

St George Girls' High School

Trial Higher School Certificate Examination

2000



Mathematics

4 Unit

*Time Allowed: Three hours
(Plus 5 minutes reading time)*

Directions to Candidates

- All questions may be attempted.
- All necessary working must be shown.
- Marks may be deducted for careless or poorly presented work.
- Begin each question on a new page.
- A list of standard integrals is included at the end of this paper.
- The mark allocated for each question is listed at the side of the question.

Students are advised that this is a Trial Examination only and does not necessarily reflect the content or format of the Higher School Certificate Examination.

Question 1 – (15 Marks)

Marks

- ✱ a) For the two complex numbers

$$z_1 = (1+i)^2 \text{ and } z_2 = \sqrt{2} \left[\cos\left(\frac{-5\pi}{6}\right) + i \sin\left(\frac{-5\pi}{6}\right) \right]$$

- (i) express z_1 in the modulus – argument form.

2

- (ii) express in the $(a+ib)$ form, where a and b are real numbers:

$$z_2, \bar{z}_2, iz_1$$

3

- (iii) find (x, y) such that $(z_2)^6 = x + iy$

2

- b) Determine the complex square roots of $2 - 2\sqrt{3}i$. Express your answer in the form $a + ib$.

Diff. relates
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 → c)

- c) Find, in simplest form, the quadratic equation whose roots are $2 - i$ and $(2 - i)^{-1}$

4

Question 2 – (15 Marks)

Marks

a) Find the indefinite integrals

(i) $\int \frac{2x}{(x+4)(x+3)} dx$ 2

(ii) $\int x^3 e^{x^4+1} dx$ 1

(iii) $\int \frac{dx}{\sqrt{8x-4x^2}}$ 2

b) Evaluate

(i) $\int_0^{\frac{\pi}{2}} \sin^2 x \cos^3 x dx$ 3

(ii) Show that $\int_0^a f(x) dx = \int_0^a f(a-x) dx$, and, hence, show that

$$\int_0^{\frac{\pi}{2}} \frac{\sin^4 x}{\sin^4 x + \cos^4 x} dx = \int_0^{\frac{\pi}{2}} \frac{\cos^4 x}{\sin^4 x + \cos^4 x} dx$$
 3

Hence, evaluate each integral

c) (i) If $U_n = \int_0^{\frac{\pi}{2}} \cos^n \theta d\theta$, prove that $U_n = \frac{n-1}{n} U_{n-2}$, for $n \geq 2$ 3

(ii) Hence, evaluate $\int_0^{\frac{\pi}{2}} \cos^9 \theta d\theta$ 1

Marks

Question 3 – (15 Marks)

- a) If one of the roots of $P(x) = x^3 - x^2 - 6x + 18 = 0$ is $2 - \sqrt{2}i$, find the other two roots. 3
- b) If α , β and γ are the roots of the equation $x^3 - 3x^2 + 6x + 7 = 0$, find
- (i) the polynomial equation whose roots are $\alpha^2, \beta^2, \gamma^2$ 3
- (ii) $\alpha^3 + \beta^3 + \gamma^3$ 3
- c) The roots of the equation $t^3 + qt - r = 0$ are a, b and c . If $S_n = a^n + b^n + c^n$, 3
where n is a positive integer, prove that: $S_{n+3} = rS_n - qS_{n+1}$
- d) The region bounded by $y = x^4$, $0 \leq x \leq 2$, and the x - axis is rotated about the line $x = 3$. Use the method of "cylindrical shells" to find the volume generated. (Leave the answer in terms of π).

Question 4 – (15 Marks)

Marks

a) If β is a complex root of $z^5 = 1$.

5

(i) Show that the roots are of the form $1, \beta, \beta^2, \beta^3, \beta^4$.

(ii) Find the value of $1 + \beta + \beta^2 + \beta^3 + \beta^4$.

(iii) Show that $\beta^{-1} = \beta^4$ and $\beta^{-2} = \beta^3$.

(iv) Hence find the quadratic equation with roots of $\beta + \beta^{-1}$ and $\beta^2 + \beta^{-2}$.

(v) Deduce that

$$\cos \frac{2\pi}{5} + \cos \frac{4\pi}{5} = -\frac{1}{2}$$

2

b) What is the maximum value of $|z|$ if $|z + 1 + 2i| \leq 1$?

3

c) If $|z_1 - z_2| = |z_1 + z_2|$, show that $\arg z_2 - \arg z_1 = \frac{\pi}{2}$.

2

d) Evaluate $\int_0^{\frac{2}{3}} \sqrt{4 - 9u^2} \, du$

3

Question 5 – (15 Marks)

Marks

a) Find the limiting sum of the series $\frac{1}{5} + \frac{2}{5^2} + \frac{3}{5^3} + \dots$

3

b) P and Q are the points t_1 and t_2 on the rectangular hyperbola $xy = c^2$.

(i) Show that the gradient of PQ is $\frac{-1}{t_1 t_2}$.

1

(ii) Hence, or otherwise, prove that if PQ subtends a right angle at a third point R on the hyperbola, then the tangent at R is perpendicular to PQ .

4

c) (i) By expanding $\cos(2\theta + \theta)$, show that $\cos^3 \theta - \frac{3}{4} \cos \theta = \frac{1}{4} \cos 3\theta$.

2

(ii) By making the substitution $x = m \cos \theta$ for a suitable value of m , and using the result in (i), find the roots of:

$$27x^3 - 9x - \sqrt{3} = 0$$

4

Express them in the form $m \cos \theta$.

(iii) Hence, prove that

$$\cos \frac{\pi}{18} \cdot \cos \frac{3\pi}{18} \cdot \cos \frac{5\pi}{18} \cdot \cos \frac{7\pi}{18} = \frac{3}{16}$$

1

Question 6 – (15 Marks)

Marks

The hyperbola H has Cartesian equation $\frac{x^2}{4} - \frac{y^2}{7} = 1$.

- a) Write down its eccentricity, the coordinates of its foci S and S' , the equations of the directrices and the equations of the asymptotes.

4

- b) Sketch the curve and include all details of part (a).

2

- c) P is an arbitrary point $(2 \sec \theta, \sqrt{7} \tan \theta)$. Show that P lies on H and prove that the tangent to H at P has equation $\frac{x \sec \theta}{2} - \frac{y \tan \theta}{\sqrt{7}} = 1$.

4

- d) This tangent cuts the asymptotes in L and M .

- α) Prove that $LP = PM$ and

2

- β) Prove the area of $\triangle OLM$ is independent of the position of P on H .
(O is the origin).

3

Marks

Question 7 – (15 Marks)

- a) Sketch the following graphs, indicating their essential features.
(Draw on separate graphs).

(i) $y = \sqrt{\ln x}$

2

(ii) $y = \tan^{-1}(\tan x)$ for $-\frac{3\pi}{2} \leq x \leq \frac{3\pi}{2}$

2

(iii) $|y| = \sin x$ for $0 \leq x \leq 3\pi$

2

(iv) $y = \frac{x-1}{x^2-4}$

3

Not covered yet

- b) The base of a solid is the circle $x^2 + y^2 = 6x$. Every plane section perpendicular to the x -axis is a rectangle whose height is one half of the distance of the plane section from the origin. Find the volume of the solid.

6

Question 8 – (cont'd)

Marks

- (i) Show that the projectile is above the x – axis for a total of $\frac{2V \sin \alpha}{g}$ seconds. **1**

- (ii) Show that the horizontal range of the projectile is $\frac{2V^2 \sin \alpha \cos \alpha}{g}$ metres. **1**

- (iii) At the instant the projectile is fired, the target T is d metres from O and it is moving away at a constant speed of u m/s.

Suppose that the projectile hits the target when fired at an angle of elevation α .

Show that:

$$u = V \cos \alpha - \frac{gd}{2V \sin \alpha} \quad \mathbf{3}$$

In parts (iv) and (v), assume that $gd = \frac{V^2}{2\sqrt{3}}$.

