

**St George Girls High School**

**Trial Higher School Certificate Examination**

**2005**



# **Mathematics Extension 2**

## **General Instructions**

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen
- Write your Student Number on every page
- All questions may be attempted.
- Begin each question in a new booklet.
- All necessary working must be shown.
- Marks may be deducted for careless or poorly presented work.
- Board-approved calculators may be used.
- A list of standard integrals is included at the end of this paper.
- The mark allocated for each question is listed at the side of the question.

**Students are advised that this is a Trial Examination only and does not necessarily reflect the content or format of the Higher School Certificate Examination.**

**Question 1 – (15 marks) – Start a new booklet**

**Marks**

a) Find  $\int \frac{e^x}{4 + e^{2x}} dx$

2

b) By completing the square and using the table of standard integrals find

$$\int \frac{dx}{\sqrt{x^2 - 4x + 9}}$$

2

c) Evaluate  $\int_5^{21} \frac{x}{x + 4 + 2\sqrt{x + 4}} dx$  using the substitution  $x = u^2 - 4$

3

d) Using the substitution  $t = \tan \frac{x}{2}$ , or otherwise, calculate

$$\int_0^{\frac{2\pi}{3}} \frac{dx}{13 + 5 \sin x + 12 \cos x}$$

4

e) Find  $\int e^{2x} \cos x dx$

4

**Question 2 – (15 marks) – Start a new booklet**

**Marks**

a)  $z_1 = 3 - 2i$  and  $z_2 = 1 - i$

2

Find in the form  $a + ib$  (where  $a$  and  $b$  are real).

(i)  $z_1 \bar{z}_2$

(ii)  $\frac{i}{z_1}$

b) If  $z = 1 - i$  and  $w = \sqrt{3} + i$

6

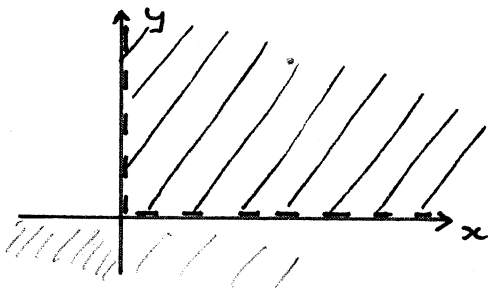
(i) Find  $\frac{w}{z}$  in the form  $a + ib$

(ii) Show that  $\arg\left(\frac{w}{z}\right) = \frac{5\pi}{12}$

(iii) Find the modulus of  $\frac{w}{z}$

(iv) Hence find the exact value of  $\cos\frac{5\pi}{12}$

c) It is given that  $z^2$  lies in the first quadrant of the complex plane, as shown in the diagram.



Shade the region in which  $z$  can lie.

2

d) Sketch the locus of  $z$  if

2

(i)  $|z - i| = |z - 2|$

(ii)  $\arg(z - i) = \arg(z - 2)$

e) It is given that  $z = 2 + i$  is a zero of  $P(z) = z^4 - 4z^3 + 20z - 25$

3

(i) Explain why  $2 - i$  is also a zero of  $P(z)$

(ii) Hence factorise  $P(z)$  over the real numbers

$(2+i)$

**Question 3 – (15 marks) – Start a new booklet**

**Marks**

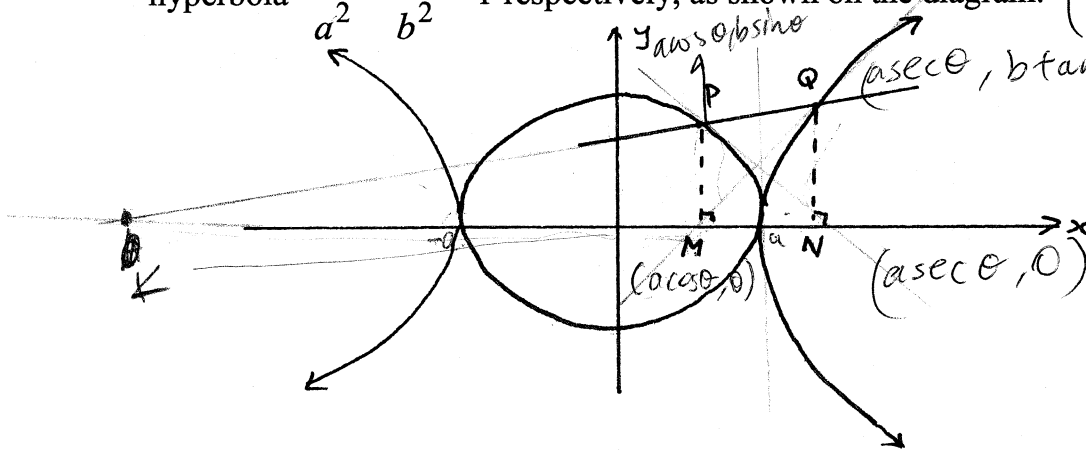
- a) The rectangular hyperbola,  $H$ , has equation  $x^2 - y^2 = 8$ . Write down
- the eccentricity
  - the coordinates of the foci
  - the equations of the directrices
  - the equations of the asymptotes and
  - sketch the curve showing the foci, directrices, asymptotes and any intercepts with the coordinate axes.

5

- b)  $P(a \cos \theta, b \sin \theta)$  and  $Q(a \sec \theta, b \tan \theta)$  lie on the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  and the

10

hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  respectively, as shown on the diagram.  $(0 < \theta < \frac{\pi}{2})$



$M$  is the foot of the perpendicular from  $P$  to the  $x$ -axis and  $N$  is the foot of the perpendicular from  $Q$  to the  $x$ -axis.  $QP$  meets the  $x$ -axis at  $K$ .  $A$  is the point  $(a, 0)$ .

- (i) Given that  $\triangle KPM \parallel \triangle KQN$ , show that  $\frac{KM}{KN} = \cos \theta$

- (ii) Hence show that  $K$  has coordinates  $(-a, 0)$

- (iii) Show that the tangent to the ellipse at  $P$  has equation  $\frac{x \cos \theta}{a} + \frac{y \sin \theta}{b} = 1$  and deduce that it passes through  $N$ .

- (iv) Given that the tangent to the hyperbola at  $Q$  has equation  $\frac{x \sec \theta}{a} - \frac{y \tan \theta}{b} = 1$

Show that it passes through  $M$ .

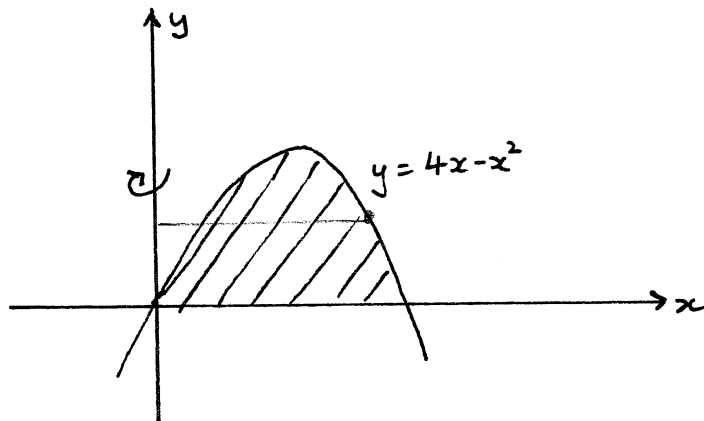
- (v) Show that the tangents  $PN$ ,  $QM$  and the common tangent at  $A$  are concurrent. Find the point of concurrence.

**Question 4 – (15 marks) – Start a new booklet**

**Marks**

- a) The shaded region is rotated about the  $y$ -axis to form a solid of revolution.

4



Using the method of cylindrical shells,

- (i) Show that the volume,  $V$ , of this solid is given by  $V = 2\pi \int_0^4 4x^2 - x^3 dx$

- (ii) Hence find the volume of the solid.

- b) The base of a solid is a circle of radius 1 unit. Parallel cross-sections perpendicular to the base are equilateral triangles. Find the volume of this solid.  
(Draw any necessary diagrams).

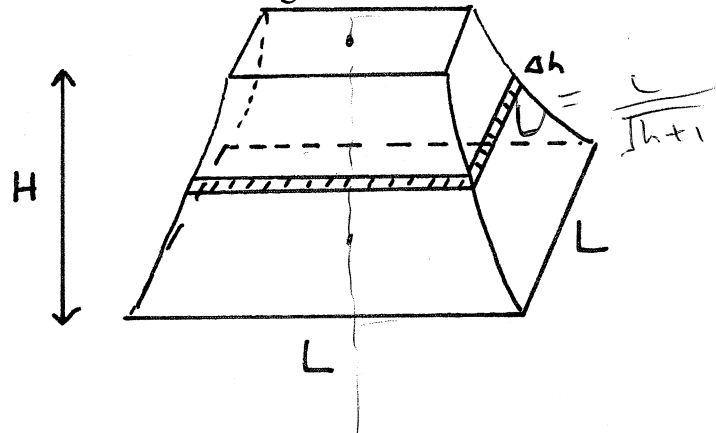
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**Part c) and d) on next page**

**Question 4 (cont'd)**

- c) A stone building of height  $H$  metres has the shape of a flat-topped square “pyramid” with curved sides as shown in the figure below.

