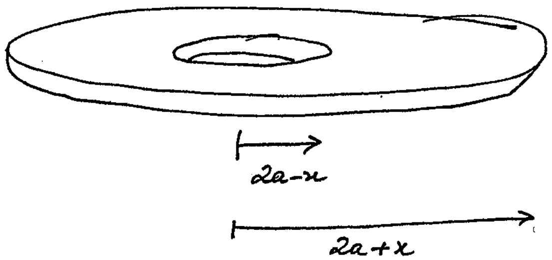
Also, $\int_0^1 \cos^{-1}x\,dx = \int_0^1 \cos^{-1}x \frac{d}{dx}x\,dx$

$= x\cos^{-1}x\Big]_0^1 + \int_0^1 \frac{x}{\sqrt{1-x^2}}$

QUESTION 6:

(a)





Volume of slice is

$$\delta V = \pi(2a+x)^2\,\delta y - \pi(2a-x)^2\,\delta y$$
$$= \pi\left[(2a+x+2a-x)(2a+x-2a+x)\right]\delta y$$
$$= \pi(4a)(2x)\,\delta y$$
$$= 8\pi a x\,\delta y$$

∴ Volume of solid is

$$V = \lim_{\delta y \to 0} \sum_{y=-b}^{b} 8\pi a x\,\delta y$$

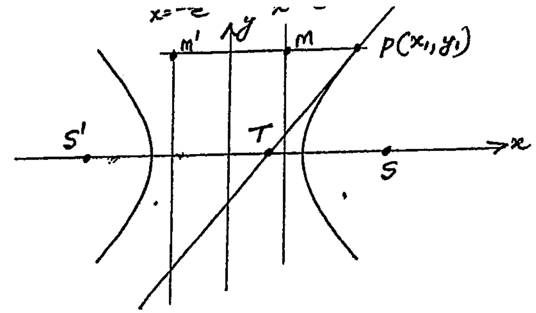
$$= 8\pi a \int_{-b}^{b} x\,dy \quad \text{where} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$= 8\pi a \int_{-b}^{b} \frac{a}{b}(b^2-y^2)^{\frac{1}{2}}\,dy \qquad \frac{x^2}{a^2} = 1 - \frac{y^2}{b^2}$$

$$\therefore x^2 = \frac{a^2}{b^2}(b^2-y^2)$$

$= 8\pi a \times \frac{a}{b} \times \frac{1}{2}\pi b^2$ (area of

(b)



(i) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Differentiating with respect to x

$$\Rightarrow \frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0$$

$$\text{ie } \frac{dy}{dx} = + \frac{b^2x}{a^2y}$$

at $P(x_1,y_1)$ $\qquad \frac{dy}{dx} = + \frac{b^2x_1}{a^2y_1}$

∴ Tangent at $P(x_1,y_1)$ is

$$y - y_1 = + \frac{b^2x_1}{a^2y_1}(x - x_1)$$

$$a^2yy_1 - a^2y_1^2 = + b^2xx_1 - b^2x_1^2$$

$$\text{ie } b^2xx_1 - a^2yy_1 = b^2x_1^2 - a^2y_1^2$$

$$\div a^2b^2 \Rightarrow \frac{xx_1}{a^2} - \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2}$$

$= 1$ since (x_1,y_1) lies on $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

(ii) at T, $y = 0$

$$\therefore \frac{xx_1}{a^2} = 1$$

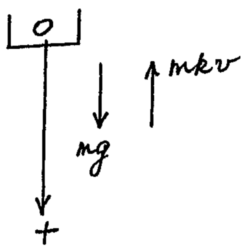
$$\text{ie } x = \frac{a^2}{x_1}$$

$$\therefore T = \left(\frac{a^2}{x_1}, 0\right)$$

(iii) $$\frac{PS}{PS'} = \frac{ePM}{ePM'} = \frac{PM}{PM'} = \frac{x_1 - \frac{a}{e}}{x_1 + \frac{a}{e}} = \frac{x_1 e - a}{x_1 e + a}$$

$$\frac{TS}{TS'} = \frac{ae - \frac{a^2}{x_1}}{\frac{a^2}{x_1} + ae} = \frac{aex_1 - a^2}{a^2 + aex_1} = \frac{\cancel{a}(x_1 e - a)}{\cancel{a}(a + x_1 e)} = \frac{PS}{PS'}$$

)



(i) $$R = m\ddot{x}$$
$$\Rightarrow mg - mkv = m\ddot{x}$$
$$g - kv = \ddot{x} \quad \text{——— (1)}$$