- (iv) By using (iii) and the graph of part (a), show that if $u > \frac{V}{\sqrt{3}}$ the target cannot be hit by the projectile, no matter at what angle of elevation α the projectile is fired. **3**

- (v) Suppose $u < \frac{V}{\sqrt{3}}$. Show that the target can be hit when it is at precisely two distances from O . **2**

The following list of standard integrals may be used:

$$\int x^n dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \quad \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln(x + \sqrt{x^2 - a^2}), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln(x + \sqrt{x^2 + a^2}).$$

Note: $\ln x = \log_e x, \quad x > 0$

TRIAL HSC
4 UNIT COURSE
SOLUTIONS.

QUESTION 1:

$$\begin{aligned} (a) \quad (i) \quad z_1 &= (1+i)^2 \\ &= 1+2i+i^2 \\ &= 2i \\ &= 2 \operatorname{cis} \frac{\pi}{2}. \end{aligned}$$

$$\begin{aligned} (ii) \quad z_2 &= \sqrt{2} \left[-\frac{\sqrt{3}}{2} - \frac{1}{2} i \right] \\ &= -\frac{\sqrt{6}}{2} - \frac{\sqrt{2}}{2} i \\ \bar{z}_2 &= -\frac{\sqrt{6}}{2} + \frac{\sqrt{2}}{2} i \\ iz_1 &= i \cdot 2i \\ &= -2 \end{aligned}$$

$$\begin{aligned} (iii) \quad z_2 &= \sqrt{2} \left[\operatorname{cis} \left(-\frac{5\pi}{6} \right) \right] \\ z_2^6 &= 8 \left[\operatorname{cis} (-5\pi) \right] \\ &= 8 \operatorname{cis} \pi \\ &= -8. \end{aligned}$$

$$(b) \quad \text{let } a+ib = \sqrt{2-2\sqrt{3}i}$$

squaring

$$\Rightarrow a^2 - b^2 + 2abi = 2 - 2\sqrt{3}i$$

$$\Rightarrow a^2 - b^2 = 2 \quad \text{--- (1)}$$

$$ab = -\sqrt{3} \quad \text{--- (2)}$$

$$\text{from (2) } b = -\frac{\sqrt{3}}{a} \text{ sub in (1)}$$

$$\therefore a^2 - \frac{3}{a^2} = 2$$

$$a^4 - 2a^2 - 3 = 0$$

$$(a^2 - 3)(a^2 + 1) = 0$$

$$a^2 = 3$$

$$a = \sqrt{3}, -\sqrt{3}$$

$$b = -1, 1$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin^4 x + \cos^4 x}{\sin^4 x + \cos^4 x} dx$$

$$= \int_0^{\frac{\pi}{2}} 1 dx$$

$$= \frac{\pi}{2}$$

and since the two integrals are of equal value, each integral is equal to $\frac{\pi}{4}$.

$$(c) (i) U_n = \int_0^{\frac{\pi}{2}} \cos^n \theta d\theta$$

$$= \int_0^{\frac{\pi}{2}} \underbrace{\cos^{n-1} \theta}_v \cdot \underbrace{\cos \theta}_{dv} d\theta$$

$$= \left[\sin \theta \cdot \cos^{n-1} \theta \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin \theta \cdot (n-1) \cos^{n-2} \theta \cdot \frac{d\theta}{d\theta}$$

$$= (n-1) \int_0^{\frac{\pi}{2}} \sin^2 \theta \cdot \cos^{n-2} \theta d\theta$$

$$= (n-1) \int_0^{\frac{\pi}{2}} (1 - \cos^2 \theta) \cdot \cos^{n-2} \theta d\theta$$

$$= (n-1) [U_{n-2} - U_n]$$

$$U_n = (n-1)U_{n-2} - (n-1)U_n$$

$$U_n(1+n-1) = (n-1)U_{n-2}$$

$$\therefore U_n = \left(\frac{n-1}{n} \right) U_{n-2}$$

$$n \geq 2$$

$$\begin{aligned}
 \text{(ii)} \quad \int_0^{\pi} \cos^9 \theta \, d\theta &= U_9 \\
 &= \frac{8}{9} \times U_7 \\
 &= \frac{8}{9} \times \frac{6}{7} \times U_5 \\
 &= \frac{8}{9} \times \frac{6}{7} \times \frac{4}{5} \times U_3 \\
 &= \frac{8}{9} \times \frac{6}{7} \times \frac{4}{5} \times \frac{2}{3} U_1 \\
 &= \frac{128}{315} \times \int_0^{\pi} \cos \theta \, d\theta \\
 &= \frac{128}{315} \times [\sin \theta]_0^{\pi} \\
 &= \frac{128}{315}
 \end{aligned}$$

2.

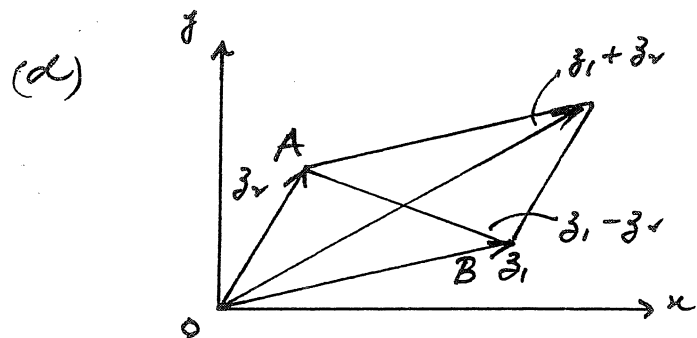
$$\begin{aligned} \text{(c) Sum of roots} &= 2-i + \frac{1}{2-i} \\ &= 2-i + \frac{2+i}{5} \\ &= \frac{12}{5} - \frac{4}{5}i \end{aligned}$$

$$\begin{aligned} \text{Product of roots} &= 2-i \times \frac{1}{2-i} \\ &= 1 \end{aligned}$$

$\therefore$ Quadratic equation is

$$x^2 - \left(\frac{12}{5} - \frac{4}{5}i\right)x + 1 = 0$$

$$\text{ie } 5x^2 - (12-4i)x + 5 = 0$$



$|z_1 + z_2| = |z_1 - z_2| \Rightarrow$ diagonals of parallelogram are equal

$\Rightarrow$ //gram is a square or rectangle

$\Rightarrow \angle AOB = \frac{\pi}{2}$

$\therefore \arg z_2 - \arg z_1 = \frac{\pi}{2}$

(2)

(e) $\int_0^{\frac{2}{3}} \sqrt{4-9u^2} du$

let $3u = 2\sin\theta$
 $3du = 2\cos\theta d\theta$

$= \int_0^{\frac{\pi}{2}} \sqrt{4-4\sin^2\theta} \cdot \frac{2\cos\theta}{3} d\theta$

$\cos 2\theta = 2\cos^2\theta - 1$

$= \frac{4}{3} \int_0^{\frac{\pi}{2}} \cos^2\theta d\theta$

$= \frac{4}{3} \cdot \frac{1}{2} \int_0^{\frac{\pi}{2}} (1 + \cos 2\theta) d\theta$

$= \frac{2}{3} \left[\theta + \frac{1}{2} \sin 2\theta \right]_0^{\frac{\pi}{2}}$

$= \frac{2}{3} \left[\frac{\pi}{2} - 0 \right]$

$= \frac{\pi}{3}$

$\underline{I} = 3 \int_0^{2/3} \sqrt{\left(\frac{2}{3}\right)^2 - u^2} du$

$= 3 \times \frac{1}{4} \pi r^2$
 $= \frac{3}{4} \pi \cdot \frac{4}{9}$
 $= \frac{\pi}{3}$

(3)