4



The cross-section at height  $h$  metres above the base is a square with sides parallel to the sides of the base and of length  $l$  where  $l$  is given by  $l = \frac{L}{\sqrt{h+1}}$

( $L$  is the length of the side of the square base in metres).

- (i) Write an expression for the volume of a slice of width  $\Delta h$  at height  $h$  metres.
- (ii) Hence find the volume of the building in terms of  $L$  and  $H$ .

- d) Prove by Mathematical Induction that  $5^n + 2 \times 11^n$  is a multiple of 3 for all positive integers,  $n$ .

4

**Question 5 – (15 marks) – Start a new booklet**

**Marks**

a) (i) Find real numbers  $A, B$  and  $C$  such that  $\frac{5x^2 - 7x + 15}{(x^2 + 4)(x - 3)} \equiv \frac{Ax + B}{x^2 + 4} + \frac{C}{x - 3}$  6

(ii) Hence find  $\int_0^2 \frac{5x^2 - 7x + 15}{(x^2 + 4)(x - 3)} dx$

b)  $P(x) = 3x^3 + 7x + 2$  6

If  $\alpha, \beta$  and  $\gamma$  are the roots of  $P(x) = 0$

(i) Find the value of  $\alpha^3 + \beta^3 + \gamma^3$

(ii) Form polynomial equations with integer coefficients whose roots are

(A)  $\alpha^2, \beta^2, \gamma^2$

(B)  $\alpha + \beta, \beta + \gamma$  and  $\gamma + \alpha$

c) Find the equation of the tangent to the curve  $x^3 + 3xy - y^2 = 3$  at the point  $(1, 2)$  3

**Question 6 – (15 marks) – Start a new booklet**

**Marks**

- a) A particle of mass 5kg is acted on by a variable force whose direction is constant and whose magnitude at time  $t$  seconds is  $(3t - 4t^2)g$  Newtons. 4

If the particle has an initial velocity of 2m/s in the direction of the force, find its velocity at the end of 1 second.

- b) A particle of mass  $m$  kg is set in motion with speed  $u$  m/s and moves in a straight line before coming to rest. At time  $t$  seconds the particle has displacement  $x$  metres from its starting point  $O$ , velocity  $v$  ms<sup>-1</sup> and acceleration  $a$  ms<sup>-2</sup>. 11

The resultant force acting on the particle directly opposes its motion and has magnitude  $m(1 + v)$  Newtons.

(i) Show that  $a = -(1 + v)$

(ii) Find expressions for

(α)  $x$  in terms of  $v$

(β)  $v$  in terms of  $t$

and (γ)  $x$  in terms of  $t$

(iii) Show that  $x + v + t = u$

(iv) Find the distance travelled and time taken by the particle in coming to rest.

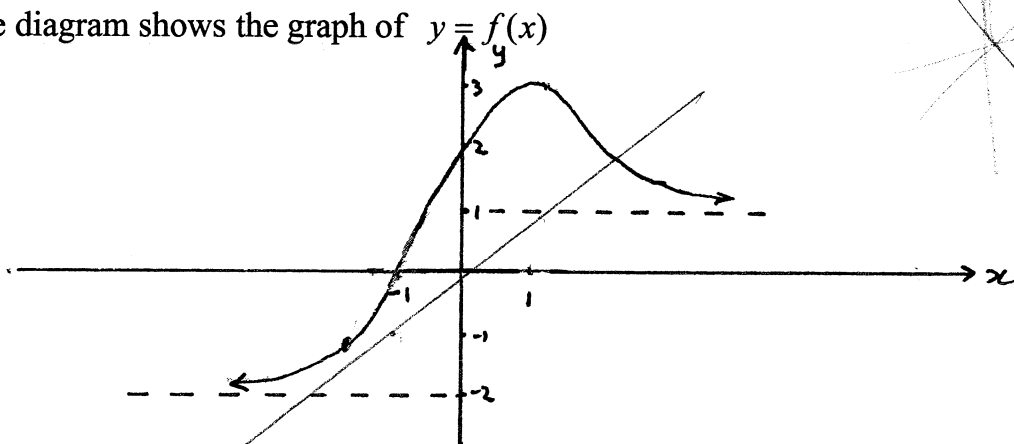


**Question 7 – (15 marks) – Start a new booklet**

Marks

a) The diagram shows the graph of  $y = f(x)$

8



Draw separate one-third page sketches of the graphs of

(i)  $y = f(|x|)$

(ii)  $y = \frac{1}{f(x)}$

(iii)  $y = 2^{f(x)}$

(iv)  $y = x f(x)$

b) (i) Show that  $\int_0^{\frac{\pi}{4}} (\tan x)^{2k} \sec^2 x \, dx = \frac{1}{2k+1}$

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(ii) By writing  $(\sec x)^{2n}$  as  $(1 + \tan^2 x)^n$  show that

$$\int_0^{\frac{\pi}{4}} (\sec x)^{2n+2} \, dx = \sum_{k=0}^n \frac{1}{2k+1} \binom{n}{k}$$

(iii) Hence or otherwise find the value of  $\int_0^{\frac{\pi}{4}} (\sec x)^8 \, dx$

**Question 8 – (15 marks) – Start a new booklet**

**Marks**

a) Let  $(3 + 2x)^{20} = \sum_{r=0}^{20} a_r x^r$

7

(i) Write an expression for  $a_r$

(ii) Show that  $\frac{a_{r+1}}{a_r} = \frac{40 - 2r}{3r + 3}$

(iii) Hence find the greatest coefficient in the expansion of  $(3 + 2x)^{20}$ .  
Give your answer in scientific notation correct to 4 significant figures.

b) A projectile is fired from ground level with an initial velocity of  $V$  m/s at an angle of elevation of  $\alpha$ . The only force acting on the particle is gravity. Acceleration due to gravity is  $g$  m/s<sup>2</sup>

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(i) Derive expressions for the horizontal and vertical components of displacement from the point of projection in terms of  $t$ , where  $t$  is the time in seconds since the projectile was fired.

(ii) Derive an expression for the time of flight, given that the projectile lands at ground level.

~~(iii)~~ At the instant the projectile is fired a target, which is initially  $b$  metres ahead of the point of projection, starts moving along the ground in the same horizontal direction as the projectile is moving. The target is moving at a speed of  $A$  m/s.

Show that if the projectile is to hit the target,  $V$  and  $\alpha$  must satisfy the equation

$$V^2 \sin 2\alpha - 2AV \sin \alpha - bg = 0$$

Solutions To Ext.2, TRIAL HSC 2005.

Q1  
a)  $I = \int \frac{e^x}{4+e^{2x}} dx$

let  $u = e^x$   
 $\therefore du = e^x dx$

$I = \int \frac{du}{4+u^2} \quad (2)$

$= \frac{1}{2} \tan^{-1} \frac{u}{2} + C$

$= \frac{1}{2} \tan^{-1} \left( \frac{e^x}{2} \right) + C$

b)  $\int \frac{dx}{\sqrt{x^2-4x+9}} = \int \frac{dx}{\sqrt{x^2-4x+4+5}}$

$= \int \frac{dx}{\sqrt{(x-2)^2+5}} \quad (2)$

$= \ln \left| x-2 + \sqrt{x^2-4x+9} \right| + C$

c)  $\int_5^{21} \frac{x}{x+4+2\sqrt{x+4}} dx$

let  $x = u^2 - 4$

$dx = 2u du$

when  $x = 5, u = 3$

$x = 21, u = 5$

$= \int_3^5 \frac{u^2-4}{u^2+2u} 2u du$

$= \int_3^5 \frac{(u+2)(u-2) 2u du}{u(u+2)}$

$= [u^2 - 4u]_3^5 \quad (3)$

$= 25 - 20 - (9 - 12)$

$= 8$

$$d) \int_0^{\frac{2\pi}{3}} \frac{dx}{13 + 5 \sin x + 12 \cos x}$$

$$= \int_0^{\sqrt{3}} \frac{2 dt}{13 + 5 \times \frac{2t}{1+t^2} + 12 \frac{(1-t^2)}{1+t^2}}$$