Terminal velocity at $\ddot{x} = 0$

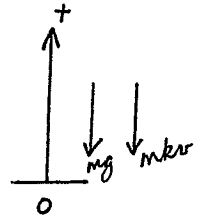
ie $g - kv = 0$

$$\therefore v_T = \frac{g}{k}$$

(ii) (1) $\Rightarrow$ $$\frac{dv}{dt} = g - kv$$
$$\therefore \frac{dt}{dv} = \frac{1}{g-kv}$$
$$t = \int_0^{\frac{v_T}{2}} \frac{1}{g-kv}\,dv$$
$$= -\frac{1}{k}\ln(g-kv)\Big]_0^{\frac{g}{2k}}$$
$$= -\frac{1}{k}\ln\left(g - \frac{g}{2}\right) + \frac{1}{k}\ln g$$
$$= \frac{1}{k}\ln\left(\frac{g}{\frac{g}{2}}\right)$$

QUESTION 7:

(a)



$$R = m\ddot{x}$$
$$\Rightarrow m\ddot{x} = -\cancel{m}g - mkv$$
$$\text{ie } \ddot{x} = -g - kv$$
$$\therefore v\frac{dv}{dx} = -g - kv$$
$$\frac{dv}{dx} = \frac{-g-kv}{v}$$
$$\therefore \frac{dx}{dv} = -\frac{v}{g+kv}$$
$$\therefore x_{max} = -\int_V^0 \frac{v}{g+kv}\,dv$$
$$= -\int_0^V \left(\frac{1}{k}\cdot\frac{g+kv}{g+kv} - \frac{g}{k}\cdot\frac{1}{g+kv}\right) dv$$
$$= -\int_0^V \left[\left(\frac{1}{k}\right)\cdot 1 - \frac{g}{k^2}\cdot\frac{k}{g+kv}\right] dv$$
$$= -\left[\frac{v}{k} - \frac{g}{k^2}\ln|g+kv|\right]_V^0$$
$$= \frac{g}{k^2}\ln g + \frac{V}{k} - \frac{g}{k^2}\ln(g+kV)$$
$$= \frac{V}{k} + \frac{g}{k^2}\ln\left(\frac{g}{g+kV}\right)$$

(b)

$$A+B+C = 180$$
$$\therefore A+B = 180-C$$
$$\therefore \tan(A+B) = \tan(180-C)$$
$$= -\tan C$$

$$\Rightarrow \frac{\tan A+\tan B}{1-\tan A\tan B} = -\tan C$$

$$\therefore \tan A+\tan B = -\tan C + \tan A\tan B\tan C$$

ie $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

(c) (i)

$$\tan 4x = \frac{2\tan 2x}{1-\tan^2 2x}$$

$$= \frac{2.\left(\frac{2\tan x}{1-\tan^2 x}\right)}{1-\left(\frac{2\tan x}{1-\tan^2 x}\right)^2}$$

$$= \frac{2.\ \frac{2t}{1-t^2}}{1-\left(\frac{2t}{1-t^2}\right)^2} \times \frac{(1-t^2)^2}{(1-t^2)^2}$$

$$= \frac{4t(1-t^2)}{(1-t^2)^2-4t^2}$$

$$= \frac{4t(1-t^2)}{t^4-6t^2+1}$$

(ii) $\tan x \tan 4x = 1$

$$\Rightarrow t.\ \frac{4t(1-t^2)}{t^4-6t^2+1} = 1$$

$$4t^2(1-t^2) = t^4-6t^2+1$$
$$4t^2-4t^4 = t^4-6t^2+1$$

(iii) $x = 18^\circ$: $\tan 18^\circ \tan 72^\circ$

$$= \cot 72^\circ .\ \tan 72^\circ$$
$$= 1$$

$x = 54^\circ$: $\tan 54^\circ \tan 216^\circ$

$$= \cot 36^\circ .\ \tan 36^\circ$$
$$= 1.$$

$\therefore$ Both $x = 18^\circ$ and $x = 54^\circ$ satisfy $\tan x \tan 4x = 1$

(iv) $5t^4 - 10t^2 + 1 = 0$ has solutions $t = \tan 18^\circ, \tan 54^\circ$

$$\Rightarrow t^2 = \frac{10 \pm \sqrt{100-20}}{10}$$

$$= \frac{10 \pm 4\sqrt{5}}{10}$$

$$= \frac{5 \pm 2\sqrt{5}}{5}$$

$$= \frac{5-2\sqrt{5}}{5},\ \frac{5+2\sqrt{5}}{5}$$

ie $t = \pm\sqrt{\frac{5-2\sqrt{5}}{5}}, \pm\sqrt{\frac{5+2\sqrt{5}}{5}}$

since $\tan 18^\circ$ and $\tan 54^\circ$ are both positive with $\tan 54^\circ > \tan 18^\circ$ Then

$$\tan 54^\circ = \left(\frac{5+2\sqrt{5}}{5}\right)^{\frac{1}{2}}$$

QUESTION 8:

