St George Girls' High School

Trial Higher School Certificate Examination

2001

Mathematics Extension 2

Time Allowed: 3 hours
(Plus 5 minutes reading time)

Directions to Candidates

- All 8 questions may be attempted.
- Begin each question on a new page.
- All necessary working must be shown.
- All questions are of equal value.

Students are advised that this is a Trial Examination only and does not necessarily reflect the content or format of the Higher School Certificate Examination.

Question 1 (15 Marks) – Start A New Page

Marks

a) Find $\int \sin^6 x \cos^3 x \, dx$ 3

b) Use completion of squares and the table of standard integrals to find 2

$$\int \frac{dx}{\sqrt{x^2+8x+17}}$$

c) Use the substitution $t = \tan\frac{\theta}{2}$ to find the exact value of 5

$$\int_0^{\frac{\pi}{3}} \frac{d\theta}{5+3\sin\theta+4\cos\theta}$$

d) $I_n = \int_0^1 x^n e^{-x} dx$ 5

(i) Show that $I_n = -\frac{1}{e} + nI_{n-1}$

(ii) Hence, or otherwise, find the exact value of $\int_0^1 x^4 e^{-x} dx$

Question 2 (15 Marks) – Start A New Page

Marks

a) (i) Find $\sqrt{8-6i}$ in the form $a+ib$ where a, b, are real and $a>0$. 4

(ii) Hence solve $2z^2+(1-3i)z-2=0$ giving answers in the form $c+id$ where c, d are real.

b) (i) Sketch the arc of a circle of all points z such that 5

$$\arg\left(\frac{z+1}{z-3}\right)=-\frac{\pi}{3}$$

(ii) Find the centre and radius of the circle.

c) (i) Prove that $z\bar{z}=|z|^2$ 6

(ii) Suppose that z_1, z_2 and z_3 are three complex numbers of modulus 1 such that $z_1+z_2+z_3=0$. If z is a complex number of modulus 3 prove that

(α) $|z-z_1|^2=10-(z\bar{z}_1+\bar{z}z_1)$

(β) $|z-z_1|^2+|z-z_2|^2+|z-z_3|^2=30$

Question 3 (15 Marks) – Start A New Page

Marks

a) Find a, b if $(x-1)^2$ is a factor of $P(x) = x^5 + 2x^4 + ax^3 + bx^2$. 3

b) (i) Prove that if the polynomial $Q(x)$ has a zero of multiplicity m then the polynomial $Q^1(x)$ has the same zero of multiplicity $m-1$. 8

(ii) Show that the polynomial $H(x) = x^n + px - q$ has a multiple root if

$$\left(\frac{p}{n}\right)^n + \left(\frac{q}{n-1}\right)^{n-1} = 0$$

and find this root.

c) The cubic equation $x^3 + px^2 + q = 0$ where p, q are real numbers has roots α, β and γ. 4

The equation $x^3 + ax^2 + bx + c = 0$ has roots α^2, β^2 and γ^2.

Find a, b, c in terms of p and q.

Question 4 (15 Marks) – Start A New Page

Marks

a) Draw a neat sketch of the ellipse 3

$$\frac{x^2}{25}+\frac{y^2}{16}=1$$

On your sketch (which should be at least <u>one third</u> of a page) you must clearly show the x and y intercepts, the coordinates of the foci and the equations of the directrices.

b) Consider the ellipse, E, with equation $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1 \quad (a>b)$ 2

(i) Show that the equation of the tangent to the ellipse, E, at the point $P(a\cos\theta, b\sin\theta)$ is $bx\cos\theta + ay\sin\theta = ab$.

(ii) Find the equation of the normal to E at P.

(iii) The tangent (in (i)) and normal (in (ii)) cut the y-axis at A and B respectively. Find the coordinates of A and B.

(iv) Show that a focus, S, lies on the circumference of the circle which has AB as diameter (for each choice of P).

Question 5 (15 Marks) – Start A New Page

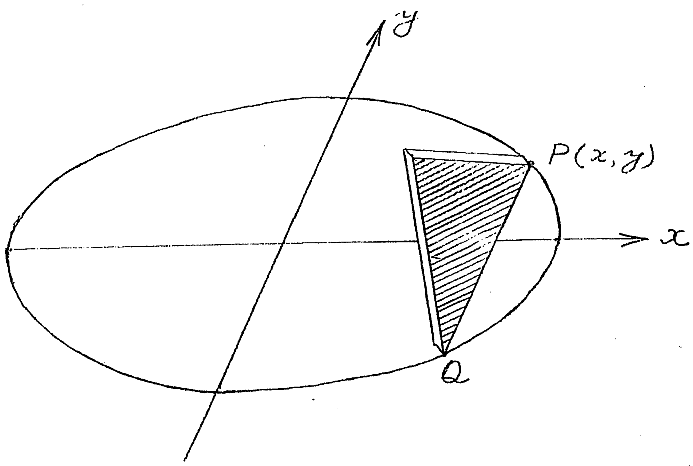
Marks

a) The vertices of ΔABC are $A(0, 2)$ $B(1, 1)$ and $C(-1, 1)$. H is the point $(0, y)$ where $1 < y < 2$. The line through H parallel to the x-axis cuts AB and AC at X and Y respectively. 6

(i) Find the length of XY as a function of y.

(ii) Hence or otherwise find the volume of the solid formed by rotating ΔABC about the x-axis through one complete revolution.

b) 7



The base of a particular solid is an ellipse with major and minor axes of 10 cm and 8 cm respectively. Every cross-section perpendicular to the major axis is an equilateral triangle one side of which lies in the base of the solid as shown above.

(i) Show that the cross-sectional area shaded above is $A(y) = y^2\sqrt{3}$.

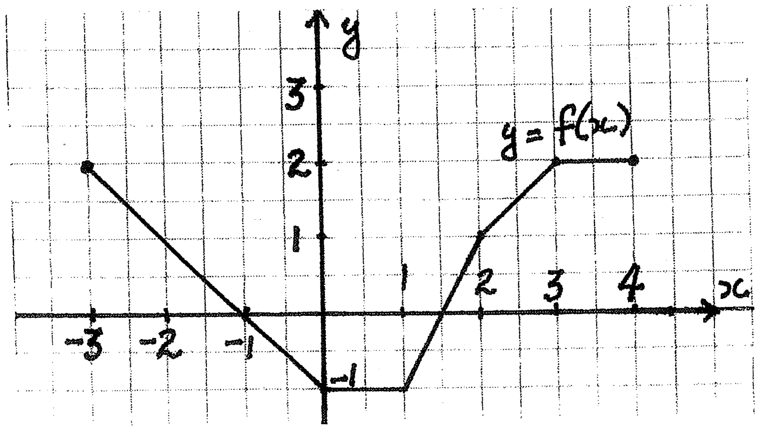
(ii) Hence find the volume of the slice of thickness Δx (as shown) as a function of x.

(iii) Find the volume of the solid.

c) Find the number of different ways of seating n people around 2 circular tables if there are to be k people at one table and the remainder at the other table. 2

Question 6 (15 Marks) – Start A New Page **Marks**

a) 8



The graph of $y = f(x)$ $-3 \le x \le 4$ is shown.

Drawn neat sketches of the graphs of the following on separate diagrams.

Each sketch should be approximately <u>one third</u> of a page and should clearly show <u>all</u> important features, including endpoints and discontinuities.

