Student Number:

St Catherine's School
Waverley

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

August 2012

Mathematics Extension 2

General Instructions

- Reading Time – 5 minutes
- Working Time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question
- Complete each section in a separate booklet

- All questions are of equal value
- **Total Marks** – 100
- Attempt Questions 1 – 16

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0,\ \textit{if } n<0$$

$$\int \frac{1}{x}\,dx = \ln x,\ x>0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\ a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\ a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\ a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\ a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\ a>0,\ -a<x<a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\ x>a>0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x,\ x>0$

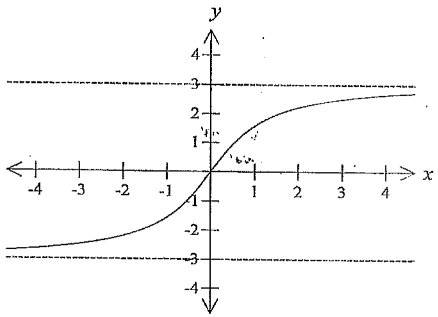
SECTION 1

Attempt Questions 1-10 All questions are of equal value

Answer each question on the Multiple Choice Answer Sheet supplied

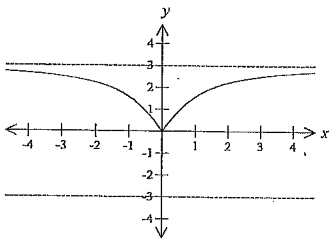
Question 1:

1 The diagram shows the graph of the function $y = f(x)$.

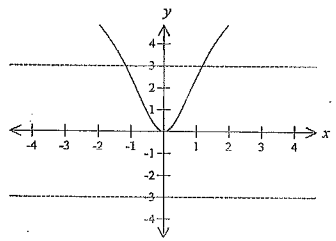


Which of the following is the graph of $y = \sqrt{f(x)}$?

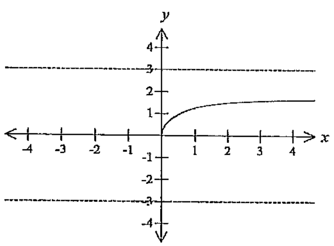
(A)



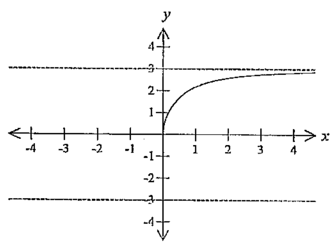
(B)



(C)



(D)



Question 2:

What is the value of $\arg \bar{z}$ given the complex number $z = 1 - i\sqrt{3}$?

(A) $-\dfrac{\pi}{3}$ (B) $\dfrac{\pi}{3}$

(C) $-\dfrac{2\pi}{3}$ (D) $\dfrac{2\pi}{3}$

Question 3:

It is given that $3+i$ is a root of $P(z) = z^3 + az^2 + bz + 10$ where a and b are real numbers. Which expression factorises $P(z)$ over the real numbers?

(A) $(z+1)(z^2 - 6z + 10)$ (B) $(z-1)(z^2 - 6z - 10)$

(C) $(z+1)(z^2 + 6z + 10)$ (D) $(z-1)(z^2 + 6z - 10)$

Question 4:

For the ellipse with the equation $\dfrac{x^2}{4} + \dfrac{y^2}{3} = 1$. What is the eccentricity?

(A) $\dfrac{1}{4}$ (B) $\dfrac{1}{2}$

(C) $\dfrac{3}{4}$ (D) $\dfrac{9}{16}$

Question 5:

Consider the hyperbola with the equation $\dfrac{x^2}{144} - \dfrac{y^2}{25} = 1$.

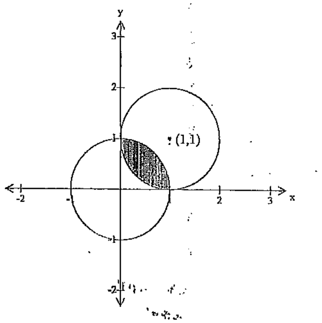
What are the equations of the directrices?