(a) (i) $\underbrace{1+10+10^2+\dots+10^n}$

Geometric series $a=1, r=10, n=n+1$

$$S = \frac{1\left[10^{n+1}-1\right]}{10-1}$$

$$= \frac{10^{n+1}-1}{9} \quad \text{———} \; (1)$$

(ii) Let $S(n)$ be the assertion that

$1\times 9^2 + 11\times 9^2 + \dots + \underbrace{(111\dots1)}_{n \text{ ones}}9^2 = 10^{n+1} - 9n - 10$

Test $S(1)$: $LHS = 1\times 9^2$
$= 81$
$RHS = 10^2 - 9 - 10$
$= 81$

$\therefore S(1)$ is true

Assume $S(k)$ is true for some integer $n=k$

ie $1\times 9^2 + 11\times 9^2 + \dots + \underbrace{111\dots1}_{k \text{ ones}} \times 9^2 = 10^{k+1} - 9k - 10$

(Now $1\times 9^2 + 11\times 9^2 + \dots + \underbrace{111\dots1}_{k \text{ ones}} \times 9^2 + \underbrace{111\dots1}_{(k+1) \text{ ones}} \times 9^2$

$= 10^{k+1} - 9k - 10 + 9^2\left[1+10+10^2+\dots+10^k\right]$

$= 10^{k+1} - 9k - 10 + 9^2 \cdot \left(\frac{10^{k+1}-1}{9}\right)$ from (1)

$= 10^{k+1} - 9k - 10 + 9 \cdot 10^{k+1} - 9$

$= 10^{k+1}\left[1+9\right] - 9(k+1) - 10$

$= 10^{k+2} - 9(k+1) - 10$

$= 10^{n+1} - 9n - 10$ where $n = k+1$

Hence if $S(n)$ is true for $n=k$ then it is also true for $n=k+1$.

But true for $n=1 \Rightarrow$ true for $n=2$ and then by the principle of mathematical induction $S(n)$ is true for all $n \geq 1$

(b) (i) $z^5 = 1 \qquad z \neq 1$

$\Rightarrow z^5 - 1 = 0$

$(z-1)(z^4+z^3+z^2+z+1) = 0$

$z \neq 1 \Rightarrow z^4 + z^3 + z^2 + z + 1 = 0$

$\Rightarrow z^2 + z + 1 + z^{-1} + z^{-2} = 0 \quad \text{———} \; (1)$

(ii) $z^5 = 1$
$= 1 \operatorname{cis}(0 + 2k\pi)$

$\therefore z = \operatorname{cis}\frac{2k\pi}{5} \qquad k = 0,1,2,3,4$

but $z \neq 1 \Rightarrow z = \operatorname{cis}\frac{2k\pi}{5} \qquad k = 1,2,3,4$

Then $z + z^{-1} = \cos\frac{2k\pi}{5} + i\sin\frac{2k\pi}{5} + \cos\left(\frac{-2k\pi}{5}\right) + i\sin\left(\frac{-2k}{5}\right.$

$= 2\cos\frac{2k\pi}{5} \qquad k = 1,2,3,4$

(iii) If $x = z + z^{-1}$

$x^2 = z^2 + 2 + z^{-2} \Rightarrow z^2 + z^{-2} = x^2 - 2$

Then (1) $\Rightarrow (z^2 + z^{-2}) + (z + z^{-1}) + 1 = 0$

$\Rightarrow x^2 - 2 + x + 1 = 0$

$x^2 + x - 1 = 0 \quad \text{———} \; (2)$

now $z + z^{-1} = 2\cos\frac{2k\pi}{5}$ $\quad k = 1, 2, 3, 4$

$k=1 \Rightarrow z + z^{-1} = 2\cos\frac{2\pi}{5}$

$k=2 \Rightarrow z + z^{-1} = 2\cos\frac{4\pi}{5} = -2\cos\frac{\pi}{5}$

$k=3 \Rightarrow z + z^{-1} = 2\cos\frac{6\pi}{5} = -2\cos\frac{\pi}{5}$

$k=4 \Rightarrow z + z^{-1} = 2\cos\frac{8\pi}{5} = 2\cos\frac{2\pi}{5}$

Hence the solutions of ② are

$x = 2\cos\frac{2\pi}{5}$ and $x = -2\cos\frac{\pi}{5}$

Product of roots

$\Rightarrow 2\cos\frac{2\pi}{5} \times -2\cos\frac{\pi}{5} = \frac{c}{a}$

$= -1$

$\therefore -4\cos\frac{\pi}{5}\cos\frac{2\pi}{5} = -1$

$\therefore \cos\frac{\pi}{5}\cos\frac{2\pi}{5} = \frac{1}{4}.$