$$= \int_0^{\sqrt{3}} \frac{2 dt}{13 + 13t^2 + 10t + 12 - 12t^2}$$

$$= \int_0^{\sqrt{3}} \frac{2 dt}{t^2 + 10t + 25}$$

$$= \int_0^{\sqrt{3}} \frac{2 dt}{(t+5)^2}$$

$$= \left[ -2(t+5)^{-1} \right]_0^{\sqrt{3}}$$

$$= -2 \left( \frac{1}{5+\sqrt{3}} - \frac{1}{5} \right)$$

$$= \frac{2}{5} - \frac{2}{5+\sqrt{3}}$$

$$e) \int e^{2x} \cos x dx$$

$$= \int e^{2x} \frac{d(\sin x)}{dx} dx$$

$$= e^{2x} \sin x - \int 2e^{2x} \sin x dx$$

$$= e^{2x} \sin x - 2 \int e^{2x} \frac{d(-\cos x)}{dx} dx$$

$$= e^{2x} \sin x + 2 \int e^{2x} \frac{d(\cos x)}{dx} dx$$

$$= e^{2x} \sin x + 2 \left[ e^{2x} \cos x - \int \cos x \cdot 2e^{2x} dx \right]$$

$$= e^{2x} \sin x + 2e^{2x} \cos x - 4 \int \cos x e^{2x} dx$$

$$\text{Let } t = \tan \frac{x}{2}$$

$$\therefore \frac{dt}{dx} = \frac{\sec^2 \frac{x}{2}}{2}$$

$$dx = \frac{2 dt}{\sec^2 \frac{x}{2}}$$

$$= \frac{2 dt}{1 + \tan^2 \frac{x}{2}}$$

$$= \frac{2 dt}{1 + t^2}$$

$$\text{when } x=0, t=0$$

$$x = \frac{2\pi}{3}, t = \tan \frac{\pi}{3}$$

$$= \sqrt{3}$$

(4)

$$\therefore 5 \int e^{2x} \cos x \, dx = e^{2x} \sin x + 2e^{2x} \cos x + C$$

$$\therefore \int e^{2x} \cos x \, dx = \frac{1}{5} (e^{2x} \sin x + 2e^{2x} \cos x) + C$$

(4)

## Question 2

a) (i)  $z_1 = 3 - 2i$ ,  $z_2 = 1 - i$

$$\begin{aligned} z_1 \bar{z}_2 &= (3 - 2i)(1 + i) \\ &= 3 + 3i - 2i + 2 \\ &= 5 + i \end{aligned} \quad (1)$$

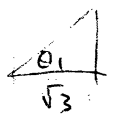
(ii)  $\frac{i}{z_1} = \frac{i}{3 - 2i} \times \frac{3 + 2i}{3 + 2i}$

$$\begin{aligned} &= \frac{3i - 2}{9 + 4} \\ &= -\frac{2}{13} + i \frac{3}{13} \end{aligned} \quad (1)$$

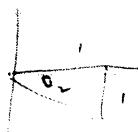
b) (i)  $\frac{w}{z} = \frac{\sqrt{3} + i}{1 - i} \times \frac{1 + i}{1 + i}$

$$\begin{aligned} &= \frac{\sqrt{3} + \sqrt{3}i + i - 1}{1 - i^2} \\ &= \frac{\sqrt{3} - 1 + i(\sqrt{3} + 1)}{2} \\ &= \frac{\sqrt{3} - 1}{2} + i \frac{\sqrt{3} + 1}{2} \end{aligned} \quad (1)$$

(ii)  $\arg\left(\frac{w}{z}\right) = \arg w - \arg z$



$\tan \theta_1 = \frac{1}{\sqrt{3}}$   
 $\therefore \theta_1 = \frac{\pi}{6}$



$\theta_2 = -\frac{\pi}{3}$

$$\begin{aligned} &= \frac{\pi}{6} - \left(-\frac{\pi}{3}\right) \\ &= \frac{2\pi}{12} + \frac{3\pi}{12} \\ &= \frac{5\pi}{12} \end{aligned} \quad (2)$$

$$(iii) \left| \frac{w}{z} \right| = \frac{|w|}{|z|}$$

$$= \frac{|\sqrt{3}+i|}{|1-i|}$$

$$= \frac{\sqrt{(\sqrt{3})^2+1^2}}{\sqrt{1^2+(-1)^2}} \quad (2)$$

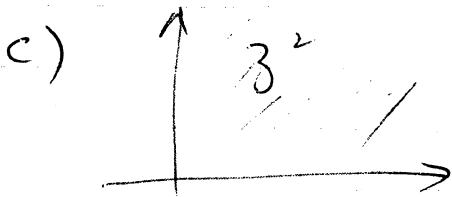
$$= \frac{2}{\sqrt{2}} = \sqrt{2}$$

$$(iv) \frac{w}{z} = \sqrt{2} \left( \cos \frac{5\pi}{12} + i \sin \frac{5\pi}{12} \right) \text{ from (ii), (iii)}$$

$$= \frac{\sqrt{3}-1}{2} + i \frac{\sqrt{3}+1}{2} \text{ from (i)}$$

$$\therefore \sqrt{2} \cos \frac{5\pi}{12} = \frac{\sqrt{3}-1}{2} \text{ (equating real parts)}$$

$$\therefore \cos \frac{5\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}} \quad (1)$$



let  $z = a+ib$ .

$$\therefore z^2 = a^2 - b^2 + 2abi$$

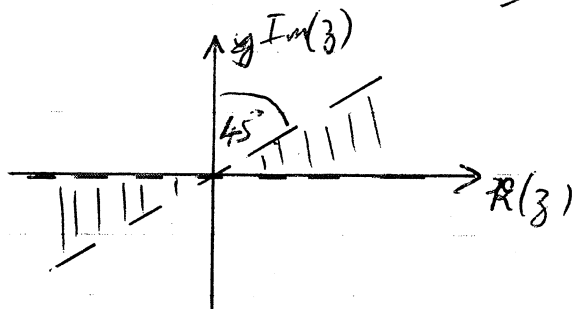
For  $z^2$  to be in 1st quadrant,

$$a^2 - b^2 > 0 \text{ and } 2ab > 0$$

$$\text{or } a^2 > b^2 \text{ and } ab > 0$$

$$\therefore a > b > 0 \text{ or } a < b < 0$$

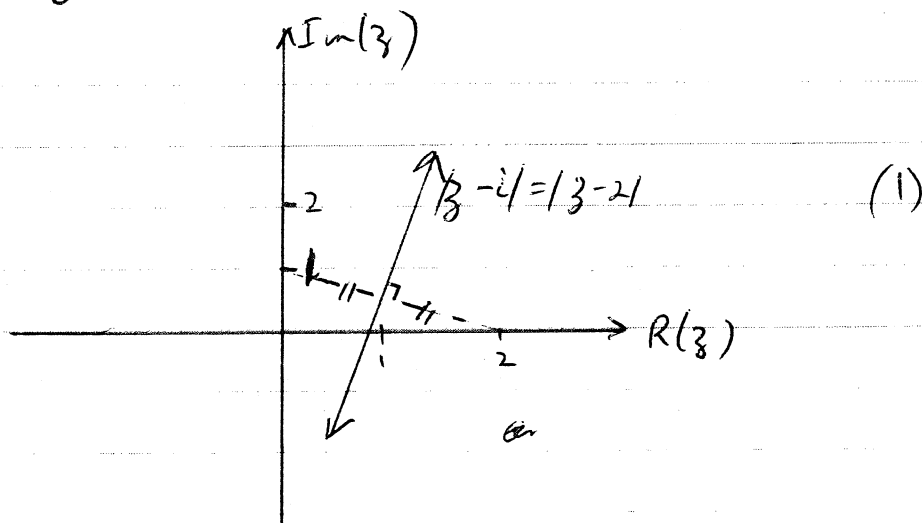
$a+ib$  (i.e.  $z$ )  
must lie in  
this region:



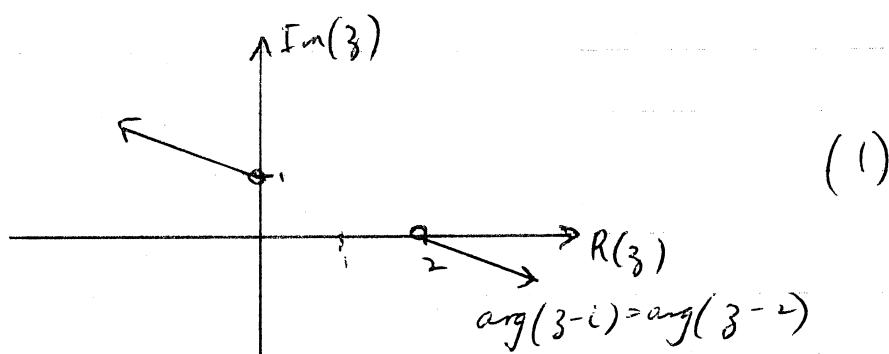
(2)

6

d) (i)



(ii)



e) (i) Since the coefficients are real, if  $2+i$  is a zero, so its conjugate  $2-i$  is a zero. (conjugate root theorem) (1)

(ii)  $[z - (2+i)][z - (2-i)]$  is a factor of  $P(z)$ .

$$\underline{=} (z - 2 - i)(z - 2 + i) \quad \text{"}$$

$$\underline{=} (z - 2)^2 + 1 \quad \text{"}$$

$$\underline{=} z^2 - 4z + 5 \quad \text{"}$$

$$P(z) = z^4 - 4z^3 + 20z^2 - 25$$

$$= (z^2 - 4z + 5)(z^2 - 5) \quad \text{by inspection}$$

$$= (z^2 - 4z + 5)(z + \sqrt{5})(z - \sqrt{5}) \quad \text{[division of polynomials]}$$

$$(z^2 - 4z + 5, \Delta < 0)$$

(over reals)

(2)



Q3

CONICS



7

a) (i)  $e = \sqrt{2}$ .