(i) $y = |f(x)|$

(ii) $y^2 = f(x)$

(iii) $y = \dfrac{1}{f(x)}$

(iv) $y = f^1(x)$

b) For the function in (a) find $\int_{-3}^{4} f(x)\,dx$. 2

c) Find the equation of the tangent to the curve $x^3 + y^3 - 3xy = 3$ at the point (2, 1). 3

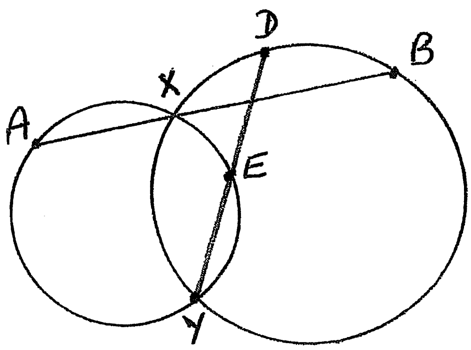
Not Done yet

d) (i) Express 42000 as the product of powers of prime numbers. 2

(ii) Hence or otherwise find the number of factors of 42000.

Question 7 (15 Marks) – Start A New Page **Marks**

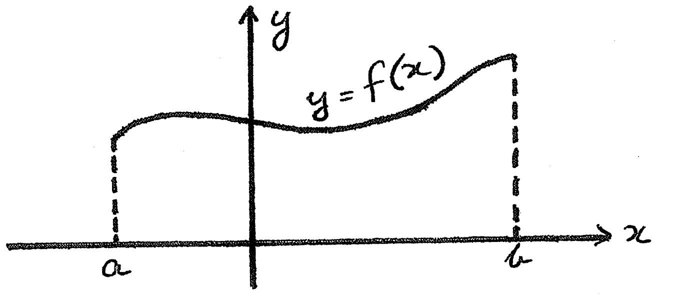
a)



The two circles intersect at X and Y. AXB and DEY are straight lines. 3

Copy the diagram into your booklet and prove that AE is parallel to DB.

b)



For $y = f(x)$ $a \le x \le b$, L, the length of the curve, is given by 4

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$

Using this formula find the exact value of the length of the curve $y = \frac{1}{2}(e^x + e^{-x})$ from $x = 0$ to $x = 1$.

c) A positive integer is said to be blue if no two adjacent digits are the same, so that 7, 30, 242, 695 are examples of blue integers. 9
Let $B(n)$ be the number of n-digit blue integers,
$C(n)$ be the number of even n-digit blue integers and
$D(n)$ be the number of odd n-digit blue integers.

(i) Explain why $B(n) = 9^n$.

(ii) Explain why $C(k+1) = 4.C(k) + 5.D(k)$ (k is a positive integer).

(iii) Using induction prove that $C(n) = \dfrac{9^n + (-1)^n}{2}$

(iv) Hence, or otherwise, find $D(5)$, the number of odd 5-digit blue integers.

Marks

Question 8 (15 Marks) – Start A New Page

a) A particle of mass m kg is dropped from rest in a medium which causes a resistance of mkv. 8

(i) Find the terminal velocity V_T.

(ii) Find the time taken to reach a velocity of $\frac{1}{2}V_T$.

(iii) Find the distance travelled in this time.

b) A mass of 1 kg moves in a straight line from the positive x-axis towards the origin. When its displacement from the origin is x metres it experiences a force of 7

$-\frac{k}{x^2}N$ (ie directed towards the origin).

If the mass starts from rest with displacement p metres, find the time required to reach the origin.

Standard Integrals

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{ if } n < 0.$$

$$\int \frac{1}{x} dx = \log_e x,\ x > 0.$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax},\ a \neq 0.$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax,\ a \neq 0.$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax,\ a \neq 0.$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax,\ a \neq 0.$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax,\ a \neq 0.$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ a \neq 0.$$

$$\int \frac{1}{\sqrt{(a^2 - x^2)}} dx = \sin^{-1}\frac{x}{a},\quad a > 0,\quad -a < x < a.$$

$$\int \frac{1}{\sqrt{(x^2 - a^2)}} dx = \log_e \left\{ x + \sqrt{(x^2 - a^2)} \right\},\ |x| > |a|.$$

$$\int \frac{1}{\sqrt{(x^2 + a^2)}} dx = \log_e \left\{ x + \sqrt{(x^2 + a^2)} \right\}.$$

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EXTENSION 2 MATHEMATICS TRIAL HIGHER SCHOOL CERT.

SOLUTIONS.

Question 1

(a) $\int \sin^6 x \cos^3 x \, dx = \int \sin^6 x . \cos^2 x . \cos x \, dx$

$= \int \sin^6 x (1-\sin^2 x) \cos x \, dx$

$= \int (\sin^6 x - \sin^8 x) \cos x \, dx$

$= \frac{\sin^7 x}{7} - \frac{\sin^9 x}{9} + C$

(b) $\int \frac{dx}{\sqrt{x^2+8x+17}} = \int \frac{dx}{\sqrt{(x+4)^2+1}}$

$= \log_e \{ x+4 + \sqrt{(x+4)^2+1} \}$

$= \log_e \{ x+4 + \sqrt{x^2+8x+17} \}$

(c) $\int_0^{\frac{\pi}{3}} \frac{d\theta}{5+3\sin\theta+4\cos\theta}$

$t = \tan\frac{\theta}{2}$

$\theta = 2\tan^{-1} t$

$d\theta = \frac{2}{1+t^2} dt$

When $\theta = 0 \quad t = 0$

$\theta = \frac{\pi}{3} \quad t = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$

$5 + 3\sin\theta + 4\cos\theta$

$= 5 + 3 \times \frac{2t}{1+t^2} + \frac{4(1-t^2)}{1+t^2}$

$= \frac{5+5t^2+6t+4-4t^2}{1+t^2}$

$= \frac{t^2+6t+9}{1+t^2}$

$= \frac{(t+3)^2}{1+t^2}$

$= \int_0^{\frac{1}{\sqrt{3}}} \frac{1}{\frac{(t+3)^2}{1+t^2}} \times \frac{2}{1+t^2} dt$

$= \int_0^{\frac{1}{\sqrt{3}}} \frac{2}{(t+3)^2} dt$

$= 2 \left[\frac{(t+3)^{-1}}{-1} \right]_0^{\frac{1}{\sqrt{3}}}$

$= 2 \left[\frac{-1}{t+3} \right]_0^{1/\sqrt{3}}$

$= -2 \left(\frac{1}{\frac{1}{\sqrt{3}}+3} - \frac{1}{3} \right)$

$= 2 \left(\frac{1}{3} - \frac{\sqrt{3}}{3\sqrt{3}+1} \times \frac{3\sqrt{3}-1}{3\sqrt{3}-1} \right)$

$= 2 \left(\frac{1}{3} - \frac{9-\sqrt{3}}{26} \right)$

$= \frac{2}{3} - \frac{9-\sqrt{3}}{13} = \frac{26-27+3\sqrt{3}}{39} = \frac{3\sqrt{3}-1}{39}$

(d) $$I_n = \int_0^1 x^n e^{-x}\,dx$$

(i) $$I_n = \int_0^1 x^n \cdot \frac{d(-e^{-x})}{dx}\,dx$$

$$= \left[-e^{-x} \cdot x^n\right]_0^1 - \int_0^1 n x^{n-1}(-e^{-x})\,dx$$

$$= (-e^{-1} \cdot 1 - -0) + n\int_0^1 x^{n-1} e^{-x}\,dx$$

$$= -\frac{1}{e} + n I_{n-1}$$

(ii) $$\cancel{e}\int_0^1 x^4 e^{-x}\,dx = I_4$$

$$I_4 = -\frac{1}{e} + 4I_3$$

$$= -\frac{1}{e} + 4\left(-\frac{1}{e} + 3I_2\right)$$

$$= -\frac{5}{e} + 12\left(-\frac{1}{e} + 2I_1\right)$$

$$= -\frac{17}{e} + 24\left(-\frac{1}{e} + I_0\right)$$

$$= -\frac{41}{e} + 24 I_0$$

$$I_0 = \int_0^1 e^{-x}\,dx$$

$$= \left[-e^{-x}\right]_0^1$$

$$= -e^{-1} + e^0$$

$$= 1 - \frac{1}{e}$$

$$I_4 = -\frac{41}{e} + 24\left(1 - \frac{1}{e}\right)$$

$$= 24 - \frac{65}{e}$$

Question 2

(a) (i) $\sqrt{8-6i} = a+ib \qquad a, b \in \mathbb{R}, \ a > 0$

$$8-6i = (a+ib)^2$$
$$= a^2-b^2+2iab$$
$$\therefore \ a^2-b^2 = 8 \qquad (1)$$
$$2ab = -6$$
$$ab = -3 \qquad (2)$$