$x^2 + y^2 = \frac{1}{2}$

$y = \sqrt{\frac{1}{4} - x^2}$

Area

QUESTION 2

$$(a) \quad (i) \int \frac{2x}{(x+4)(x+3)} dx$$

$$\text{let } \frac{2x}{(x+4)(x+3)} = \frac{a}{x+4} + \frac{b}{x+3}$$

$$\text{ie } 2x = a(x+3) + b(x+4)$$

$$x = -3 \Rightarrow -6 = b$$

$$x = -4 \Rightarrow -8 = -a$$

$$\therefore a = 8$$

$$\therefore \frac{2x}{(x+4)(x+3)} = \frac{8}{x+4} - \frac{6}{x+3}$$

$$\therefore \int \frac{2x}{(x+4)(x+3)} dx = 8 \ln|x+4| - 6 \ln|x+3| + c$$

$$(ii) \int x^3 e^{x^4+1} dx$$

$$u = e^{x^4+1}$$

$$du = 4x^3 e^{x^4+1} dx$$

$$= \frac{1}{4} \int 4x^3 e^{x^4+1} dx$$

$$= \frac{1}{4} \int e^u du$$

$$= \frac{1}{4} \cdot e^u + c$$

$$= \frac{1}{4} \cdot e^{x^4+1} + c$$

$$(iii) 8x - 4x^2 = -4(x^2 - 2x)$$

$$= -4[(x^2 - 2x + 1) - 1]$$

$$= 4 - 4(x-1)^2$$

$$\int \frac{dx}{\sqrt{8x-4x^2}} = \int \frac{dx}{2\sqrt{1-(x-1)^2}}$$

$$= \frac{1}{2} \sin^{-1}(x-1) + c$$

QUESTION 3:

(a) $P(x) = x^3 - x^2 - 6x + 18$

Since co-effs of $P(x)$ are real

$x = 2 - \sqrt{2}i$ a zero $\Rightarrow 2 + \sqrt{2}i$ also a zero

$\therefore x^2 - [2 - \sqrt{2}i + 2 + \sqrt{2}i]x + (2 - \sqrt{2}i)(2 + \sqrt{2}i)$ is a factor of $P(x)$

i.e. $x^2 - 4x + 6$ is a factor

$\therefore P(x) = (x^2 - 4x + 6)(x + 3)$

$\therefore$ Other roots are $2 + \sqrt{2}i, -3$

(b) $P(x) = x^3 - 3x^2 + 6x + 7$

(i) $y = x^2 \Rightarrow x = \sqrt{y}$

$\therefore P(\sqrt{x}) = 0$

$\Rightarrow (\sqrt{x})^3 - 3(\sqrt{x})^2 + 6\sqrt{x} + 7 = 0$

$x\sqrt{x} - 3x + 6\sqrt{x} + 7 = 0$

$\sqrt{x}(x+6) = 3x-7$

$x(x+6)^2 = (3x-7)^2$

$x^3 + 12x^2 + 36x = 9x^2 - 42x + 49$

i.e. $x^3 + 3x^2 + 78x - 49 = 0$ ——— (1)

(ii) Since α, β, γ are roots of $P(x) = 0$

$\alpha^3 - 3\alpha^2 + 6\alpha + 7 = 0$

$\beta^3 - 3\beta^2 + 6\beta + 7 = 0$

$\gamma^3 - 3\gamma^2 + 6\gamma + 7 = 0$

Adding $\Rightarrow \Sigma \alpha^3 - 3\Sigma \alpha^2 + 6\Sigma \alpha + 21 = 0$

$\Sigma \alpha^3 = 3(-3) - 6(3) - 21$

$= -48$

$\Sigma \alpha^2$ from (1)

$$(c) \quad t^3 + qt - r = 0$$

a, b, c roots

$$\Rightarrow a^3 + qa - r = 0$$

$$b^3 + qb - r = 0$$

$$c^3 + qc - r = 0$$

$$\text{adding} \Rightarrow a^3 + b^3 + c^3 = -q(a+b+c) + 3r \\ = 3r$$

$$\text{Then } S_{n+3} = a^{n+3} + b^{n+3} + c^{n+3}$$

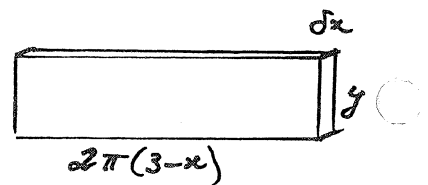
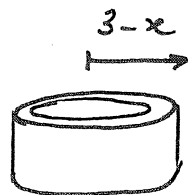
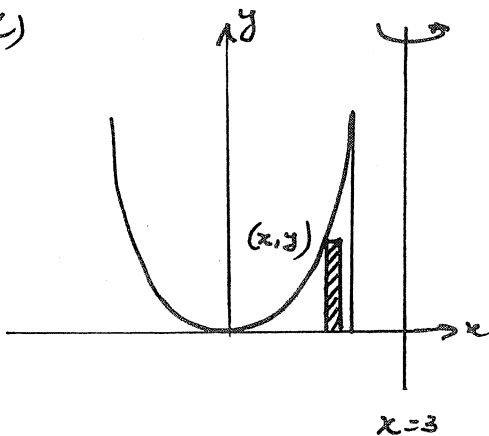
$$= a^n \cdot a^3 + b^n \cdot b^3 + c^n \cdot c^3$$

$$= a^n(r - qa) + b^n(r - qb) + c^n(r - qc)$$

$$= r(a^n + b^n + c^n) - q(a^{n+1} + b^{n+1} + c^{n+1})$$

$$= r S_n - q S_{n+1}$$

(d)



ignoring terms
in $(\delta x)^2$

Volume of shell is $\delta V = 2\pi(3-x)y \delta x$

$\therefore$ Volume of solid is $V = \lim_{\delta x \rightarrow 0} \sum_{x=0}^3 2\pi(3-x)y \delta x$

$$= 2\pi \int_0^3 (3-x) \cdot x^2 dx$$

$$= 2\pi \left[\frac{3x^3}{3} - \frac{x^4}{4} \right]_0^3$$

$$= 2\pi \left[\frac{96}{3} - \frac{64}{4} - 0 \right]$$

$$= \frac{256\pi}{15} \text{ units}^3$$

QUESTION 4:

(a) $\omega^3 = 1$ and $1 + \omega + \omega^2 = 0$

Then $\frac{a + b\omega + c\omega^2}{c + a\omega + b\omega^2} = \frac{a + \frac{b}{\omega^2} + \frac{c}{\omega}}{c + \frac{a}{\omega} + \frac{b}{\omega^2}}$

$= \frac{a\omega^2 + b + c\omega}{c\omega^2 + a + b\omega}$

$= \frac{\omega^2(a + b\omega + c\omega^2)}{a + b\omega + c\omega^2}$

since $\omega^3 = 1$

$= \omega^2$

(b) $z^5 = 1$
 $= 1 \operatorname{cis}(0 + 2k\pi)$ k integer

$\therefore z = 1 \operatorname{cis} \frac{2k\pi}{5}$

$k=0 \Rightarrow z = 1$

$k=1 \Rightarrow z = \operatorname{cis} \frac{2\pi}{5} = \cos \frac{2\pi}{5} + i \sin \frac{2\pi}{5}$

$k=2 \Rightarrow z = \operatorname{cis} \frac{4\pi}{5} = -\cos \frac{\pi}{5} + i \sin \frac{\pi}{5}$

$k=3 \Rightarrow z = \operatorname{cis} \frac{6\pi}{5} = -\cos \frac{\pi}{5} - i \sin \frac{\pi}{5}$

$k=4 \Rightarrow z = \operatorname{cis} \frac{8\pi}{5} = \cos \frac{2\pi}{5} - i \sin \frac{2\pi}{5}$

(ii) If $\beta = \operatorname{cis} \frac{2\pi}{5}$, $\beta^{-1} = \operatorname{cis}(-\frac{2\pi}{5})$

$= \cos \frac{2\pi}{5} - i \sin \frac{2\pi}{5} = \beta^4$

$\beta \beta^4 = \beta^5 = 1$

$\beta^{-1} \beta^4 = \beta^3 = 1$

$\beta^2 \beta^3 = \beta^5 = 1$

$\beta + \beta^{-1} = 2 \cos \frac{2\pi}{5}$

and $\beta^2 = \operatorname{cis} \frac{4\pi}{5} = -\cos \frac{4\pi}{5} + i \sin \frac{4\pi}{5}$