(ii)  $x^2 - y^2 = 8$

$\therefore a^2 = b^2 = 8$

$\therefore a = 2\sqrt{2}$ .

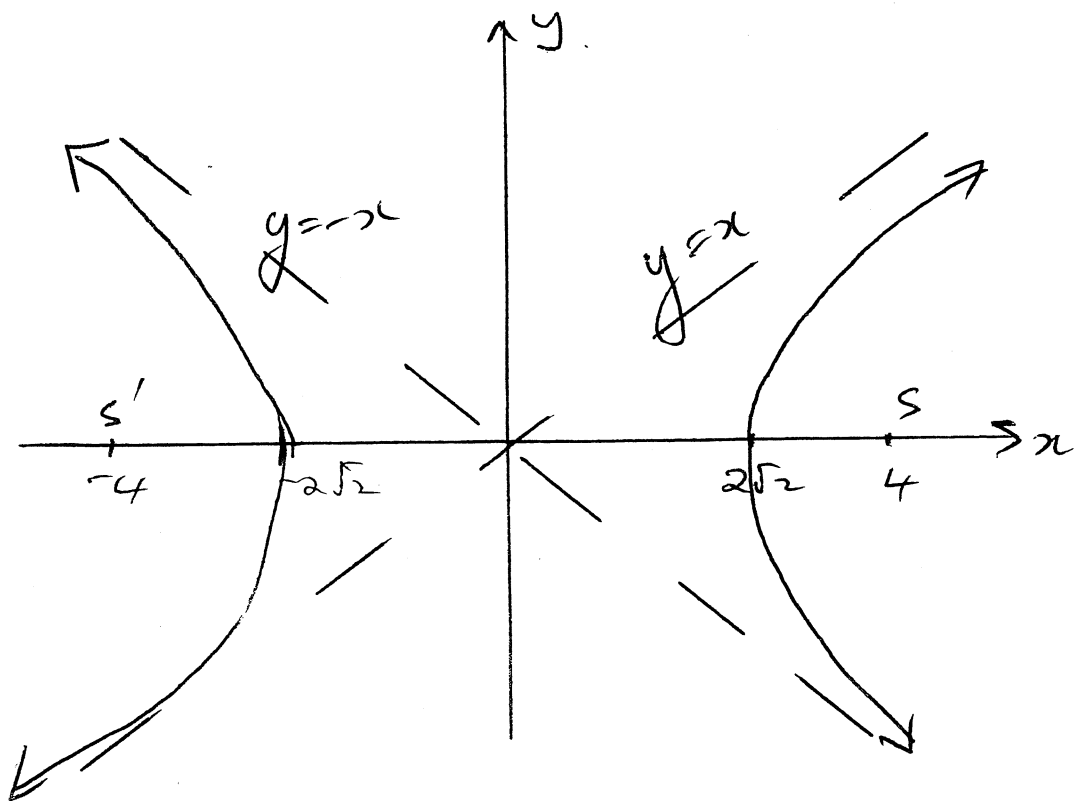
Foci are  $(\pm ae, 0)$

ie  $(\pm 4, 0)$

(iii)  $x = \pm \frac{a}{e} \Rightarrow x = \pm 2$  are the directrices

(iv)  $y = \pm \frac{bx}{a} \Rightarrow y = \pm x$  are the asymptotes.

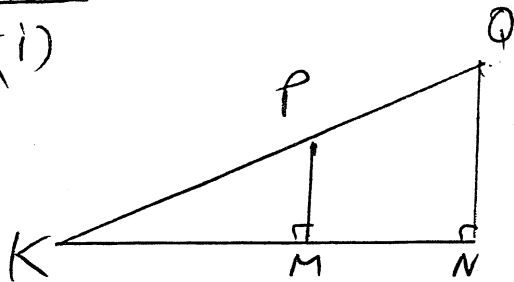
(v)



8

## CONICS

b) (i)



By similar  $\Delta$ 's,

$$\frac{KM}{KN} = \frac{b \sin \theta}{b \tan \theta}$$

$$= \cos \theta$$

①

(ii) Let K be  $(k, 0)$ 

Using result above:  $KM = a \cos \theta - k$   
 $KN = a \sec \theta - k$

$$\therefore \frac{a \cos \theta - k}{a \sec \theta - k} = \cos \theta$$

$$\therefore a \cos \theta - k = (a \sec \theta - k) \cos \theta$$

$$= a - k \cos \theta$$

$$a \cos \theta - a = k(1 - \cos \theta)$$

$$a(\cos \theta - 1) = -k(\cos \theta - 1)$$

$$\therefore a = -k$$

②

$$\therefore K \text{ is } (-a, 0)$$

(iii)  $P(a \cos \theta, b \sin \theta)$  is a point on the ellipse  $\frac{x \cos \theta}{a} + \frac{y \sin \theta}{b} = 1$ .

Now,  $x = a \cos \theta$

$$y = b \sin \theta$$

$$\therefore \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

$$= \frac{b \cos \theta}{-a \sin \theta}$$

③

$\therefore$  equation of tangent at P given by  $y = -\frac{b \cos \theta}{a \sin \theta} x + B$  (B const.)

i.e.  $b \cos \theta x + a \sin \theta y = C$  (C const.)

Since P lies on this tangent, it satisfies this equation.

$$\text{i.e. } ab \cos^2 \theta + ab \sin^2 \theta = c$$

$$\therefore c = ab.$$

$\therefore$  tangent at P has equation

$$b \cos \theta x + a \sin \theta y = ab.$$

$$\therefore x \frac{\cos \theta}{a} + y \frac{\sin \theta}{b} = 1 \quad *$$

Does N  $(a \sec \theta, 0)$  satisfy  $*$  ?

$$a \sec \theta \cdot \frac{\cos \theta}{a} + 0 = 1 \quad \checkmark$$

$\therefore$  N lies on tangent at P.

(iv) M has coords  $(a \cos \theta, 0)$  and it has to satisfy  $x \frac{\sec \theta}{a} - y \frac{\tan \theta}{b} = 1$

if it is to lie on tangent at Q.

$$(2) \quad \text{LHS} = a \cos \theta \frac{\sec \theta}{a} - 0$$

$$= 1$$

$$= \text{RHS.} \quad \checkmark$$

$\therefore$  M lies on tangent at Q

(v) Common tangent at A has equation  $x = a$   
Required to find the point of intersection of  $x = a$  with tangent PN.

$$\text{Put } x = a \text{ into } x \frac{\cos \theta}{a} + y \frac{\sin \theta}{b} = 1$$

$$\text{i.e. } \cos \theta + y \frac{\sin \theta}{b} = 1$$

$$\therefore y = b \frac{(1 - \cos \theta)}{\sin \theta}$$

$\therefore$  PN meets common tangent at

$$\left(a, b \frac{(1 - \cos \theta)}{\sin \theta}\right)$$

Similarly, find point of intersection of  $x=a$  with tangent  $QM$ .

e put  $x=a$  into

$$\frac{x}{a} \sec \theta - \frac{y}{b} \tan \theta = 1$$

$$\Rightarrow \sec \theta - y \frac{\tan \theta}{b} = 1 \quad (x \cos \theta)$$

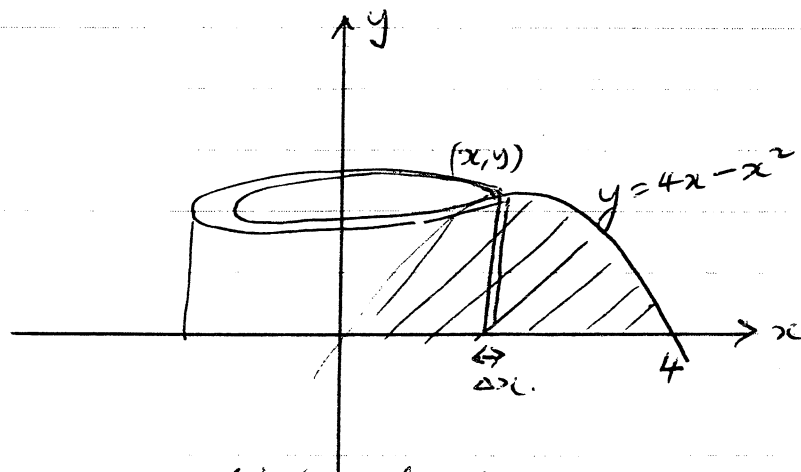
$$\Rightarrow 1 - y \frac{\sin \theta}{b} = \cos \theta$$

$$\text{e} \quad \& \ y = b \frac{(1 - \cos \theta)}{\sin \theta}$$

e 3 tangents concurrent at  $\left(a, b \frac{(1 - \cos \theta)}{\sin \theta}\right)$ .