(A) $x = \pm\dfrac{144}{13}$ (B) $x = \pm\dfrac{13}{25}$

(C) $x = \pm\dfrac{25}{13}$ (D) $x = \pm\dfrac{13}{144}$

Question 6:

Consider the Argand diagram below.



Which inequality could define the shaded area?

(A) $|z|\leq 1$ and $|z-(1-i)|\geq 1$ (B) $|z|\leq 1$ and $|z-(1+i)|\geq 1$

(C) $|z|\leq 1$ and $|z-(1-i)|\leq 1$ (D) $|z|\leq 1$ and $|z-(1+i)|\leq 1$

Question 7:

Which of the following is an expression for $\int \frac{2x}{\sqrt{16-x^2}}dx$?

(A) $-2\sqrt{16-x^2}+c$ (B) $-\sqrt{16-x^2}+c$

(C) $\frac{1}{2}\sqrt{16-x^2}+c$ (D) $-\frac{1}{2}\sqrt{16-x^2}+c$

Question 8:

The polynomial equation $x^3-5x^2+6=0$ has roots α, β and γ.

Which of the following polynomial equations have roots $\alpha-1$, $\beta-1$ and $\gamma-1$?

(A) $x^3-8x^2-7x=0$ (B) $x^3-8x^2+13x=0$

(C) $x^3-3x^2-7x+2=0$ (D) $x^3-2x^2-7x+2=0$

Question 9:

The region bounded by $y\leq 4x^2-x^4$ and $0\leq x\leq 2$ is rotated about the y axis to form a solid.

What is the volume of this solid using the method of cylindrical shells

(A) $\frac{16\pi}{3}$ units^3 (B) $\frac{8\pi}{3}$ units^3

(C) $\frac{32\pi}{3}$ units^3 (D) $\frac{20\pi}{3}$ units^3

Question 10:

Which of the following is an expression for $\int\frac{1}{\sqrt{7-6x-x^2}}dx$?

(A) $\sin^{-1}\left(\frac{x-3}{2}\right)+c$ (B) $\sin^{-1}\left(\frac{x+3}{2}\right)+c$

(C) $\sin^{-1}\left(\frac{x-3}{4}\right)+c$ (D) $\sin^{-1}\left(\frac{x+3}{4}\right)+c$

End of Section 1

Marks

Question 11 (15 Marks) Start a new booklet

(a) Using $t=\tan\frac{x}{2}$, evaluate $\int_0^{\frac{\pi}{2}}\frac{dx}{1+\sin x}$. **2**

(b) Use the substitution $u=e^x$, or otherwise, find $\int\frac{e^x dx}{\sqrt{1-e^{2x}}}$. **2**

(c) Find $\int\frac{4x^3-2x^2+1}{2x-1}\,dx$. **3**

(d) Show, using integration by parts, that **3**

$$\int_0^{\frac{\pi}{3}} x\sec^2 x\,dx=\frac{\pi\sqrt{3}}{3}-\ln 2$$

(e) Let $I_n=\int_0^{\pi}x^n\sin x\,dx$, where $n=0,1,2,\ldots\ldots.$

(i) Use integration by parts to show that $I_n=\pi^n-n(n-1)I_{n-2}$ **3**

for $n=2,3,4,\ldots.$

(ii) Hence, evaluate $\int_0^{\pi}x^4\sin x\,dx$ **2**

St Catherine's School Waverley
HSC MX2 Trial HSC Examination 2012

Marks

Question 12 (15 Marks) Start a new booklet

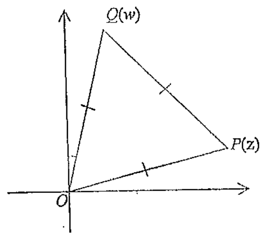
(a) If $A=3+4i$ and $B=5-13i$ write the following in the form $x+iy$

(i) AB **1**

(ii) $\frac{A}{B}$ **1**

(iii) $\sqrt{A}$ **2**

(b)



In the Argand diagram, OPQ is an equilateral triangle. P represents the complex number z and Q represents the complex number w.

(i) Explain why $w=z\,cis\frac{\pi}{3}$ **2**

(ii) Show that $w^3+z^3=0$ **2**

(c) The complex number Z moves such that $\text{Im}\left(\frac{1}{\overline{Z}-i}\right)=1$. **3**

Show that the locus of Z is a circle and find its centre and radius.