$\beta^{-2} = \operatorname{cis}(-\frac{4\pi}{5}) = -\cos \frac{4\pi}{5} - i \sin \frac{4\pi}{5}$

$$\therefore \beta^2 + \beta^{-2} = -2\cos\frac{\pi}{5} \quad \text{--- (2)}$$

$\therefore$ Quadratic equation

$$\text{ie } z^2 - (2\cos\frac{2\pi}{5} - 2\cos\frac{\pi}{5})z - 4\cos\frac{\pi}{5}\cos\frac{2\pi}{5} = 0 \quad \text{--- (3)}$$

now $\beta^5 = 1$

$$\Rightarrow \beta^5 - 1 = 0$$

$$(\beta - 1)(\beta^4 + \beta^3 + \beta^2 + \beta + 1) = 0$$

$$\beta \neq 1 \Rightarrow \beta^4 + \beta^3 + \beta^2 + \beta + 1 = 0$$

$$\therefore \beta + \beta^2 + \beta^3 + \beta^4 = -1$$

If roots are $\beta + \beta^{-1}$, $\beta^2 + \beta^{-2}$ then quadratic is

$$z^2 - (\beta + \beta^{-1} + \beta^2 + \beta^{-2})z + (\beta + \beta^{-1})(\beta^2 + \beta^{-2}) = 0$$

$$\text{ie } z^2 - (\beta + \beta^4 + \beta^2 + \beta^3)z + (\beta^3 + \beta^4 + \beta + \beta^2) = 0 \quad \text{--- (1)}$$

$$\text{ie } z^2 + z - 1 = 0 \quad \text{--- (4)}$$

Comparing (3) and (4)

$$\rightarrow 2\cos\frac{2\pi}{5} - 2\cos\frac{\pi}{5} = -1$$

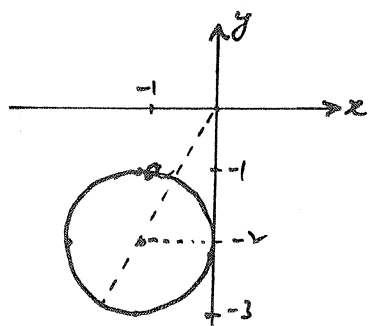
$$\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2} \quad \text{Q.E.D.}$$

$$(c) b) |z + 1 + 2i| \leq 1$$

$$\vec{a} = 2\vec{i} + 1\vec{j}$$

$$\therefore a = \sqrt{5}$$

maximum value of $|z|$ is $\sqrt{5} + 1$



1.5-
circle
center
1.5- radius

(3)

QUESTION 5:

$$(a) \quad \frac{1}{5} + \frac{2}{5^2} + \frac{3}{5^3}$$

$$= \frac{1}{5} + \left(\frac{1}{5^2} + \frac{1}{5^2} \right) + \left(\frac{1}{5^3} + \frac{1}{5^3} + \frac{1}{5^3} \right) + \left(\frac{1}{5^4} + \frac{1}{5^4} + \frac{1}{5^4} + \frac{1}{5^4} \right) + \dots$$

$$= \left(\frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \dots \right) + \left(\frac{1}{5^2} + \frac{1}{5^3} + \frac{1}{5^4} + \dots \right)$$

$$+ \left(\frac{1}{5^3} + \frac{1}{5^4} + \frac{1}{5^5} + \dots \right) + \left(\frac{1}{5^4} + \frac{1}{5^5} + \dots \right) + \dots$$

$$= \frac{\frac{1}{5}}{\frac{1}{5}} + \frac{\frac{1}{5^2}}{\frac{1}{5}} + \frac{\frac{1}{5^3}}{\frac{1}{5}} + \dots$$

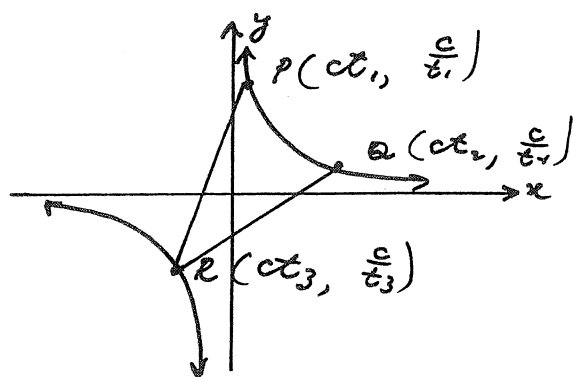
$$= \frac{1}{4} + \frac{1}{20} + \frac{1}{100} + \dots$$

the common ratio clearly being $\frac{1}{5}$

$$= \frac{\frac{1}{4}}{\frac{1}{5}}$$

$$= \frac{5}{16}$$

(b)



$$(i) \quad m_{PQ} = \frac{\frac{c}{t_2} - \frac{c}{t_1}}{ct_2 - ct_1} \times \left(\frac{t_1, t_2}{t_1, t_2} \right)$$

$$= \frac{C(t_1 - t_2)}{Ct_1 t_2 (t_2 - t_1)}$$

$$= -\frac{1}{t_1 t_2}$$

$$(ii) \quad PR \perp QR \Rightarrow -\frac{1}{t_1 t_3} \times \frac{-1}{t_2 t_3} = -1$$

$$\therefore t_1 t_2 t_3 = -1 \quad \text{--- (1)}$$

$$\begin{aligned} \text{now } xy &= c^2 \\ \Rightarrow y &= c^2 x^{-1} \\ y' &= -c^2 x^{-2} \end{aligned}$$

$$\text{at } R(ct_3, \frac{c}{t_3}) \quad y' = \frac{-c^2}{c^2 t_3^2}$$

$$\text{ie } m_R = -\frac{1}{t_3^2}$$

$$\text{Then } m_R \times m_{PA} = -\frac{1}{t_3^2} \times \frac{-1}{t_1 t_v}$$

$$= \frac{1}{t_1 t_v t_3^2}$$

$$= -1 \text{ from } \textcircled{1}$$

$\therefore$ Tangent at R must be $\perp$ PA.

$$(c) \quad (i) \quad \cos 3\theta = \cos(2\theta + \theta)$$

$$= \cos 2\theta \cos \theta - \sin 2\theta \sin \theta$$

$$= (2\cos^2 \theta - 1) \cos \theta - 2\sin \theta \cos \theta \sin \theta$$

$$= 2\cos^3 \theta - \cos \theta - 2\cos \theta (1 - \cos^2 \theta)$$

$$= 4\cos^3 \theta - 3\cos \theta$$

$$\Rightarrow \cos^3 \theta - \frac{3}{4} \cos \theta = \frac{1}{4} \cos 3\theta \quad \text{--- } \textcircled{1}$$

$$(ii) \quad \text{let } x = m \cos \theta \text{ in } 27x^3 - 9x - \sqrt{3} = 0$$

$$\text{ie } 27m^3 \cos^3 \theta - 9m \cos \theta - \sqrt{3} = 0 \quad \text{--- } \textcircled{2}$$

$$\therefore \cos^3 \theta - \frac{1}{3m^2} \cos \theta = \frac{\sqrt{3}}{27m^3} \quad \text{--- } \textcircled{2}$$

$$\text{Comparing with } \textcircled{1}, \quad \frac{1}{3m^2} = \frac{3}{4}$$

$$\text{ie } m^2 = \frac{4}{9}$$

$$\therefore m = \frac{2}{3} \text{ will suffice}$$

② then becomes

$$\begin{aligned} \cos^3 \theta - \frac{3}{4} \cos \theta &= \frac{\sqrt{3}}{27 \times \frac{8}{27}} \\ &= \frac{\sqrt{3}}{8} \end{aligned}$$

$$\Rightarrow \frac{1}{4} \cos 3\theta = \frac{\sqrt{3}}{8}$$

$$\therefore \cos 3\theta = \frac{\sqrt{3}}{4}$$

$$3\theta = 2n\pi \pm \frac{\pi}{6}$$

$$\therefore 3\theta = \pm \frac{\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}, \frac{23\pi}{6}, \frac{25\pi}{6}, \dots$$

$$\text{ie } \theta = \pm \frac{\pi}{18}, \frac{11\pi}{18}, \frac{13\pi}{18}, \frac{23\pi}{18}, \frac{25\pi}{18}, \dots$$