From (2) $b = -\frac{3}{a}$

Subst in (1)

$$a^2 - \frac{9}{a^2} = 8$$

$$a^4 - 8a^2 - 9 = 0$$
$$(a^2-9)(a^2+1) = 0$$
$$(a+3)(a-3)(a^2+1) = 0$$
$$a = -3, 3$$

But $a > 0$

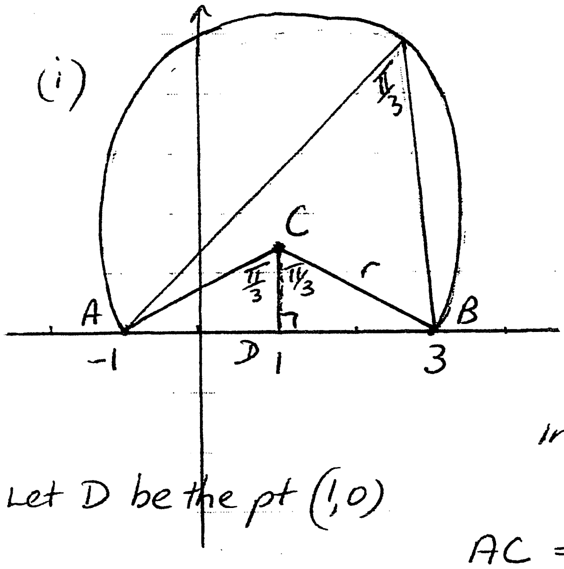
$$\therefore \ a = 3$$
$$\therefore \ b = -\frac{3}{3} = -1$$

$$\sqrt{8-6i} = 3-i$$

(ii) $2z^2 + (1-3i)z - 2 = 0$

$$z = \frac{-(1-3i) \pm \sqrt{(1-3i)^2 - 4\times 2\times -2}}{2\times 2}$$
$$= \frac{-1+3i \pm \sqrt{1-6i+9i^2+16}}{4}$$
$$= \frac{-1+3i \pm \sqrt{8-6i}}{4}$$
$$= \frac{-1+3i \pm (3-i)}{4} = \frac{2+2i}{4}, \frac{-4+4i}{4} = \cancel{1+i}$$
$$= \tfrac{1}{2}+\tfrac{1}{2}i, \ -1+i$$

(b) (i)



$\arg(z+1) - \arg(z-3) = -\frac{\pi}{3}$
$\arg(z-3) - \arg(z+1) = \frac{\pi}{3}$

(ii) Let C be the centre of circle
C lies on perp bisector of interval joining A(-1,0) and B(3,0)
ie C lies on $x = 1$

Let D be the pt (1,0)

$AC = BC = r$

$B\hat{C}A = \frac{2\pi}{3}$ (angle at centre = twice angle at circumf standing on same arc)

$\therefore B\hat{C}D = A\hat{C}D = \frac{\pi}{3}$

$\frac{BD}{CD} = \tan\frac{\pi}{3}$ $\qquad$ $\frac{BD}{r} = \sin\frac{\pi}{3}$

$\frac{2}{CD} = \sqrt{3}$ $\qquad$ $\frac{2}{r} = \frac{\sqrt{3}}{2}$

$CD = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$ $\qquad$ $r = \frac{4}{\sqrt{3}}$

$\therefore$ Centre is $\left(1, \frac{2\sqrt{3}}{3}\right)$

Radius $= \frac{4\sqrt{3}}{3}$

(c) (i) Let $z = x + iy$

$|z| = \sqrt{x^2+y^2}$

$$z\bar{z} = (x+iy)(x-iy)$$
$$= x^2 - i^2y^2$$
$$= x^2 + y^2$$
$$= |z|^2$$

(ii) $|z_1| = |z_2| = |z_3| = 1$

$z_1 + z_2 + z_3 = 0$

$|z| = 3$

(α) $|z - z_1|^2 = (z - z_1)(\overline{z - z_1})$

$= (z - z_1)(\bar{z} - \bar{z}_1)$

$= z\bar{z} - z\bar{z}_1 - z_1\bar{z} + z_1\bar{z}_1$

$= z\bar{z} + z_1\bar{z}_1 - (z\bar{z}_1 + z_1\bar{z})$

$= |z|^2 + |z_1|^2 - (z\bar{z}_1 + z_1\bar{z})$

$= 9 + 1 - (z\bar{z}_1 + z_1\bar{z})$

$= 10 - (z\bar{z}_1 + z_1\bar{z})$

(β) $|z - z_1|^2 + |z - z_2|^2 + |z - z_3|^2$

$= 10 - (z\bar{z}_1 + z_1\bar{z}) + 10 - (z\bar{z}_2 + z_2\bar{z}) + 10 - (z\bar{z}_3 + z_3\bar{z})$

$= 30 - (z(\bar{z}_1 + \bar{z}_2 + \bar{z}_3) + \bar{z}(z_1 + z_2 + z_3))$

$= 30 - (z(\overline{z_1 + z_2 + z_3}) + \bar{z} \times 0)$

$= 30 - z \times \bar{0}$

$= 30$

QUESTION 3:

[3]

(a) $P(x) = x^5 + 2x^4 + ax^3 + bx^2$

$P(1) = 1 + 2 + a + b = 0$

$\Rightarrow a + b = -3$ ——— ①

$P'(1) = 5 + 8 + 3a + 2b = 0$

$\Rightarrow 3a + 2b = -13$ ——— ②

② $- 2 \times$ ① : $\left.\begin{array}{l} a = -7 \\ \therefore b = 4 \end{array}\right\}$

(b) (i) Let $Q(x) = (x-\alpha)^m . P(x)$ where $P(\alpha) \neq 0$ [8]

Then $Q'(x) = (x-\alpha)^m . P'(x) + P(x) . m(x-\alpha)^{m-1}$

$= (x-\alpha)^{m-1}\left[(x-\alpha) P'(x) + mP(x)\right]$

$= (x-\alpha)^{m-1} R(x)$ where $R(\alpha) = mP(\alpha) \neq 0$

$\therefore Q'(x)$ has root of multiplicity $(m-1)$

(ii) $H(x) = x^n + px - q$ ——— ①

$\Rightarrow H'(x) = nx^{n-1} + p$

now $H'(x) = 0 \Longrightarrow x^{n-1} = -\frac{p}{n}$

and if there is to be a multiple root this must be it and hence it is a root of $H(x) = 0$

ie $x^n + px - q = 0$ where $x^{n-1} = -\frac{p}{n}$

ie $x^{n-1} + p - \frac{q}{x} = 0$

ie $-\frac{p}{n} + p = \frac{q}{x}$

$\frac{p(n-1)}{n} = \frac{q}{x}$

$$\text{ie} \quad \left[\frac{p(n-1)}{n}\right]^{n-1} = \frac{q^{n-1}}{x^{n-1}}$$

$$= \frac{q^{n-1}}{\left(-\frac{p}{n}\right)}$$

$$= -\frac{nq^{n-1}}{p}$$

$$\text{ie} \quad \frac{p^{n-1}\cdot(n-1)^{n-1}}{n^{n-1}} = -\frac{nq^{n-1}}{p}$$

$$\text{ie} \quad \frac{p^n}{n^n} = -\frac{q^{n-1}}{(n-1)^{n-1}}$$

$$\text{ie} \quad \left(\frac{p}{n}\right)^n + \left(\frac{q}{n-1}\right)^{n-1} = 0 \qquad \text{QED}$$

and the root is $x = \left(-\frac{p}{n}\right)^{\frac{1}{n-1}}$

(c) $\quad x^3 + px^2 + q = 0 \quad \text{———} \ (1)$

has roots α, β, γ

$$\text{ie} \quad \left.\begin{aligned} \alpha+\beta+\gamma &= -p \\ \alpha\beta+\alpha\gamma+\beta\gamma &= 0 \\ \alpha\beta\gamma &= -q \end{aligned}\right\}$$