Question 4

a) (i)



Take a strip of thickness  $\Delta x$ , parallel to  $y$ -axis as shown - rotate about  $y$ -axis

$$\begin{aligned} \Delta V &= \pi(R^2 - r^2)h \\ &= \pi((x + \Delta x)^2 - x^2)y \\ &\doteq 2\pi x \Delta x y \end{aligned}$$

$\{(\Delta x)^2$  is disregarded - too insignificant]

(3)

$\therefore \Delta V \doteq 2\pi x (4x - x^2) \Delta x$  since

$$\begin{aligned} \therefore V &= \lim_{\Delta x \rightarrow 0} \sum_{x=0}^4 2\pi (4x^2 - x^3) \Delta x \\ &= 2\pi \int_0^4 (4x^2 - x^3) dx \end{aligned}$$

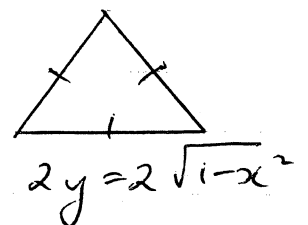
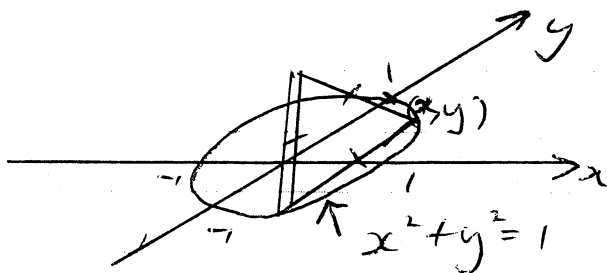
(ii)  $\therefore V = 2\pi \left[ \frac{4}{3}x^3 - \frac{x^4}{4} \right]_0^4$

$$= 2\pi \left( \frac{4}{3} \times 64 - 64 - 0 \right)$$

(1)

$$\begin{aligned} &= 2\pi \times \frac{64}{3} \\ &= \frac{128\pi}{3} \text{ units}^3 \end{aligned}$$

b)



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(\*) Area of each triangular cross-section  
 $\frac{1}{2} ab \sin C$

$$= \frac{1}{2} \cdot 2\sqrt{1-x^2} \cdot 2\sqrt{1-x^2} \cdot \sin 60^\circ$$

$$= 2(1-x^2) \cdot \frac{\sqrt{3}}{2}$$

$$= \sqrt{3}(1-x^2)$$

$$\therefore \Delta V = \sqrt{3}(1-x^2) \Delta x$$

$$\therefore V = \lim_{\Delta x \rightarrow 0} \sum_{x=-1}^1 \sqrt{3}(1-x^2) \Delta x$$

$$= \sqrt{3} \int_{-1}^1 (1-x^2) dx$$

$$= 2\sqrt{3} \int_0^1 (1-x^2) dx$$

$$= 2\sqrt{3} \left[ x - \frac{x^3}{3} \right]_0^1$$

$$= 2\sqrt{3} \times \frac{2}{3}$$

$$= \frac{4}{\sqrt{3}}$$

$$= \frac{4\sqrt{3}}{3} \text{ units}^3$$

(\*) (i) Consider a slice of thickness  $\Delta h$  at height  $h$ .

$$\text{Area of cross-section} = \frac{L^2}{h+1} \text{ m}^2$$

$$\therefore \text{Volume of slice} = \frac{L^2}{h+1} \Delta h \text{ m}^3$$

$$(ii) \therefore V = \lim_{\Delta h \rightarrow 0} \sum_{h=0}^H \frac{L^2}{h+1} \Delta h \text{ m}^3$$

$$= L^2 \int_0^H \frac{1}{h+1} dh$$

(\*) (3)

$$= [L^2 \ln(H+1)]_0^H$$

$$\textcircled{2} \quad = L^2 \ln(H+1)$$

d) Step 1. Show true for  $n=1$ .  
 $5^1 + 2(11)^1 = 27$ , which is  
 a multiple of 3

Step 2. Let  $n=k$  be a value for  
 which the result is true.

$$\text{i.e. } 5^k + 2(11)^k = 3M, \quad M \in \mathbb{J}$$

Consider now  $n=k+1$ .

$$\begin{aligned} & 5^{k+1} + 2(11)^{k+1} \\ &= 5 \times 5^k + 2 \times 11^k \times 11 \\ &= 5(5^k + 2 \times 11^k) + 6 \times 2 \times 11^k \\ &= 5(3M) + 3(4(11)^k) \\ &= 3[15 + 4 \times 11^k] \\ &= 3N, \quad N \in \mathbb{J} \end{aligned}$$

i.e. if result holds for  $n=k$ , it  
 also holds for  $n=k+1$ .

Step 3

Since result holds for  $n=1$ , it must  
 also hold for  $n=1+1=2$  (from step 2),  
 and hence for  $n=3$  etc  
i.e. result true for all integral  $n \geq 1$ .

Question 5

$$a)(i) \frac{5x^2 - 7x + 15}{(x^2 + 4)(x - 3)} \equiv \frac{Ax + B}{x^2 + 4} + \frac{C}{x - 3}$$

$$\therefore 5x^2 - 7x + 15 \equiv (Ax + B)(x - 3) + C(x^2 + 4)$$

$$\text{let } x = 3$$

$$45 - 21 + 15 = 13C$$

$$\therefore C = 3$$

$$\text{let } x = 0$$

$$15 = -3B + 4C$$

$$3B = 12 - 15$$

$$\therefore B = -1$$

coeff. of  $x^2$ :

$$5 = A + C$$

$$\therefore A = 2$$

$$(ii) \int_0^2 \frac{5x^2 - 7x + 15}{(x^2 + 4)(x - 3)} dx = \int_0^2 \frac{2x - 1}{x^2 + 4} dx + \frac{3}{x - 3} dx$$

$$= \left[ \ln(x^2 + 4) - \frac{1}{2} \tan^{-1} \frac{x}{2} + 3 \ln|x - 3| \right]_0^2$$

$$= \ln 8 - \frac{1}{2} \tan^{-1} 1 + 3 \ln|-1| - \ln 4 + \frac{1}{2} \tan^{-1} 0 - 3 \ln|-3|$$

$$= \ln 8 - \frac{1}{2} \cdot \frac{\pi}{4} + 0 - \ln 4 + 0 - 3 \ln 3$$

$$= \ln \frac{8}{4 \times 27} - \frac{\pi}{8}$$

$$= \ln \frac{2}{27} - \frac{\pi}{8}$$



$$b)(i) P(x) = 3x^3 + 7x + 2$$

$$P(\alpha) = P(\beta) = P(\gamma) = 0$$

$$\therefore 3\alpha^3 + 7\alpha + 2 = 0$$

$$3\beta^3 + 7\beta + 2 = 0$$

$$3\gamma^3 + 7\gamma + 2 = 0$$

Adding gives

$$3(\alpha^3 + \beta^3 + \gamma^3) + 7(\alpha + \beta + \gamma) + 6 = 0$$

(2)

$$3(\alpha^3 + \beta^3 + \gamma^3) + 7 \times 0 + 6 = 0$$

$$\therefore \alpha^3 + \beta^3 + \gamma^3 = -2$$

(ii) (A) We want an equation in  $x$  such that  $x = \alpha^2$

$$\text{i.e. } \alpha = \sqrt{x}$$

$$\text{Now, } P(\alpha) = 0$$

$$\therefore P(\sqrt{x}) = 0 \text{ is}$$

required equation.

$$\text{i.e. } 3(\sqrt{x})^3 + 7\sqrt{x} + 2 = 0$$

$$3x\sqrt{x} + 7\sqrt{x} + 2 = 0$$

$$\sqrt{x}(3x + 7) = -2$$

$$x(9x^2 + 42x + 49) = 4$$

$$9x^3 + 42x^2 + 49x - 4 = 0$$

(2)

$$(B) \quad \alpha + \beta = \alpha + \beta + \gamma - \gamma = -\gamma$$

$$\beta + \gamma = \alpha + \beta + \gamma - \alpha = -\alpha$$

$$\gamma + \alpha = \alpha + \beta + \gamma - \beta = -\beta$$

(2)