(d) On the Argand diagram, shade the region where both $|z-1-i|\le 2$ and $0\le\arg z<\frac{\pi}{4}$ **2**

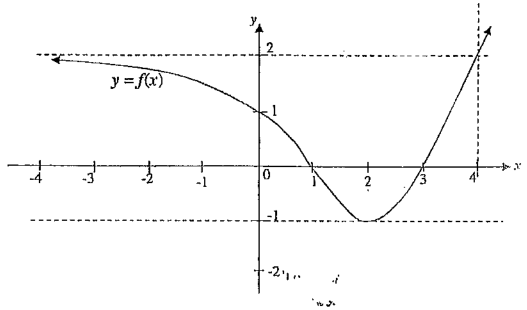
(e) Show that the complex number $z=\frac{1-t^2+2it}{1+t^2}$ lies on a unit circle centre the origin, **2**

for all values of t

Marks

Question 13 (15 Marks) Start a new booklet

(a) The diagram shows the graph of $y = f(x)$. It has a horizontal asymptote at $y = 2$.



Draw separate one-third page sketches of the graphs of the following:

(i) $y = |f(x)|$ 2

(ii) $y = \dfrac{1}{f(x)}$ 2

(iii) $y^2 = f(x)$ 2

(iv) $y = \ln(f(x))$ 2

(b) Find the equation, in general form, of the tangent to the curve defined by 3

$$x^3 + y - 3xy = 3$$

at the point $(1,2)$

(c) The hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1,\ a > b > 0$, has eccentricity e

(i) Show that the line through the focus $S(ae,0)$ that is perpendicular to the asymptote $y = \dfrac{bx}{a}$ has an equation $ax + by - a^2e = 0$. 1

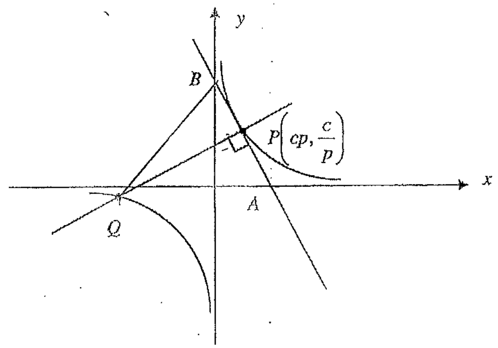
(ii) Show that this line meets the asymptote at a point on the corresponding directrix 3

Marks

Question 14 (15 Marks) Start a new booklet

(a) The point $P\left(cp, \dfrac{c}{p}\right)$ is a point on the hyperbola $xy = c^2$.

The tangent to the hyperbola at P intersects the x and y axes at A and B respectively, and the normal at to the hyperbola at P intersects the second 'branch' of the hyperbola at Q.



(i) Show that the equation of the normal at P is $py - c = p^3(x - cp)$ 2

(ii) Show that the x coordinates of P and Q are the roots of the equation 2

$$x^2 - c\left(p - \frac{1}{p^3}\right)x - \frac{c^2}{p^2} = 0$$

and hence find the coordinates of Q

(iii) Given the distance $AB = 2c\sqrt{p^2 + \dfrac{1}{p^2}}$, show that the area of $\Delta ABQ = c^2\left(p^2 + \dfrac{1}{p^2}\right)^2$ 2

(iv) Given the fact that the sum of any two reciprocals is ≥ 2, find the minimum area of ΔABQ 1

Question 14 continues on page 11

Marks

Question 14 (continued)

(b) The polynomial $P(x)=x^3-6x^2+9x+c$ has a double zero. Find the possible values of the real number c. **2**

(c) Given that $z=\cos\theta+i\sin\theta$ prove that:

(i) $z^n+\dfrac{1}{z^n}=2\cos n\theta$ **1**

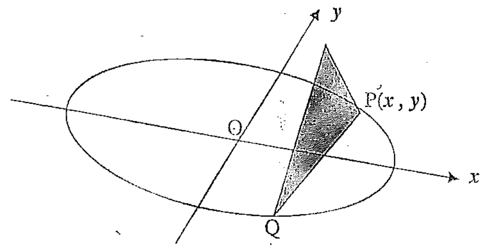
(ii) Express x^5-1 as the product of three factors each containing real coefficients. **3**