Then polynomial with roots $\alpha^2, \beta^2, \gamma^2$ is

$$P(\sqrt{x}) = 0$$

$$\text{ie} \quad (\sqrt{x})^3 + p(\sqrt{x})^2 + q = 0$$

$$x\sqrt{x} + px + q = 0$$

$$x\sqrt{x} = -px - q$$

$$x^3 = p^2x^2 + 2pqx + q^2$$

$$\text{ie} \quad x^3 - p^2x^2 - 2pqx - q^2 = 0$$

$$\therefore \quad \left.\begin{aligned} a &= -p^2 \\ b &= -2pq \\ c &= -q^2 \end{aligned}\right\}$$

Question 4

(a) $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$

$a = 5 \qquad b = 4$

$b^2 = a^2(1-e^2)$

$16 = 25(1-e^2)$

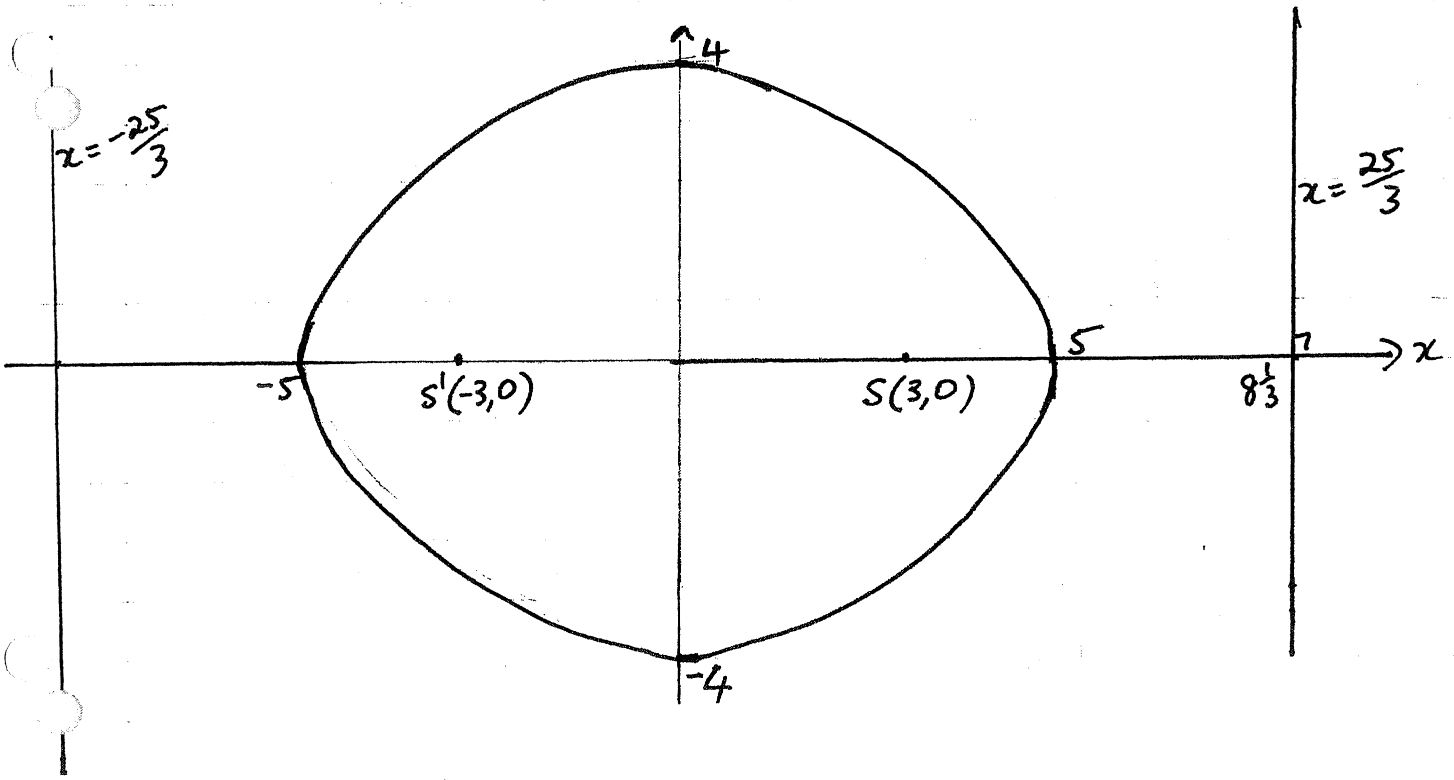
$e^2 = 1 - \dfrac{16}{25}$

$= \dfrac{9}{25}$

$e = \dfrac{3}{5} \quad (e > 0)$

$ae = 5 \times \dfrac{3}{5} = 3$

$\dfrac{a}{e} = \dfrac{5}{3/5} = \dfrac{25}{3}$



(b) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

(i) $\dfrac{2x}{a^2} + \dfrac{2y}{b^2}\dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx} = -\dfrac{2x}{a^2} \times \dfrac{b^2}{2y}$

$= -\dfrac{b^2}{a^2} \cdot \dfrac{x}{y}$

At $P(a\cos\theta, b\sin\theta)$

$$\frac{dy}{dx} = -\frac{b^2}{a^2} \cdot \frac{a\cos\theta}{b\sin\theta}$$

$$= -\frac{b\cos\theta}{a\sin\theta}$$

$\therefore$ Eq^n of tangent at P is

$$y - b\sin\theta = -\frac{b\cos\theta}{a\sin\theta}(x - a\cos\theta)$$

$$ay\sin\theta - ab\sin^2\theta = -bx\cos\theta + ab\cos^2\theta$$

$$bx\cos\theta + ay\sin\theta = ab(\sin^2\theta + \cos^2\theta)$$

$$= ab$$

(ii) Grad of normal at $P = \dfrac{a\sin\theta}{b\cos\theta}$

Eq^n of normal is

$$y - b\sin\theta = \frac{a\sin\theta}{b\cos\theta}(x - a\cos\theta)$$

$$b\cos\theta y - b^2\sin\theta\cos\theta = a\sin\theta x - a^2\sin\theta\cos\theta$$

$$\therefore a\sin\theta x - b\cos\theta y = (a^2 - b^2)\sin\theta\cos\theta$$

(iii) For tangent in (i)

When $x = 0$

$$ay\sin\theta = ab$$

$$y = \frac{ab}{a\sin\theta} = \frac{b}{\sin\theta}$$

$\therefore$ A has coords $\left(0, \dfrac{b}{\sin\theta}\right)$

For normal in (ii)