$P(-x) = 0$  has roots  $-\alpha, -\beta, -\gamma$

$$\text{i.e. } 3(-x)^3 + 7(-x) + 2 = 0$$

$$-3x^3 - 7x + 2 = 0$$

$$\text{i.e. } 3x^3 + 7x - 2 = 0$$

c)

$$x^3 + 3xy - y^2 = 3$$

Differentiate w.r.t  $x$  gives

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$$3x^2 + 3y + 3x \frac{dy}{dx} - 2y \frac{dy}{dx} = 0.$$

$$3x^2 + 3y = (2y - 3x) \frac{dy}{dx}$$

(3)

$$\frac{dy}{dx} = \frac{3(x^2 + y)}{2y - 3x}$$

$$\text{at } (1, 2), \quad \frac{dy}{dx} = \frac{3(1+2)}{4-3}$$

$$= 9$$

$$\therefore \text{eqn of tangent is}$$

$$y - 2 = 9(x - 1)$$

$$y = 9x - 7$$

## Question 6

a)

$$m = 5.$$

$$F = m \frac{dv}{dt} = 5 \frac{dv}{dt} = (3t - 4t^2)g$$

$$\therefore \frac{dv}{dt} = \frac{3t - 4t^2}{5} g$$

$$v = \frac{g}{5} \int (3t - 4t^2) dt.$$

$$= \frac{g}{5} \left( \frac{3}{2} t^2 - \frac{4}{3} t^3 \right) + c.$$

when  $t = 0$ ,  $v = 2$

$$\therefore 2 = 0 + c$$

i.e.  $c = 2$

④

$$\therefore v = \frac{g}{5} \left( \frac{3}{2} t^2 - \frac{4}{3} t^3 \right) + 2$$

when  $t = 1$ ,  $v = \frac{g}{5} \left( \frac{3}{2} - \frac{4}{3} \right) + 2$

$$= \frac{g}{30} + 2.$$

i.e. particle travelling at  $\left(2 + \frac{g}{30}\right)$  m/s  
at end of first second.

b)

Mechanics question.

① a) (i) By Newton's 2nd law,  $m\ddot{x} = -m(1+v)$

$$\therefore a = -(1+v)$$

ⓐ (ii) (a)  $a = v \frac{dv}{dx} = -(1+v)$

$$\therefore \frac{dv}{dx} = -\frac{1+v}{v}$$

$$\frac{dx}{dv} = \frac{-v}{1+v}$$

$$v+1 \overline{) -v} \\ \underline{-v-1} \\ 1$$

$$= -1 + \frac{1}{v+1}$$

$$\therefore x = \int -1 + \frac{1}{v+1} dv$$

$$= -v + \ln(1+v) + C$$

at  $t=0$ ,  $x=0$ ,  $v=u$

$$\therefore 0 = -u + \ln(1+u) + C$$

$$\therefore C = u - \ln(1+u)$$

$$\therefore x = -v + \ln(1+v) + u - \ln(1+u)$$

$$= u - v + \ln\left(\frac{1+v}{1+u}\right)$$

(b)  $a = \frac{dv}{dt} = -(1+v)$

$$\frac{dt}{dv} = -\frac{1}{1+v}$$

$$\therefore t = -\int \frac{dv}{1+v}$$

$$= -\ln(1+v) + C$$

when  $t=0$ ,  $v=u$

$$\therefore 0 = -\ln(1+u) + C$$

$$C = \ln(1+u)$$

$$\therefore t = \ln(1+u)$$

②

$$\therefore e^t = \frac{1+u}{1+v}$$

$$1+v = (1+u)e^{-t}$$

$$v = (1+u)e^{-t} - 1$$

$$(1) \quad v = \frac{dx}{dt} = (1+u)e^{-t} - 1$$

$$\therefore x = \int (1+u)e^{-t} - 1 dt$$

$$= -(1+u)e^{-t} - t + k$$

when  $t=0$ ,  $x=0$

$$\therefore 0 = -(1+u)e^0 + k$$

$$\therefore k = 1+u$$

$$\therefore x = -(1+u)e^{-t} - t + 1+u$$

$$(ii) \quad x = u - v + \ln\left(\frac{1+v}{1+u}\right) \quad \# \text{ from (i)(2)}$$

from (i)(3)  $v = (1+u)e^{-t} - 1$

$$\therefore \frac{v+1}{u+1} = e^{-t}$$

$$\therefore e^t = \frac{u+1}{v+1}$$

$$\therefore t = \ln\left(\frac{u+1}{v+1}\right)$$

$$\therefore -t = \ln\left(\frac{v+1}{u+1}\right)$$

$\therefore$  from  $\#$ ,  $x = u - v - t$

$$\therefore x + v + t = u$$

$$(iv) \quad x = u - v + \ln\left(\frac{1+v}{1+u}\right)$$

when  $x=0$ ,  $v=0$ ,  $x = u + \ln\left(\frac{1}{1+u}\right)$   
 $= u - \ln(1+u)$

20

i.e. particle travels  $\{u - \ln(1+u)\}$  metres before coming to rest.

Also,  $v = (1+u)e^{-t} - 1$

$\therefore$  when  $v = 0$ ,  $0 = (1+u)e^{-t} - 1$

$\therefore \frac{1}{1+u} = e^{-t}$

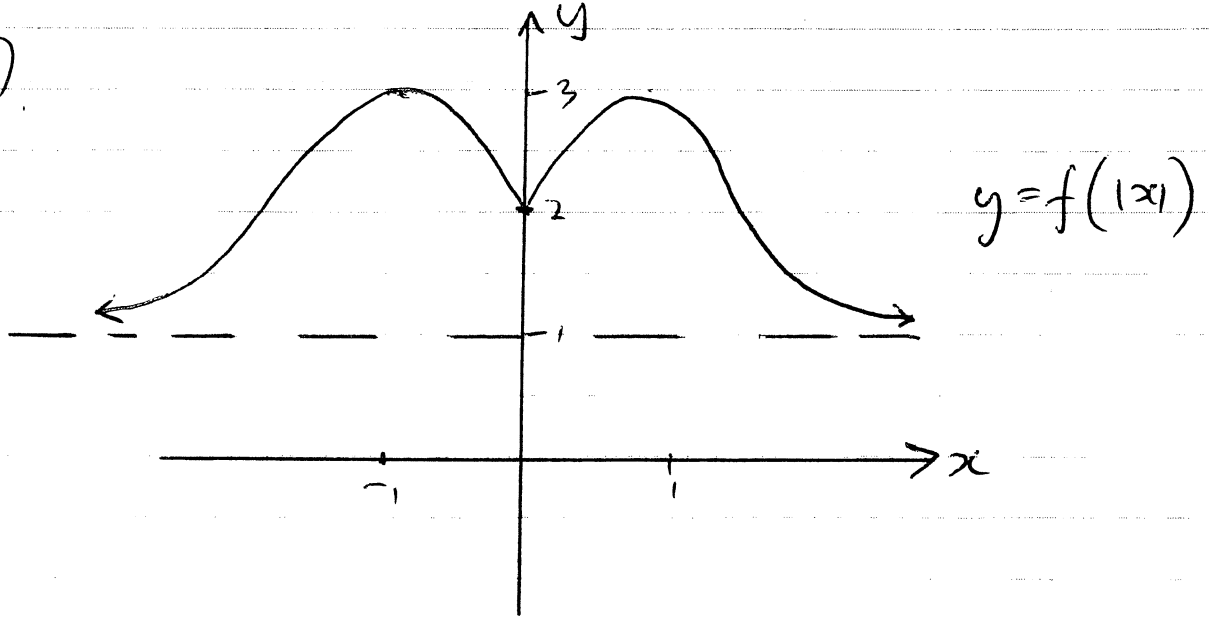
$1+u = e^t$

(2)