The 3 solutions of

$$27x^3 - 9x - \sqrt{3} = 0 \text{ will be}$$

$$\frac{2}{3} \cos \frac{\pi}{18}, \frac{2}{3} \cos \frac{11\pi}{18}, \frac{2}{3} \cos \frac{13\pi}{18}, \frac{2}{3} \cos \frac{23\pi}{18}, \frac{2}{3} \cos \frac{25\pi}{18}, \dots$$

$$= \frac{2}{3} \cos \frac{\pi}{18}, -\frac{2}{3} \cos \frac{7\pi}{18}, -\frac{2}{3} \cos \frac{5\pi}{18}, -\frac{2}{3} \cos \frac{5\pi}{18}, -\frac{2}{3} \cos \frac{7\pi}{18}, \dots$$

$$\text{ie } \frac{2}{3} \cos \frac{\pi}{18}, -\frac{2}{3} \cos \frac{7\pi}{18}, -\frac{2}{3} \cos \frac{5\pi}{18}.$$

Product of roots

$$\Rightarrow \frac{2}{3} \cos \frac{\pi}{18} \cdot -\frac{2}{3} \cos \frac{7\pi}{18} \cdot -\frac{2}{3} \cos \frac{5\pi}{18} = \frac{\sqrt{3}}{27}$$

$$\text{ie } \frac{8}{27} \cos \frac{\pi}{18} \cdot \cos \frac{5\pi}{18} \cdot \cos \frac{7\pi}{18} = \frac{\sqrt{3}}{27}$$

$$\cos \frac{\pi}{18} \cdot \cos \frac{5\pi}{18} \cdot \cos \frac{7\pi}{18} = \frac{\sqrt{3}}{8}$$

$$(b) \quad (i) \int_0^{\frac{\pi}{2}} \sin^2 x \cos^3 x \, dx$$

$$= \int_0^{\frac{\pi}{2}} \sin^2 x \cdot \cos^2 x \cdot \cos x \, dx$$

$$u = \sin x \\ du = \cos x \, dx$$

$$= \int_0^1 u^2 (1-u^2) \, du$$

$$= \int_0^1 (u^2 - u^4) \, du$$

$$= \left[\frac{u^3}{3} - \frac{u^5}{5} \right]_0^1$$

$$= \frac{1}{3} - \frac{1}{5}$$

$$= \frac{2}{15}$$

$$(ii) \quad \text{Consider } \int_0^a f(a-x) \, dx$$

$$\text{let } u = a-x \\ du = -dx$$

$$= \int_a^0 f(u) \cdot -du$$

$$= \int_0^a f(u) \, du$$

$$= \int_0^a f(x) \, dx$$

$$\int_0^{\frac{\pi}{2}} \frac{\sin^4 x}{\sin^4 x + \cos^4 x} \, dx = \int_0^{\frac{\pi}{2}} \frac{\sin^4(\frac{\pi}{2}-x)}{\sin^4(\frac{\pi}{2}-x) + \cos^4(\frac{\pi}{2}-x)} \, dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos^4 x}{\cos^4 x + \sin^4 x} \, dx$$

$$\text{Then } \int_0^{\frac{\pi}{2}} \frac{\sin^4 x}{\sin^4 x + \cos^4 x} \, dx + \int_0^{\frac{\pi}{2}} \frac{\cos^4 x}{\sin^4 x + \cos^4 x} \, dx$$

$$\begin{aligned}
 \therefore \cos \frac{\pi}{18} \cdot \cos \frac{3\pi}{18} \cdot \cos \frac{5\pi}{18} \cdot \cos \frac{7\pi}{18} &= \frac{\sqrt{3}}{8} \cdot \cos \frac{3\pi}{18} \\
 &= \frac{\sqrt{3}}{8} \cdot \frac{\sqrt{3}}{2} \\
 &= \frac{3}{16}
 \end{aligned}$$

QUESTION 6:

$$\frac{x^2}{4} - \frac{y^2}{7} = 1$$

$$\Rightarrow a=2$$

$$b=\sqrt{7}$$

$$b^2 = a^2(e^2 - 1)$$

$$7 = 4(e^2 - 1)$$

$$e^2 - 1 = \frac{7}{4}$$

$$e^2 = \frac{11}{4}$$

$$\therefore e = \frac{\sqrt{11}}{2}$$

(a)

$$e = \frac{\sqrt{11}}{2}$$

Foci at $(\pm ae, 0)$ i.e. $(\pm \sqrt{11}, 0)$

$\therefore$ Directrices are $x = \pm \frac{a}{e}$

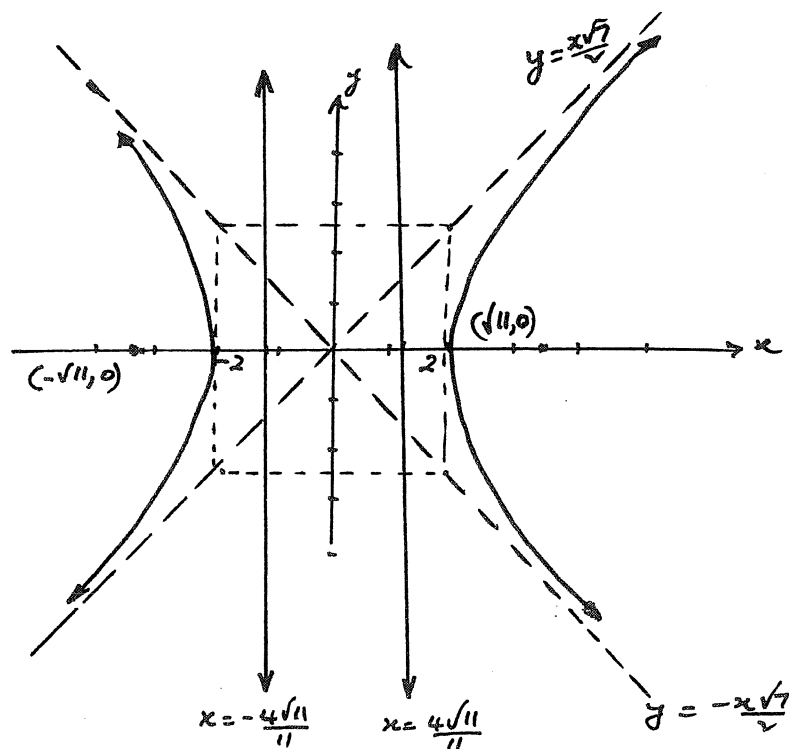
$$= \pm \frac{2}{\left(\frac{\sqrt{11}}{2}\right)}$$

$$= \pm \frac{4}{\sqrt{11}} \quad \text{OR} \quad \pm \frac{4\sqrt{11}}{11}$$

Asymptotes are $y = \pm \frac{bx}{a}$

$$= \pm \frac{\sqrt{7}x}{2}$$

(b)



(c) P is $(2\sec\theta, \sqrt{7}\tan\theta)$

$$\begin{aligned}\text{Then } \frac{x^2}{4} - \frac{y^2}{7} &= \frac{4\sec^2\theta}{4} - \frac{7\tan^2\theta}{7} \\ &= \sec^2\theta - \tan^2\theta \\ &= 1\end{aligned}$$

$\therefore$ P lies on H.

$$\frac{x^2}{4} - \frac{y^2}{7} = 1$$

differentiating w.r.t. x

$$\Rightarrow \frac{x}{2} - \frac{2yy'}{7} = 0$$

$$\text{ie } \frac{2yy'}{7} = \frac{x}{2}$$

$$y' = \frac{7x}{4y}$$

at P:

$$\begin{aligned}y' &= \frac{14\sec\theta}{4\sqrt{7}\tan\theta} \\ &= \frac{\sqrt{7}\sec\theta}{2\tan\theta}\end{aligned}$$