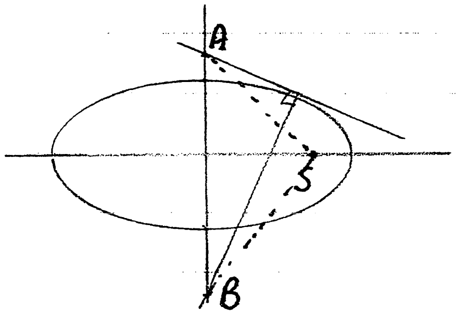
When $x = 0$

$$-b\cos\theta y = (a^2 - b^2)\sin\theta\cos\theta$$

$$y = \frac{a^2 - b^2}{-b}\sin\theta$$

$\therefore$ B has coords $\left(0, \frac{b^2-a^2}{b}\sin\theta\right)$

(iv)



If $S(ae, 0)$ lies on the circumference of the circle with AB as diameter then AB subtends a right angle at S

ie $AS \perp BS$

$$\text{Grad } AS = \frac{\frac{b}{\sin\theta} - 0}{0 - ae} = \frac{-b}{ae\sin\theta}$$

$$\text{Grad } BS = \frac{\frac{(b^2-a^2)}{b}\sin\theta - 0}{0 - ae}$$

$$= \frac{(b^2-a^2)\sin\theta}{-abe}$$

$$\text{Grad } AS \times \text{Grad } BS = \frac{-b}{ae\sin\theta} \times \frac{(b^2-a^2)\sin\theta}{-abe}$$

$$= \frac{-(b^2-a^2)}{-a^2e^2}$$

$$= \frac{b^2-a^2}{a^2e^2}$$

$$b^2 = a^2(1-e^2)$$
$$= a^2 - a^2e^2$$
$$b^2 - a^2 = -a^2e^2$$

$$= \frac{-a^2e^2}{a^2e^2}$$

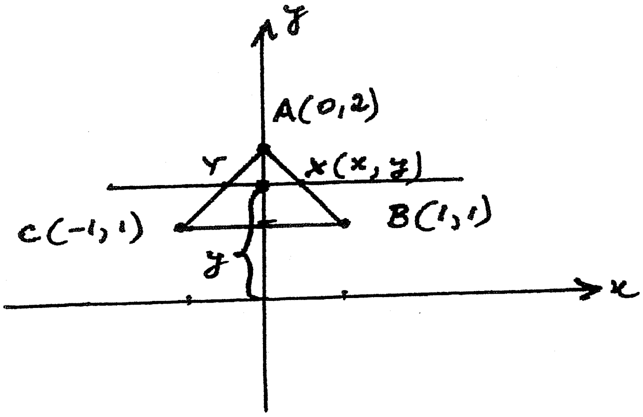
$$= -1$$

$\therefore$ $AS \perp BS$

Hence S lies on circle with AB as diameter

QUESTION 5:

(a)



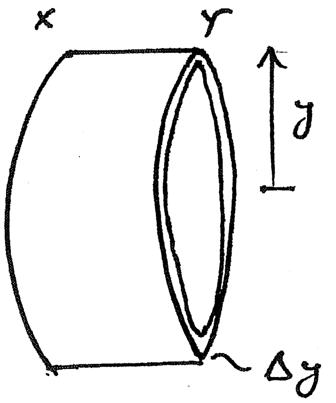
Equation AB : $y = mx + b$

$y = -x + 2$

$\therefore x = 2 - y$

(i) By symmetry $XY = 4 - 2y$

(ii) By shells



Volume of shell

$$\Delta V = 2\pi y\,(4-2y)\,\Delta y$$

$\therefore$ Volume of solid is

$$V = \lim_{\Delta y \to 0} \sum_{y=1}^{2} 2\pi y\,(4-2y)\,\Delta y$$

$$= 2\pi \int_1^2 y(4-2y)\,dy$$

$$= 2\pi \int_1^2 (4y - 2y^2)\,dy$$

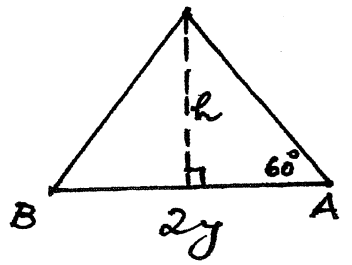
$$= 2\pi \left[2y^2 - \tfrac{2}{3}y^3\right]_1^2$$

$$= 8\pi \text{ units}^3$$

(b) Equation of ellipse is

$$\left(\frac{x}{5}\right)^2 + \left(\frac{y}{4}\right)^2 = 1 \quad \text{———} \ ①$$

(i)



To find h

$$\tan 60° = \frac{h}{y}$$

$$\therefore h = y\sqrt{3}$$

$\therefore$ area of Δ

$$A(y) = \frac{1}{2} \cdot 2y \cdot y\sqrt{3}$$

$$= \underline{y^2\sqrt{3}}$$

(ii) Volume of slice is

$$\Delta V = y^2\sqrt{3}\,\Delta x$$

$$= \frac{16}{25}(25 - x^2) \cdot \sqrt{3}\,\Delta x \quad \text{from ①}$$

(iii) Then

$$V = \lim_{\Delta x \to 0} \sum_{x=-5}^{5} \frac{16\sqrt{3}}{25}(25 - x^2)\,\Delta x$$

$$= \frac{32\sqrt{3}}{25}\int_0^5 (25 - x^2)\,dx$$

$$= \frac{32\sqrt{3}}{25}\left[25x - \frac{x^3}{3}\right]_0^5$$

$$= \frac{32\sqrt{3}}{25}\left[125 - \frac{125}{3}\right]$$

$$= \frac{32\sqrt{3}}{25} \cdot \frac{250}{3}$$

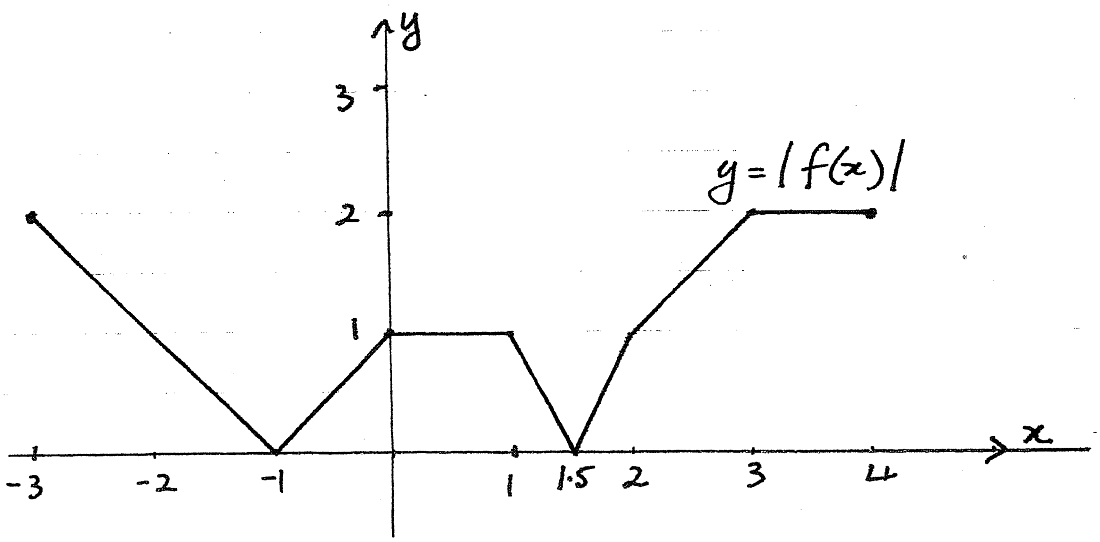
$$= \frac{320\sqrt{3}}{3} \text{ units}^3$$