i.e. particle takes  $\ln(1+u)$  seconds to come to rest.  $t = \ln(1+u)$

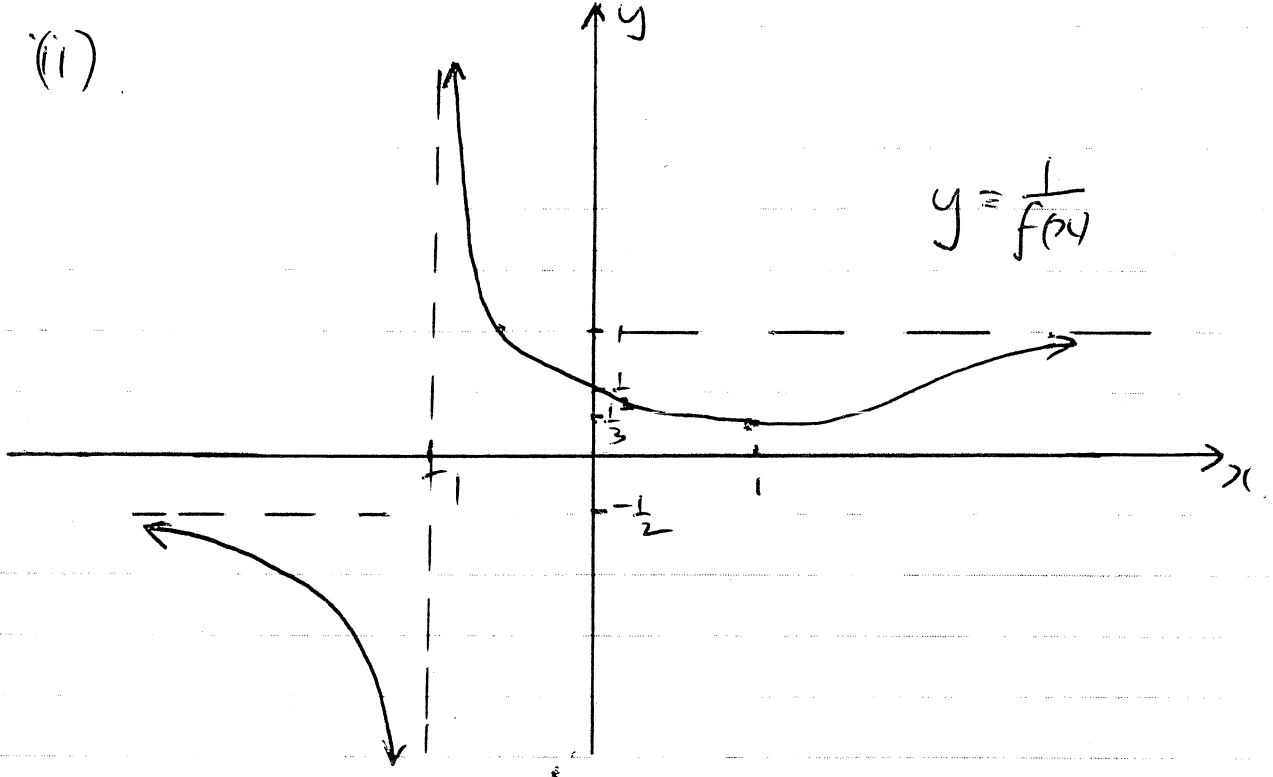
Question 7  
 a)  
 (i)

②



(ii)

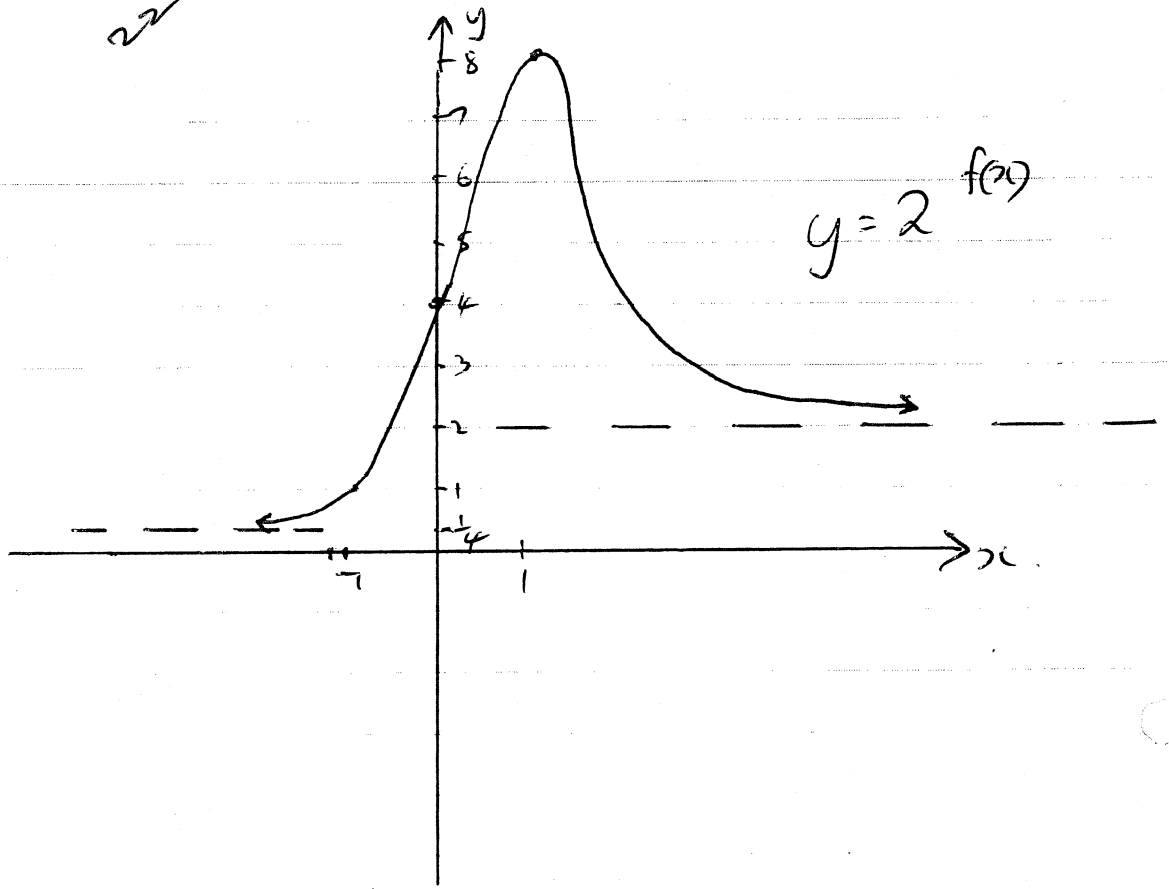
②



(iii)

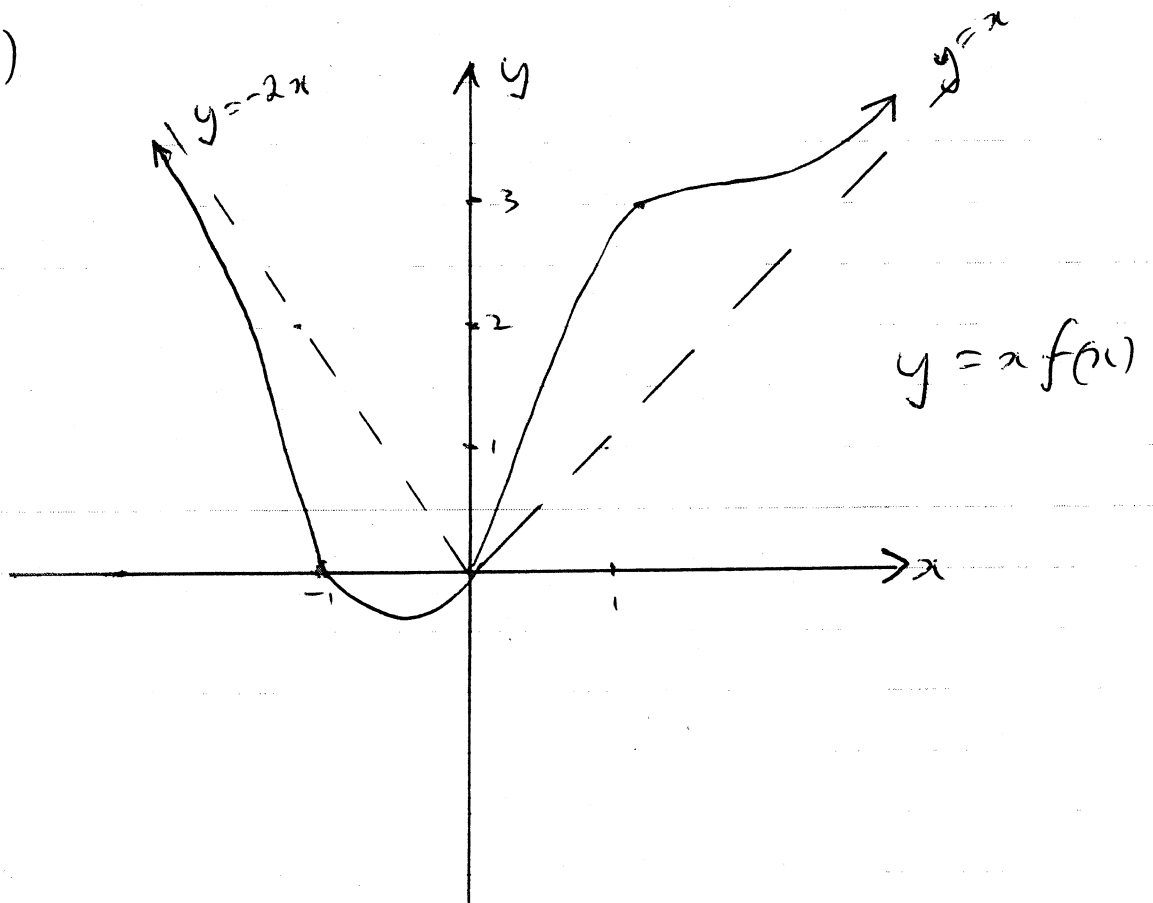
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②