Tangent at P is

$$y - \sqrt{7} \tan \theta = \frac{\sqrt{7} \sec \theta}{2 \tan \theta} (x - 2 \sec \theta) \quad \text{--- (1)}$$

$$\text{ie } 2 \tan \theta \cdot y - 2\sqrt{7} \tan^2 \theta = \sqrt{7} \sec \theta \cdot x - 2\sqrt{7} \sec^2 \theta$$

$$\text{ie } x\sqrt{7} \sec \theta - 2y \tan \theta = 2\sqrt{7} (\sec^2 \theta - \tan^2 \theta) \\ = 2\sqrt{7}$$

$$\therefore \frac{x \sec \theta}{2} - \frac{y \tan \theta}{\sqrt{7}} = 1 \quad \text{--- (1)}$$

(i) solving (1) with $y = \frac{x\sqrt{7}}{2}$

$$\Rightarrow \frac{x \sec \theta}{2} - \frac{x\sqrt{7}}{2\sqrt{7}} \tan \theta = 1$$

$$x\sqrt{7} (\sec \theta - \tan \theta) = 2\sqrt{7}$$

$$\Rightarrow x = \frac{2}{\sec \theta - \tan \theta}$$

$$= \frac{2(\sec \theta + \tan \theta)}{\sec^2 \theta - \tan^2 \theta}$$

$$= 2(\sec \theta + \tan \theta)$$

ie L is $(2(\sec \theta + \tan \theta), \sqrt{7}(\sec \theta + \tan \theta))$

Solving (1) with $y = -\frac{x\sqrt{7}}{2}$

$$\Rightarrow \frac{x \sec \theta}{2} + \frac{x\sqrt{7}}{2\sqrt{7}} \tan \theta = 1$$

$$x\sqrt{7} (\sec \theta + \tan \theta) = 2\sqrt{7}$$

$$\therefore x = \frac{2}{\sec \theta + \tan \theta}$$

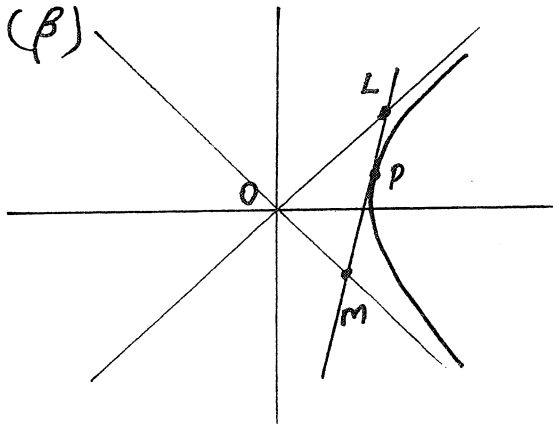
$$= 2(\sec \theta - \tan \theta)$$

M is $(2(\sec \theta - \tan \theta), -\sqrt{7}(\sec \theta - \tan \theta))$

(α) Then the mid-point of LM is
 $(2 \sec \theta, \sqrt{7} \tan \theta)$

which is P

$$\therefore LP = MP.$$



Equation LO is $y = \frac{x\sqrt{7}}{2}$ i.e. $x\sqrt{7} - 2y = 0$

$\perp$ distance from M to LO is

$$d = \frac{|2\sqrt{7}(\sec \theta - \tan \theta) + 2\sqrt{7}(\sec \theta - \tan \theta)|}{\sqrt{11}}$$

$$= \frac{|4\sqrt{7}(\sec \theta - \tan \theta)|}{\sqrt{11}}$$

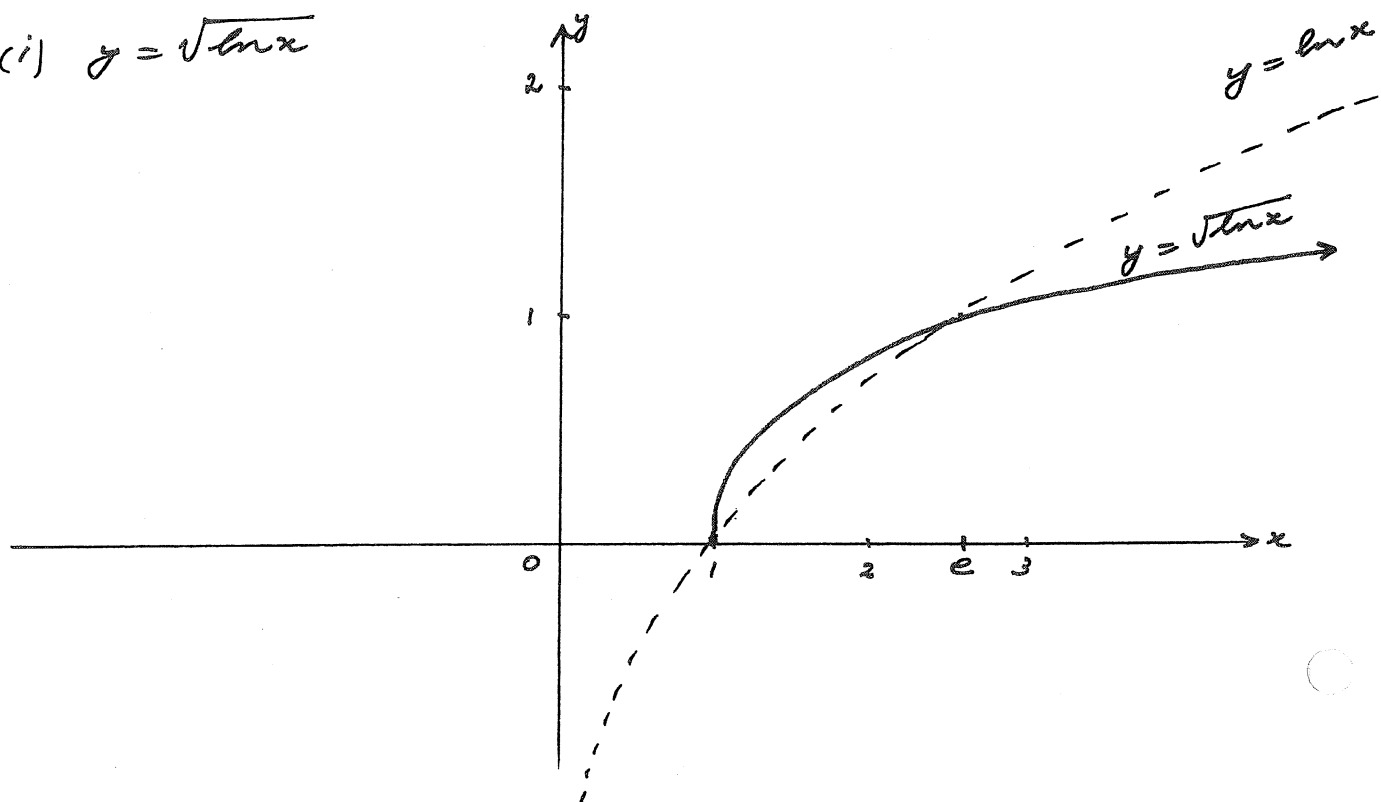
$$\begin{aligned} \text{Distance OL} &= \sqrt{4(\sec \theta + \tan \theta)^2 + 7(\sec \theta + \tan \theta)^2} \\ &= \sqrt{11 \sec^2 \theta + 22 \sec \theta \tan \theta + 11 \tan^2 \theta} \\ &= \sqrt{11} |\sec \theta + \tan \theta| \end{aligned}$$

$$\text{Area } \triangle OLM = \frac{1}{2} \cdot OL \cdot d$$

$$\begin{aligned} &= \frac{1}{2} \cdot \sqrt{11} (\sec \theta + \tan \theta) \cdot \frac{4\sqrt{7}}{\sqrt{11}} (\sec \theta - \tan \theta) \\ &= 2\sqrt{7} \text{ units (ie independent of } \theta) \end{aligned}$$