(c) # of seating arrangements $= \binom{n}{k} \times (k-1)! \times (n-k-1)!$

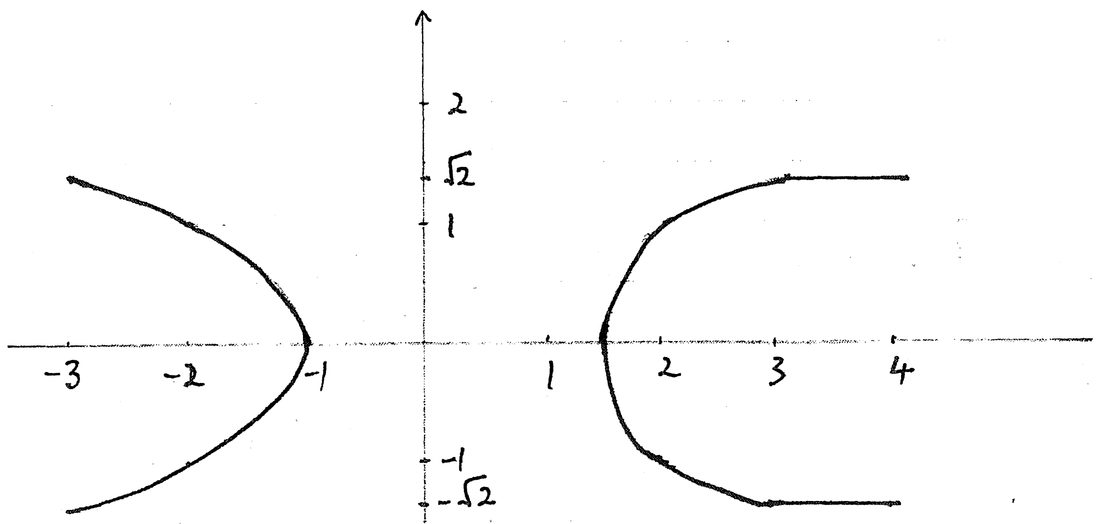
Question 6

(a)

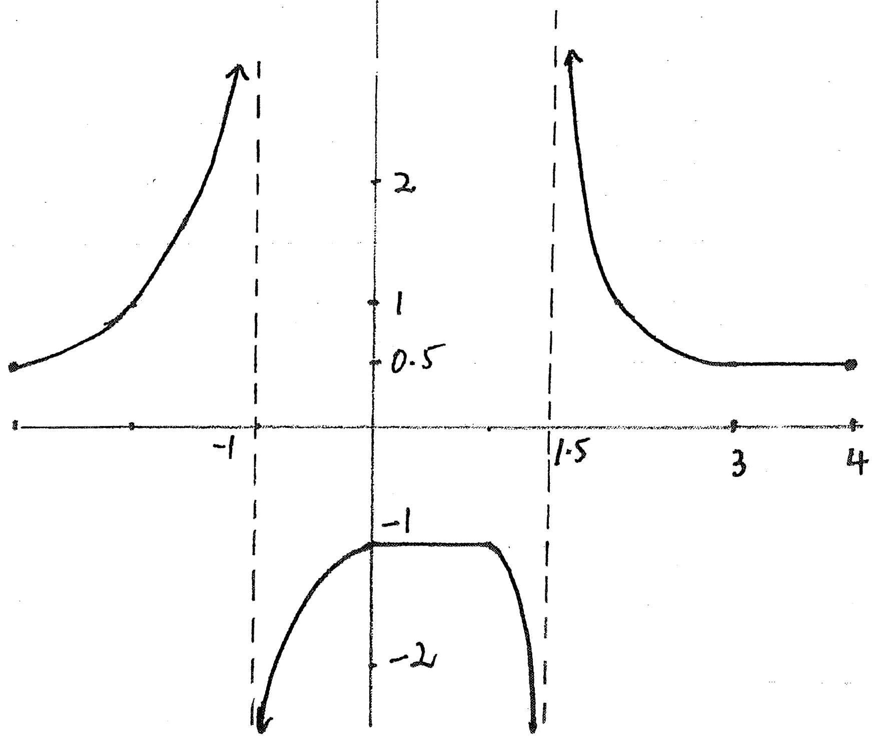
(i)



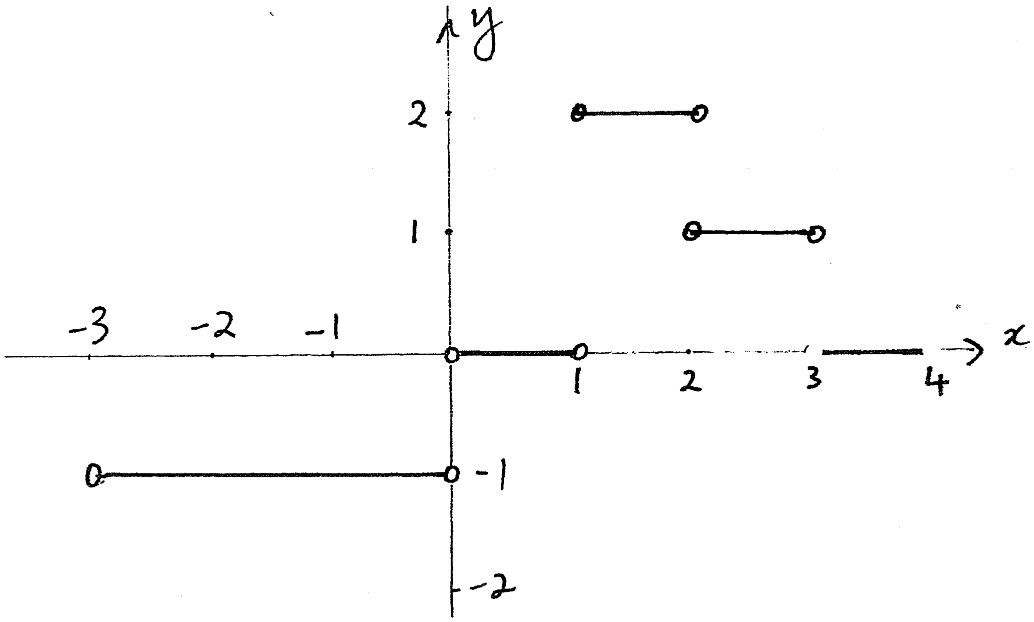
(ii)



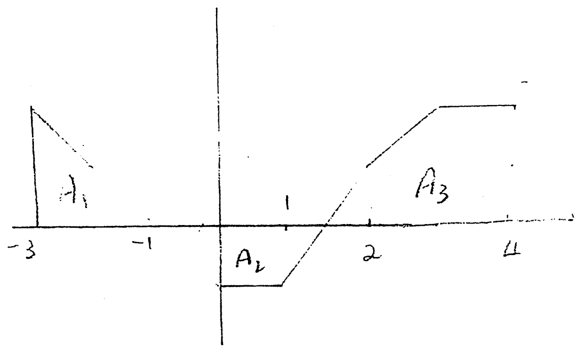
(iii)



(iv)