(iii) Prove that $\left(1-\cos\dfrac{2\pi}{5}\right)\left(1-\cos\dfrac{4\pi}{5}\right)=\dfrac{5}{4}$ **2**

Marks

Question 15 (15 Marks) Start a new booklet

(a)



The base of a certain solid is the ellipse $\dfrac{x^2}{25}+\dfrac{y^2}{16}=1$

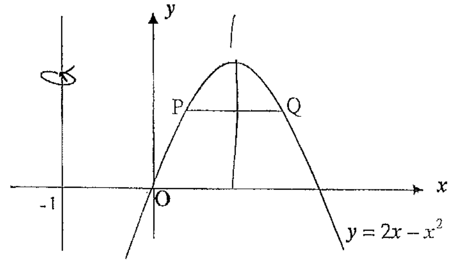
Every cross-section perpendicular to the x-axis is an equilateral triangle. The shaded cross-section shown with base PQ is a typical slice of the solid.

(i) Show that the shaded cross-sectional area is given by $A=\sqrt{3}y^2$ **1**

(ii) Find the cross sectional area as a function of x. **1**

(iii) Hence find the volume of the solid. **2**

(b)



A solid is formed by rotating the region bounded by $y=2x-x^2$ and the x-axis about the line $x=-1$

(i) When the segment PQ of the region is rotated about $x=-1$, it will form an annulus. **2**

Show that the area of this annulus is given by $A=8\pi\sqrt{1-y}$

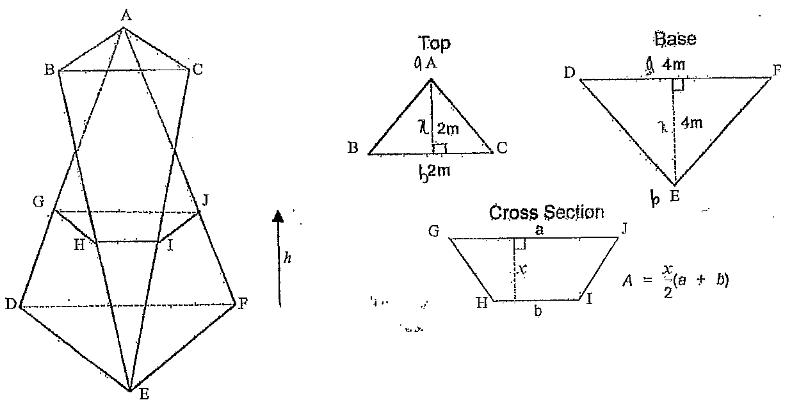
(ii) Hence find the volume of the solid. **3**

Question 15 continues on page 13

Marks

Question 15 (continued)

(c)



A large sculpture has a triangular base and top as shown. The height of the sculpture is 10 metres. Each cross section parallel to the base is an isosceles trapezium, and all other dimensions are shown in the diagram above.

(i) Show, with working, that the cross-sectional area of the slice at h metres above the base, is given by **4**

$$A = 8 - \frac{4h}{5} + \frac{h^2}{50}$$

(ii) Hence by considering the typical slice $GHIJ$ of thickness δh, find the volume of the sculpture. **2**

Marks

Question 16 (15 Marks) Start a new booklet

(a) Show that the condition for the roots of the cubic $ax^3 + bx^2 + cx + d = 0$ to be in the ratio $1:2:3$ is that $bc = 11ad$ **3**

(b) The tangent at P on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ cuts the x-*axis* at M while the normal at P cuts the x-axis at N. If O is the centre of the ellipse; **4**

Prove that $OM.\, ON = a^2e^2$

(c) A stone is projected from a point O on a horizontal plane at an angle of elevation α and with initial velocity U metres per second. The stone reaches a point A in its trajectory, and at that instant it is moving in a direction perpendicular to the angle of projection with a speed V metres per second.

Air resistance is neglected throughout the motion and g is the acceleration due to gravity.