QUESTION 7

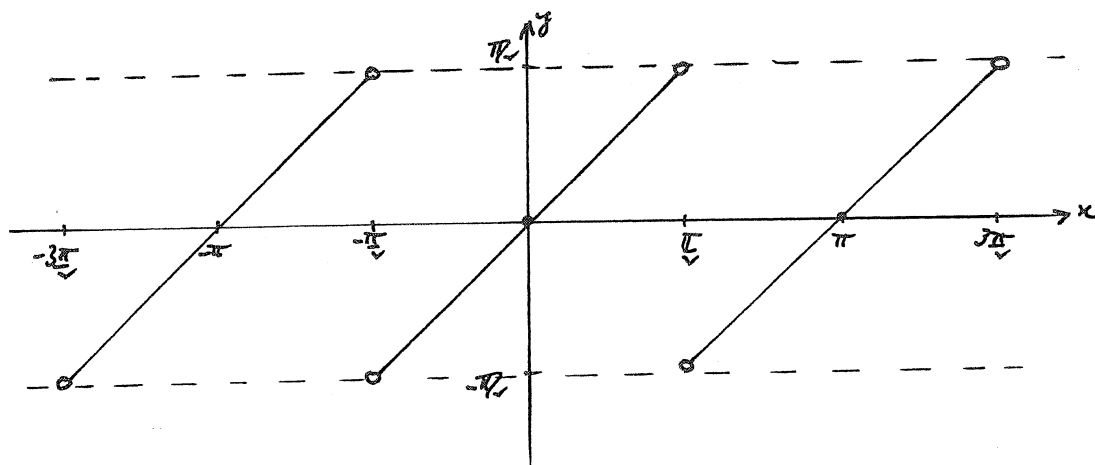
(a) (i) $y = \sqrt{\ln x}$



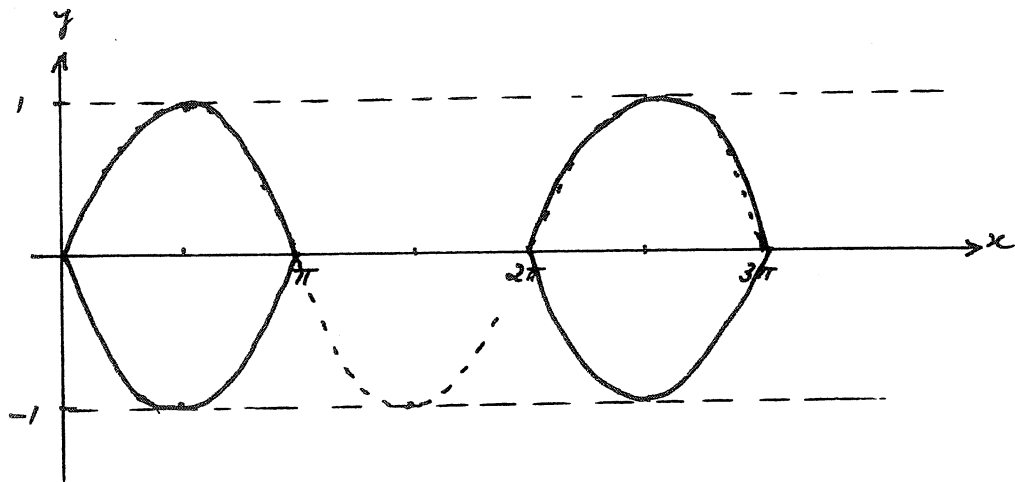
(ii) $y = \tan^{-1}(\tan x)$ for $-\frac{3\pi}{2} \leq x \leq \frac{3\pi}{2}$

$$\frac{dy}{dx} = \frac{1}{1 + \tan^2 x} \cdot \sec^2 x$$

$$= 1 \text{ for all } x$$



(iii)

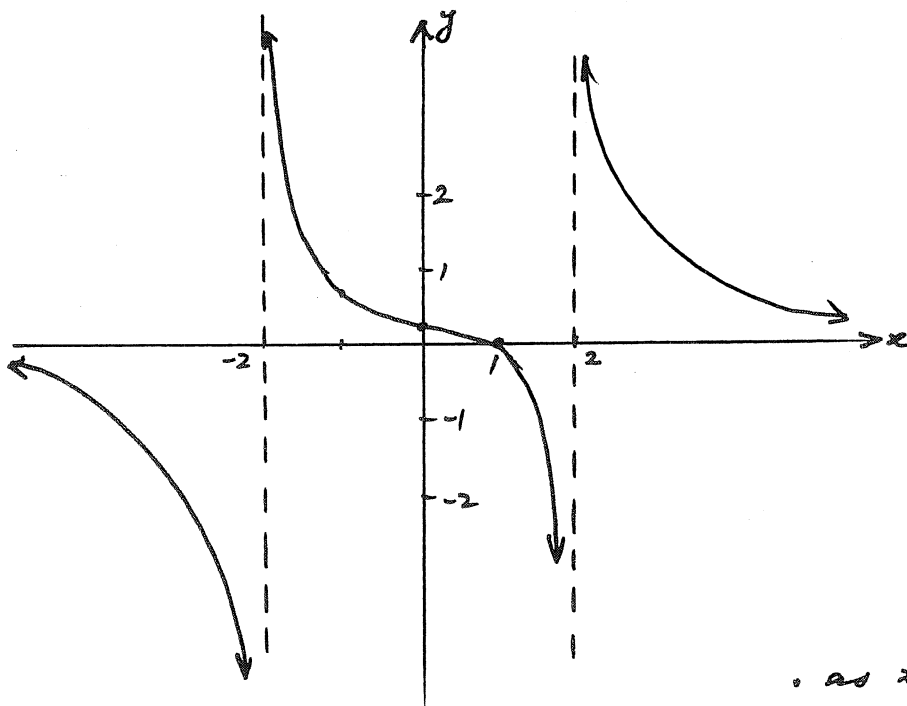


$$|y| = \sin x \Rightarrow \sin x \geq 0$$

(iv)

$$y = \frac{x-1}{x^2-4}$$

- x intercept at $(1, 0)$
- y intercept at $(0, \frac{1}{4})$
- vert. asymptote at $x = \pm 2$
- horiz. asymptote at $y = 0$
- $\lim_{x \rightarrow 2^+} \frac{x-1}{x^2-4} = +\infty$



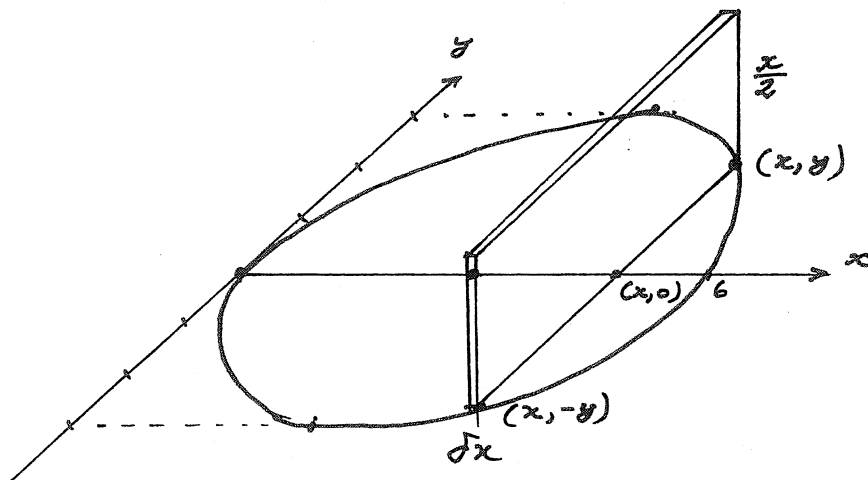
• as $x \rightarrow -2^-$, $y \rightarrow -\infty$

$$(b) \quad x^2 + y^2 = 6x$$

$$\Rightarrow x^2 - 6x + y^2 = 0$$

$$(x-3)^2 + y^2 = 9$$

$\therefore$ Centre is $(3, 0)$ radius is 3.