(b)



$$\int_{-3}^{4} f(x)\,dx = A_1 - A_2 + A_3$$

$A_1 = \frac{1}{2} \times 2 \times 2$
$\quad = 2$

$A_2 = \frac{1}{2}(1 + 2.5) \times 1$
$\quad = 1.75$

$A_3 = \frac{1}{2} \times \frac{1}{2} \times 1 + \frac{1}{2}(1+2) \times 1 + 1 \times 2$
$\quad = 0.25 + 1.5 + 2$
$\quad = 3.75$

$$\therefore \int_{-3}^{4} f(x)\,dx = 2 - 1.75 + 3.75$$
$$= 4$$

(c) $x^3 + y^3 - 3xy = 3$

Differentiate wrt x

$$3x^2 + 3y^2\frac{dy}{dx} - 3\left(1.y + x\frac{dy}{dx}\right) = 0$$

$$(3x^2 - 3y) + (3y^2 - 3x)\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{3(y - x^2)}{3(y^2 - x)}$$

$$= \frac{y - x^2}{y^2 - x}$$

When $x = 2$ $y = 1$

$$\frac{dy}{dx} = \frac{1-4}{1-2}$$

$$= 3$$

$\therefore$ Eq^n of tangent at $(2,1)$ is

$$y - 1 = 3(x-2)$$

$$y = 3x - 5$$

(d) (i) $42000 = 7 \times 6 \times 10^3$

$$= 7 \times 3 \times 2 \times (5 \times 2)^3$$

$$= 2^4 \times 3 \times 5^3 \times 7$$

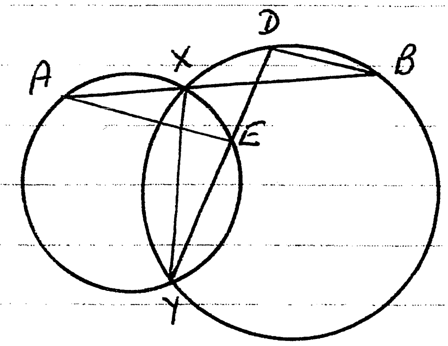
(ii) # of factors $= 5 \times 2 \times 4 \times 2$

$$= 80$$

(can include 0, 1, 2, 3, or 4 factors of 2; 0 or 1 factor of 3, 0, 1, 2 or 3 factors of 5, 0 or 1 factors of 7)

Question 7

(a)



Join AE, DB and XY

$X\hat{A}E = X\hat{Y}E$ (angles subtended by arc XE at circumf. are equal)

$X\hat{Y}D = X\hat{B}D$ (angles subtended by arc XD at circumf are equal)

$X\hat{Y}D = X\hat{Y}E$ (same angle)

$\therefore X\hat{A}E = X\hat{B}D \; (= X\hat{Y}D)$

$AE \parallel DB$ (since alt $\angle$s are equal)

(b)

$$y = \frac{1}{2}(e^x + e^{-x})$$

$$\frac{dy}{dx} = \frac{1}{2}(e^x - e^{-x})$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{1}{4}\left(e^{2x} - 2e^x e^{-x} + e^{-2x}\right)$$

$$= \frac{4 + e^{2x} - 2 + e^{-2x}}{4}$$

$$= \frac{e^{2x} + 2 + e^{-2x}}{4}$$

$$= \frac{(e^x + e^{-x})^2}{4}$$

$$\therefore L = \int_0^1 \sqrt{\frac{(e^x + e^{-x})^2}{4}}\,dx$$

$$= \int_0^1 \frac{e^x + e^{-x}}{2}\,dx$$

$$= \frac{1}{2}\left[e^x - e^{-x}\right]_0^1$$

$$= \frac{1}{2}\left\{(e - e^{-1}) - (e^0 - e^0)\right\}$$

$$= \frac{e}{2} - \frac{1}{2e}$$

(c) (i) Starting from LH digit

\# of choices for 1st digit $= 9$ (cannot choose 0)

\# of choices for 2nd ~~each~~ $= 9$ (cannot choose previous digit but now 0 can be chosen)

\# of choices for each subsequent digit $= 9$ (")

$$\therefore \quad B(n) = 9 \times 9 \times \ldots \times 9 \quad (n \text{ times})$$
$$= 9^n$$

(ii) A blue integer with $k+1$ digits can be formed by attaching a digit at the RH end of a k-digit blue integer

If the k-digit integer is even then there are 4 choices for the final digit

$\therefore$ There are $4 \times C(k)$ even blue integers of this type

If the k-digit integer is odd then there are 5 choices for this final digit.

$\therefore$ There are $5 \times D(k)$ blue integers of this type

Hence,

$$C(k+1) = 4 \times C(k) + 5 \times D(k)$$

(since the k-digit integers must be either even or odd)

(iii) Aim to prove that $C(n) = \dfrac{9^n + (-1)^n}{2}$

For $n = 1$

The even blue 1-digit integers are 2, 4, 6, 8

$$\therefore \; C(1) = 4$$
$$\frac{9^1 + (-1)^1}{2} = \frac{9-1}{2} = 4$$

$\therefore$ Proposition is true for $n = 1$

Let k be a value for which proposition is true

ie $C(k) = \dfrac{9^k + (-1)^k}{2}$

Aim to show that proposition is true for $n=k+1$ whenever it is true for $n=k$

ie $C(k+1) = \dfrac{9^{k+1}+(-1)^{k+1}}{2}$

Now
$$\begin{aligned}
C(k+1) &= 4C(k) + 5D(k) \\
&= 4C(k) + 5(B(k) - C(k)) \\
&= 5B(k) - C(k) \\
&= 5\times 9^k - \frac{9^k+(-1)^k}{2} \quad \text{(by inductive assumption)} \\
&= \frac{10\times 9^k - 9^k - (-1)^k}{2} \\
&= \frac{(10-1)\times 9^k + (-1)(-1)^k}{2} \\
&= \frac{9^{k+1}+(-1)^{k+1}}{2}
\end{aligned}$$

which is of the required form

$\therefore$ Proposition is true for $n=k+1$ whenever it is true for $n=k$

Since it is true for $n=1$ it is also true for $n=2$ and hence, by induction, it is true for all positive integers

(iv)
$$\begin{aligned}
D(5) &= B(5) - C(5) \\
&= 9^5 - \frac{9^5+(-1)^5}{2} \\
&= \frac{2\times 9^5 - 9^5 - (-1)}{2} \\
&= \frac{9^5+1}{2} \\
&= 29525
\end{aligned}$$

Question 8

(a) (i) $+\downarrow$ $\quad \uparrow R = mkv$, $\downarrow mg$

$$F = mg - mkv$$
$$m\ddot{x} = m(g - kv)$$
$$\ddot{x} = g - kv$$
$$v \to V_T \text{ as } \ddot{x} \to 0$$
$$0 = g - kV_T$$
$$V_T = \frac{g}{k}$$

(ii)
$$\frac{dv}{dt} = g - kv$$
$$\frac{dt}{dv} = \frac{1}{g-kv}$$

Let T be the time taken for velocity to reach $\frac{1}{2}V_T$

$$\therefore \quad T = \int_0^{\frac{1}{2}V_T} \frac{1}{g-kv}\,dv$$
$$= -\frac{1}{k}\Big[\ln(g-kv)\Big]_0^{\frac{1}{2}V_T}$$
$$= -\frac{1}{k}\left(\ln\left(g - \frac{k}{2}V_T\right) - \ln g\right)$$
$$= -\frac{1}{k}\left(\ln\left(g - \frac{k}{2}\cdot\frac{g}{k}\right) - \ln g\right)$$
$$= -\frac{1}{k}\ln\left(\frac{g}{2} \div g\right)$$
$$= -\frac{1}{k}\ln\left(\frac{1}{2}\right)$$
$$= \frac{1}{k}\ln 2$$

(iii)
$$\frac{dt}{dv} = \frac{1}{g-kv}$$
$$\therefore \quad t = -\frac{1}{k}\ln(g-kv) + c$$