If t is the time in seconds at any instant, show that when the stone is at A:

(i) $V = U\cot\alpha$ **2**

(ii) $t = \dfrac{U}{g\sin\alpha}$ **2**

(d) A sequence $u_1, u_2, u_3, \ldots$ is defined as follows; **4**

$$u_1 = 1\ ,\ u_2 = -12 \ \text{ and } \ u_n = u_{n-1} + 6u_{n-2} \ \text{ for } n \geq 3.$$

Prove by mathematical induction that

$$u_n = -6\left[(-2)^{n-2} + 3^{n-2}\right] \text{ for all positive integers } n.$$

Q	Solutions	Marks	Comments
11(a)	$t = \tan\frac{x}{2}$ $\therefore \sin x = \frac{2t}{1+t^2}$ $\cos x = \frac{1-t^2}{1+t^2}$		
	$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$		
	$= \frac{1}{2}(1+t^2)$ $\int_0^{\pi/2} \frac{dx}{1+\sin x}$		
	$\therefore dx = \frac{2dt}{1+t^2}$	1	
	$x=0$ $t=0$ $= \int_0^1 \frac{2dt}{(1+\frac{2t}{1+t^2})(1+t^2)}$		
	$x=\frac{\pi}{2}$ $t=1$		
	$= 2\int_0^1 \frac{dt}{1+2t+t^2}$		
	$= 2\int_0^1 \frac{dt}{(1+t)^2}$		
	$= 2\left[\frac{-1}{1+t}\right]_0^1$		
	$= 2\left[-\frac{1}{2}+1\right]$		
	$= 1.$	1	
b)	$\int \frac{e^x}{\sqrt{1-e^{2x}}}dx = \int \frac{du}{\sqrt{1-u^2}}$ $u = e^x$, $du = e^x dx$	1	
	$= \sin^{-1}u + c$		
	$= \sin^{-1}e^x + c$	1	
c)	$\int \frac{4x^3-2x^2+1}{2x-1}$ $4x^3-2x^2+1 = (2x-1)(2x^2)+1$	1	
	$= \int \left[2x^2 + \frac{1}{2x-1}\right]dx$	1	
	$= \frac{2x^3}{3} + \frac{1}{2}\ln\lvert 2x-1\rvert + c$	1	

St Catherines's School Waverley

Course: Page no. of

Marking Scheme for Task: Academic Year: 2011-12

Q	Solutions	Marks	Comments
11d)	$\int_0^{\pi/3} x\sec^2 x\, dx$ $u = x$ $v = \tan x$, $u' = 1$ $v' = \sec^2 x$	1	
	$= \left[x\tan x\right]_0^{\pi/3} - \int_0^{\pi/3}\tan x\, dx$		
	$= \frac{\sqrt{3}\pi}{3} + \left[\log_e \cos x\right]_0^{\pi/3}$	1	
	$= \frac{\sqrt{3}\pi}{3} + \ln\frac{1}{2} - 0$		
	$= \frac{\pi\sqrt{3}}{3} + \ln 1 - \ln 2$		
	$= \frac{\pi\sqrt{3}}{3} - \ln 2$	1	
11e	$I_n = \int_0^{\pi} x^n \sin x\, dx$ $u = x^n$ $v = -\cos x$, $u' = nx^{n-1}$ $v' = \sin x$	1	
	$= \left[-x^n\cos x\right]_0^{\pi} + \int_0^{\pi} nx^{n-1}\cos x\, dx$		
	$= \pi^n + n\int_0^{\pi} x^{n-1}\cos x\, dx$ $u = x^{n-1}$ $v = \sin x$, $u' = (n-1)x^{n-2}$ $v' = \cos x$	1	
	$= \pi^n + n\left[\left[x^{n-1}\sin x\right]_0^{\pi} - \int_0^{\pi}(n-1)x^{n-2}\sin x\, dx\right]$		
	$= \pi^n + n\left[0 - (n-1)I_{n-2}\right]$		
	$= \pi^n - n(n-1)I_{n-2}$	1	