Volume of slice shown is

$$\delta V = 2y \cdot \frac{x}{2} \cdot \delta x$$

$$= xy \delta x \quad \text{where } y = \sqrt{6x - x^2}$$

$\therefore$ Volume of solid is

$$V = \lim_{\delta x \rightarrow 0} \sum_{x=0}^6 xy \delta x$$

$$= \int_0^6 x \sqrt{6x - x^2} dx$$

$$= \int_0^6 x \sqrt{9 - (x-3)^2} dx$$

$$\begin{aligned} 6x - x^2 &= -1(x^2 - 6x) \\ &= -1[(x-3)^2 - 9] \\ &= 9 - (x-3)^2 \end{aligned}$$

$$\begin{aligned} \text{let } x-3 &= 3 \sin \theta \\ dx &= 3 \cos \theta d\theta \end{aligned}$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (3+3 \sin \theta) \cdot \sqrt{9-9 \sin^2 \theta} \cdot 3 \cos \theta d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (27 \cos^2 \theta + 27 \underbrace{\sin \theta \cos^2 \theta}_{\text{odd function } \therefore \text{zero}}) d\theta$$

$$= 54 \int_0^{\frac{\pi}{2}} \cos^2 \theta d\theta$$

$$\cos 2\theta = 2 \cos^2 \theta - 1$$

$$= \frac{54}{2} \int_0^{\frac{\pi}{2}} (1 + \cos 2\theta) d\theta$$

$$= 27 \left[\theta + \frac{1}{2} \sin 2\theta \right]$$

$$= 27 \left[\frac{\pi}{2} - 0 \right]$$

$$= \frac{27\pi}{2}$$

$\therefore$ Volume is $\frac{27\pi}{2}$ units³.

QUESTION 8:

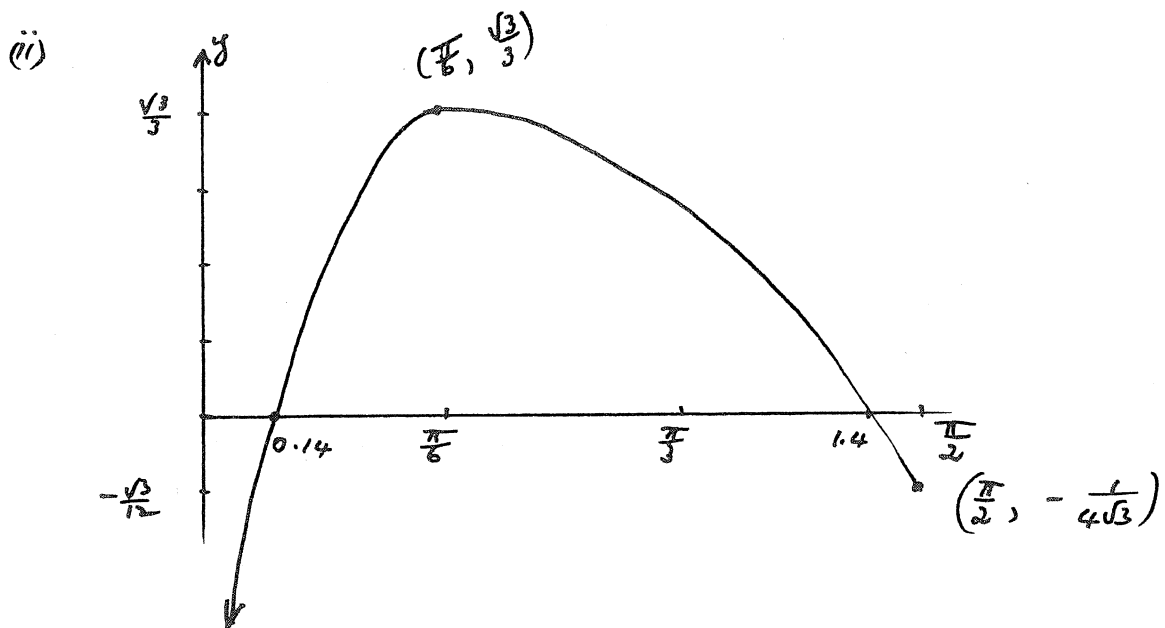
$$(a) \quad f(\theta) = \cos \theta - \frac{1}{4\sqrt{3} \sin \theta}$$

$$(i) \quad f'(\theta) = -\sin \theta - \left[\frac{-1 \cdot 4\sqrt{3} \cos \theta}{(4\sqrt{3})^2 \sin^2 \theta} \right]$$
$$= -\sin \theta + \frac{\cos \theta}{4\sqrt{3} \sin^2 \theta}$$

$$\therefore f'\left(\frac{\pi}{6}\right) = -\frac{1}{2} + \frac{\frac{\sqrt{3}}{2}}{4\sqrt{3} \cdot \frac{1}{4}}$$

$$= -\frac{1}{2} + \frac{\frac{\sqrt{3}}{2}}{\sqrt{3}}$$

$$= 0$$



$$\text{as } \theta \rightarrow 0 \quad f(\theta) \rightarrow -\infty$$

$$f\left(\frac{\pi}{6}\right) = -\frac{1}{4\sqrt{3}}$$

$$f\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} - \frac{1}{2\sqrt{3}}$$
$$= \sqrt{3}$$

max. turning point at $(\frac{\pi}{6}, \frac{\sqrt{3}}{3})$ since $f''(\theta) < 0$

x intercept at $y = 0$

$$\text{ie } \cos \theta - \frac{1}{4\sqrt{3}\sin \theta} = 0$$

$$\text{ie } 4\sqrt{3}\sin \theta \cos \theta = 1$$

$$2\sqrt{3}\sin 2\theta = 1$$

$$\sin 2\theta = \frac{1}{2\sqrt{3}}$$

$$2\theta = 0.29, 2.85$$

$$\theta = 0.14, 1.4$$

(b) (i) $x = Vt \cos \alpha$ ————— ①

$$y = -\frac{1}{2}gt^2 + Vt \sin \alpha$$
 ————— ②

$$y = 0 \Rightarrow -\frac{1}{2}gt^2 + Vt \sin \alpha = 0$$

$$-\frac{1}{2}t(gt - 2V \sin \alpha) = 0$$

$$\therefore t = \frac{2V \sin \alpha}{g}$$

$\therefore$ Hits ground at $\frac{2V \sin \alpha}{g}$ s.

(ii) Range is $x = Vt \cos \alpha$

$$= V \cos \alpha \cdot \frac{2V \sin \alpha}{g}$$

$$= \frac{2V^2 \sin \alpha \cos \alpha}{g} \text{ m}$$

(iii) Position of target is given by

$$x = d + ut$$
 ————— ③

When projectile hits the target we have $t = \frac{2V \sin \alpha}{g}$

and $x = \frac{2V^2 \sin \alpha \cos \alpha}{g}$

} sub in ③

$$\therefore \frac{2V^2 \sin \alpha \cos \alpha}{g} = d + u \cdot \left(\frac{2V \sin \alpha}{g} \right)$$

$$\text{ie } u \cdot \frac{2V \sin \alpha}{g} = \frac{2V^2 \sin \alpha \cos \alpha}{g} - d$$

$$\Rightarrow u = V \cos \alpha - \frac{gd}{2V \sin \alpha} \quad \text{--- (4)}$$

(iv) We now assume that $gd = \frac{V^2}{2\sqrt{3}}$

Then (4) becomes

$$u = V \cos \alpha - \frac{V^2}{4V \cdot \sqrt{3} \sin \alpha}$$

$$= V \left[\cos \alpha - \frac{1}{4\sqrt{3} \sin \alpha} \right] \quad \text{is the same function as in part (a)}$$

$$\text{ie } \frac{u}{V} = \cos \alpha - \frac{1}{4\sqrt{3} \sin \alpha}$$

$$\text{now if } u > \frac{V}{\sqrt{3}}$$

$$\text{ie } \frac{u}{V} > \frac{1}{\sqrt{3}}$$

$$\text{ie } \frac{u}{V} > \frac{\sqrt{3}}{3}$$

and from the graph in part (a) we see that no α gives a value of $\frac{u}{V}$ bigger than $\frac{\sqrt{3}}{3}$.

$$(v) \text{ If } u < \frac{V}{\sqrt{3}}$$

ie $\frac{u}{V} < \frac{1}{\sqrt{3}}$ we see from the graph that two α give a solution ($u, V, > 0$)