When $t=0 \quad v=0$

$$\therefore \quad 0 = -\frac{1}{k}\ln g + c$$

$$c = \frac{1}{k}\ln g$$

$$\therefore \quad t = \frac{1}{k}\ln g - \frac{1}{k}\ln(g-kv)$$

$$= -\frac{1}{k}\ln\left(\frac{g-kv}{g}\right)$$

$$-kt = \ln\left(1-\frac{k}{g}v\right)$$

$$1-\frac{k}{g}v = e^{-kt}$$

$$\frac{k}{g}v = 1-e^{-kt}$$

$$v = \frac{g}{k}\left(1-e^{-kt}\right)$$

$$\therefore \quad \frac{dx}{dt} = \frac{g}{k}\left(1-e^{-kt}\right)$$

Let X be the distance travelled until velocity $= \frac{1}{2}V_T$

ie in time $t = \frac{1}{k}\ln 2$

$$X = \int_0^{\frac{1}{k}\ln 2} \frac{g}{k}\left(1-e^{-kt}\right)dt$$

$$= \frac{g}{k}\left[t+\frac{1}{k}e^{-kt}\right]_0^{\frac{1}{k}\ln 2}$$

$$= \frac{g}{k}\left\{\left(\frac{1}{k}\ln 2+\frac{1}{k}e^{-\ln 2}\right)-\left(0+\frac{1}{k}e^{0}\right)\right\}$$

$$= \frac{g}{k^2}\left(\ln 2 + e^{\ln\left(\frac{1}{2}\right)} - 1\right)$$

$$= \frac{g}{k^2}\left(\ln 2 + \frac{1}{2} - 1\right)$$

$$= \frac{g}{2k^2}\left(\ln 4 - 1\right)$$

OR (a) (iii) $\displaystyle v\frac{dv}{dx} = g - kv$

$$\frac{dv}{dx} = \frac{g-kv}{v}$$

$$\frac{dx}{dv} = \frac{v}{g-kv}$$

$$= -\frac{1}{k}\frac{(g-kv-g)}{g-kv}$$

$$= -\frac{1}{k}\left(1 - \frac{g}{g-kv}\right)$$

$$X = \int_0^{\frac{1}{2}V_T} -\frac{1}{k}\left(1 - \frac{g}{g-kv}\right) dv$$

$$= -\frac{1}{k}\left[v + \frac{g}{k}\ln(g-kv)\right]_0^{\frac{1}{2}V_T}$$

$$= -\frac{1}{k}\left\{\frac{V_T}{2} + \frac{g}{k}\ln\left(g - k.\frac{V_T}{2}\right) - \frac{g}{k}\ln g\right\}$$

$$= -\frac{1}{k}\left\{\frac{g}{2k} + \frac{g}{k}\ln\left(g - k.\frac{g}{2k}\right) - \frac{g}{k}\ln g\right\}$$

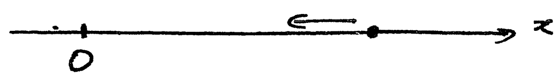
$$= -\frac{1}{k}\left\{\frac{g}{2k} + \frac{g}{k}\ln\left(\frac{g}{2}\right) - \frac{g}{k}\ln g\right\}$$

$$= -\frac{g}{2k^2} + \frac{g}{k^2}\ln\left(g \div \frac{g}{2}\right)$$

$$= \frac{g}{k^2}\left(\ln 2 - \frac{1}{2}\right)$$

$$= \frac{g}{2k^2}\left(2\ln 2 - 1\right)$$

(b)



$$F = -\frac{k}{x^2}$$

$$m\ddot{x} = -\frac{k}{x^2}$$

$$\ddot{x} = -\frac{k}{x^2} \quad (m=1)$$

$$\frac{d\left(\frac{1}{2}v^2\right)}{dx} = -\frac{k}{x^2}$$

$$\frac{1}{2}v^2 = \frac{k}{x} + c$$

When $t=0 \quad v=0 \quad x=p$

$$0 = \frac{k}{p} + c$$

$$c = -\frac{k}{p}$$

$$\therefore \quad \frac{1}{2}v^2 = \frac{k}{x} - \frac{k}{p}$$

$$v^2 = 2k\frac{(p-x)}{px}$$

$$= \frac{2k}{p}\frac{(p-x)}{x}$$

$$v = -\sqrt{\frac{2k}{p}}\,\frac{\sqrt{p-x}}{\sqrt{x}}$$ (since directed towards O from +ve x axis)

$$\frac{dx}{dt} = -\sqrt{\frac{2k}{p}}\,\frac{\sqrt{p-x}}{\sqrt{x}}$$

$$\frac{dt}{dx} = -\sqrt{\frac{p}{2k}}\,\frac{\sqrt{x}}{\sqrt{p-x}}$$

Let T be the time taken to reach $x=0$ from $x=p$

$$T = \int_p^0 -\sqrt{\frac{p}{2k}}\cdot\frac{\sqrt{x}}{\sqrt{p-x}}\,dx$$

$$= -\sqrt{\frac{p}{2k}}\int_p^0 \frac{x}{\sqrt{x(p-x)}}\,dx$$

$$= -\sqrt{\frac{p}{2k}} \times -\frac{1}{2}\int_p^0 \frac{-2x+p-p}{\sqrt{px-x^2}}\,dx$$

$$T = \sqrt{\frac{\rho}{8k}} \int_{\rho}^{0} (\rho - 2x)(\rho x - x^2)^{-\frac{1}{2}} - \frac{\rho}{\sqrt{\rho x - x^2}} \quad dx$$

$$= \sqrt{\frac{\rho}{8k}} \left[2(\rho x - x^2)^{1/2} \right]_{\rho}^{0} - \rho\sqrt{\frac{\rho}{8k}} \int_{\rho}^{0} \frac{1}{\sqrt{\frac{\rho^2}{4} - (x-\frac{\rho}{2})^2}} \, dx$$

$$= 0 - \sqrt{\frac{\rho^3}{8k}} \left[\sin^{-1}\left(\frac{x - \frac{\rho}{2}}{\rho/2} \right) \right]_{\rho}^{0}$$

$$= -\sqrt{\frac{\rho^3}{8k}} \left(\sin^{-1}(-1) - \sin^{-1}(1) \right)$$

$$= \sqrt{\frac{\rho^3}{8k}} \left(\sin^{-1}(1) - \sin^{-1}(-1) \right)$$

$$= \sqrt{\frac{\rho^3}{8k}} \left(\frac{\pi}{2} + \frac{\pi}{2} \right)$$

$$= \frac{\pi\sqrt{\rho^3}}{\sqrt{8k}}$$

OR $$T = -\int_{\rho}^{0} \sqrt{\frac{\rho}{2k}} \sqrt{\frac{x}{\rho - x}} \, dx$$

Let $x = \rho \sin^2\theta$ $\qquad \theta = \sin^{-1}\sqrt{\frac{x}{\rho}}$

$dx = 2\rho \sin\theta \cos\theta \, d\theta$

$\rho - x = \rho - \rho\sin^2\theta$

$= \rho(1 - \sin^2\theta)$

$= \rho\cos^2\theta$

When $x = \rho \quad \theta = \frac{\pi}{2}$

$x = 0 \quad \theta = 0$

$$\therefore \; T = -\int_{\frac{\pi}{2}}^{0} \sqrt{\frac{\rho}{2k}} \sqrt{\frac{\rho\sin^2\theta}{\rho\cos^2\theta}} \cdot 2\rho\sin\theta\cos\theta \, d\theta$$

$$= \int_{0}^{\pi/2} \sqrt{\frac{\rho}{2k}} \cdot 2\rho \cdot \frac{\sin\theta}{\cos\theta} \cdot \sin\theta\cos\theta \, d\theta$$

$$= \rho\sqrt{\frac{\rho}{2k}} \int_{0}^{\pi/2} 2\sin^2\theta \, d\theta$$

$$= \rho\sqrt{\frac{\rho}{2k}} \int_{0}^{\pi/2} 1 - \cos 2\theta \, d\theta$$

$$T = \rho\sqrt{\frac{\rho}{2k}}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}}$$

$$= \sqrt{\frac{\rho^3}{2k}}\left\{\left(\frac{\pi}{2} - 0\right) - (0 - 0)\right\}$$

$$= \pi\sqrt{\frac{\rho^3}{8k}}$$