(iv)

②



$$\begin{aligned}
 \text{b) (i)} \quad & \int_0^{\frac{\pi}{4}} (\tan x)^{2k} \sec^2 x \, dx \\
 & = \left[ \frac{(\tan x)^{2k+1}}{2k+1} \right]_0^{\frac{\pi}{4}}
 \end{aligned}$$



$$= \frac{(\tan \frac{\pi}{4})^{2k+1}}{2k+1} - \frac{(\tan 0)^{2k+1}}{2k+1} \quad 2^{\circ}$$

$$\textcircled{2} \quad = \frac{1^{2k+1}}{2k+1} - 0$$

$$= \frac{1}{2k+1}$$

$$\textcircled{\text{ii}} \quad \int_0^{\frac{\pi}{4}} (\sec x)^{2n+2} dx$$

$$= \int_0^{\frac{\pi}{4}} (\sec^2 x)^n \sec^2 x dx$$

$$= \int_0^{\frac{\pi}{4}} (1 + \tan^2 x)^n \sec^2 x dx$$

$$= \int_0^{\frac{\pi}{4}} \left[ \binom{n}{0} + \binom{n}{1} \tan^2 x + \binom{n}{2} \tan^4 x + \dots + \binom{n}{n} \tan^{2n} x \right] \sec^2 x dx$$

$$= \int_0^{\frac{\pi}{4}} \sum_{k=0}^n \binom{n}{k} \tan^{2k} x \cdot \sec^2 x dx$$

$\textcircled{3}$

$$= \sum_{k=0}^n \binom{n}{k} \int_0^{\frac{\pi}{4}} \tan^{2k} x \sec^2 x dx$$

$$= \sum_{k=0}^n \binom{n}{k} \frac{1}{2k+1} \quad \text{from (i)}$$

$$\textcircled{\text{iii}} \quad \int_0^{\frac{\pi}{4}} (\sec x)^8 dx$$

$$= \int_0^{\frac{\pi}{4}} \underbrace{\sec^6 x}_3 \sec^2 x dx \quad \begin{array}{l} 2n+2=8 \\ \therefore n=3 \end{array}$$

$$= \int_0^{\frac{\pi}{4}} \sum_{k=0}^3 \binom{3}{k} \frac{1}{2k+1}$$

$\textcircled{2}$

$$= \binom{3}{0} \times \frac{1}{1} + \binom{3}{1} \times \frac{1}{3} + \binom{3}{2} \times \frac{1}{5} + \binom{3}{3} \times \frac{1}{7}$$

$$= 1 + 3 \times \frac{1}{3} + 3 \times \frac{1}{5} + \frac{1}{7}$$

$$= 2\frac{1}{7} + \frac{3}{5} = 2\frac{26}{35}$$

Question 8

$$a) (i) \frac{(a+b)^n}{(a+b)^n} = \sum_{r=0}^n {}^n C_r a^{n-r} b^r$$

$$\therefore (3+2x)^{20} = \sum_{r=0}^{20} {}^{20} C_r 3^{20-r} (2x)^r$$

$$= \sum_{r=0}^{20} {}^{20} C_r 3^{20-r} 2^r x^r$$

①

$$\therefore a_r = {}^{20} C_r 3^{20-r} 2^r$$

$$(ii) \frac{a_{r+1}}{a_r} = \frac{{}^{20} C_{r+1} 3^{20-r-1} 2^{r+1}}{{}^{20} C_r 3^{20-r} 2^r}$$

$$= \frac{20!}{(r+1)!(20-r-1)!} \times \frac{2}{3}$$

$$\frac{20!}{r!(20-r)!}$$

$$= \frac{r!(20-r)!}{(r+1)!(20-r-1)!} \times \frac{2}{3}$$

③

$$= \frac{20-r}{r+1} \times \frac{2}{3}$$

or ②, ④?

$$= \frac{40-2r}{3r+3}$$

$$(iii) \text{ Consider } \frac{a_{r+1}}{a_r} > 1$$

$$\frac{40-2r}{3r+3} > 1$$

$$40-2r > 3r+3$$

$$5r < 37$$

$$r < 7\frac{4}{5}$$

∴ when  $r = 1, 2, \dots, 7$ ,  $a_{r+1} > a_r$ .

∴  $a_8 > a_7 > a_6 \dots > a_1$

Also, when  $r > 7^{1/2}$ ,  $a_{r+1} < a_r$ .

∴ when  $r = 8, 9, \dots$ ,  $a_r > a_{r+1}$ .

∴  $a_8 > a_9 > a_{10} \dots$

∴  $a_8$  is greatest coefficient.

③

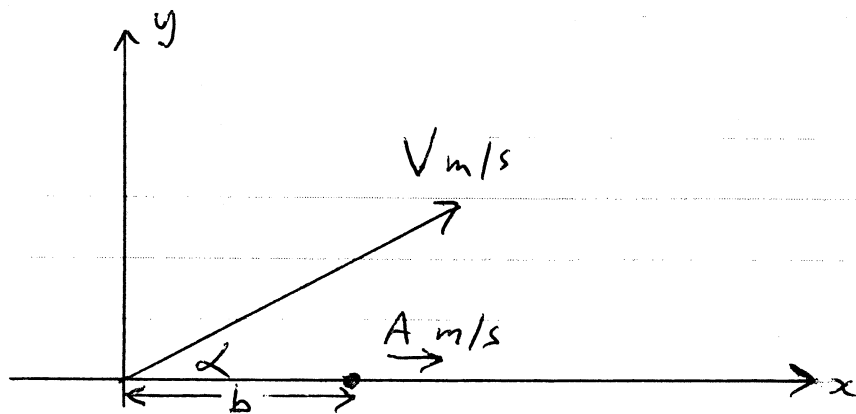
Now,  $a_r = {}^{20}C_r \cdot 3^{20-r} \cdot 2^r$

∴  $a_8 = {}^{20}C_8 \cdot 3^{20-8} \cdot 2^8$

$= \frac{20!}{8! \cdot 12!} \times 3^{12} \times 2^8$

$= 1.714 \times 10^{13}$  (to 4 sig figs)

b)



(i) initial conditions: when  $t=0$ ;  $x=0, y=0$ ,  
 $\dot{x} = V \cos \alpha$ ,  $\dot{y} = V \sin \alpha$ ,  $\ddot{x} = 0$ ,  $\ddot{y} = -g$ .

Horizontally:

$\ddot{x} = 0$

∴  $\dot{x} = C_1$

when  $t=0$ ,  $\dot{x} = V \cos \alpha$

∴  $\dot{x} = V \cos \alpha$

∴  $x = V \cos \alpha t + C_2$

when  $t=0$ ,  $x=0$

∴  $C_2 = 0$

∴  $x = V \cos \alpha t$

Vertically :

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$$\ddot{y} = -g$$

$$\therefore \dot{y} = -gt + C_3$$

$$\text{when } t=0, \dot{y} = V \sin \alpha$$

$$\therefore C_3 = V \sin \alpha$$

$$\therefore \dot{y} = V \sin \alpha - gt$$

$$\therefore y = V \sin \alpha t - \frac{gt^2}{2} + C_4$$

$$\text{when } t=0, y=0$$

$$\therefore C_4 = 0$$

$$\therefore y = V \sin \alpha t - \frac{gt^2}{2}$$

(3)

(ii) When particle hits ground,  $y=0$ .

$$\therefore t \left( V \sin \alpha - \frac{gt}{2} \right) = 0$$

$$\text{ie } t=0 \text{ or } \frac{2V \sin \alpha}{g}$$

(2)

$$\text{ie } \text{time of flight} = \frac{2V \sin \alpha}{g}$$

(iii) Range of projectile =  $V \cos \alpha \times \text{time of flight}$

$$= V \cos \alpha \cdot \frac{2V \sin \alpha}{g}$$

$$= \frac{V^2 \sin 2\alpha}{g}$$

The distance travelled by the target during the time of flight equal  $A \times \frac{2V \sin \alpha}{g}$

$\therefore$  Displacement of target from the origin is  $b + A \times \frac{2V \sin \alpha}{g}$

This must equal the range of the projectile.

$$\textcircled{3} \quad \underline{e} \quad \frac{V^2 \sin 2\alpha}{g} = b + \frac{2AV \sin \alpha}{g}.$$

$$\underline{e} \quad V^2 \sin 2\alpha - 2AV \sin \alpha - bg = 0.$$

QED