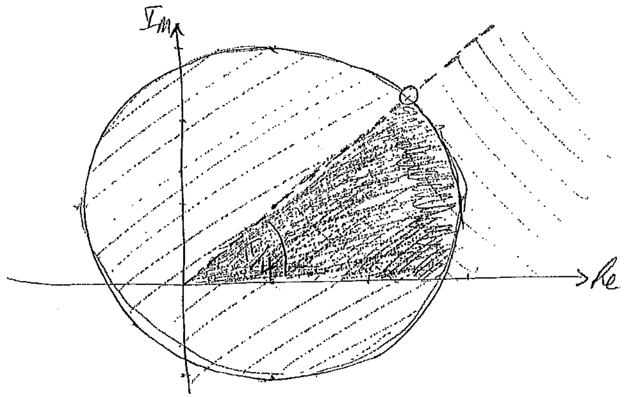
Q	Solutions	Marks	Comments
11e	(ii) $I_4 = \int_0^{\pi} x^4 \sin x \, dx$		
	$= \pi^4 - 4.3 I_2$		
	$= \pi^4 - 12[\pi^2 - 2I_0]$		
	$= \pi^4 - 12\pi^2 + 24 I_0$	1	
	Now $I_0 = \int_0^{\pi} x^0 \sin x \, dx$		
	$= \int_0^{\pi} \sin x \, dx$		
	$= [-\cos x]_0^{\pi}$		
	$= 1 + 1$		
	$= 2$		
	$\therefore I_4 = \pi^4 - 12\pi^2 + 48$	1	

St Catherines's School Waverley

Course: Page no. of

Marking Scheme for Task: Academic Year: 2011-12

Q	Solutions	Marks	Comments
Q12 a)	(i) $AB = (3+4i)(5-3i)$		
	$= 15 - 39i + 20i - 52i^2$	1	
	$= 15 - 19i + 52$		
	$= 67 - 19i$		
	(ii) $\frac{A}{B} = \frac{3+4i}{5+13i} \times \frac{5+13i}{5+13i}$		
	$= \frac{15 - 52 + 59i}{25+169}$	1	
	$= -\frac{37}{194} + \frac{59i}{194}$		
	(iii) $\sqrt{A} = \sqrt{3+4i} = x+iy$		
	$x^2 - y^2 = 3$ — ①		
	also $x^2 + y^2 = 5$ — ②		
	①+② $2x^2 = 8$		
	$x = \pm 2$	2	
	$y = \pm 1$		
	$\therefore \sqrt{A} = \pm(2+i)$		
b)	(i) ω is the complex number z rotated through $60^\circ \left(\frac{\pi}{3}\right)$ in an anticlockwise direction. Also ω and z have equal moduli	2	
	(ii) $\omega^3 + z^3 = \left(z \operatorname{cis}\frac{\pi}{3}\right)^3 + z^3$		
	$= z^3 \operatorname{cis}\pi + z^3$		
	$= -z^3 + z^3$	2	
	$= 0$		

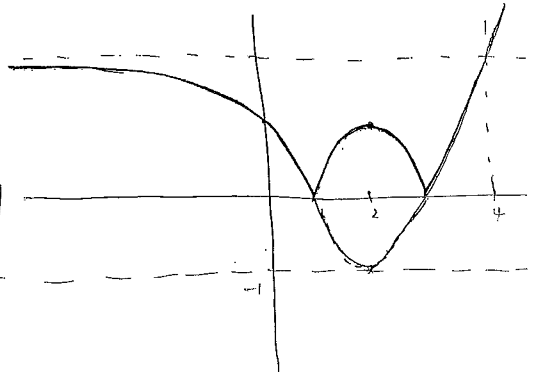
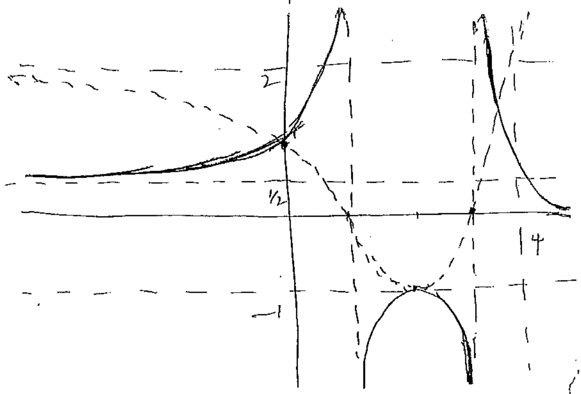
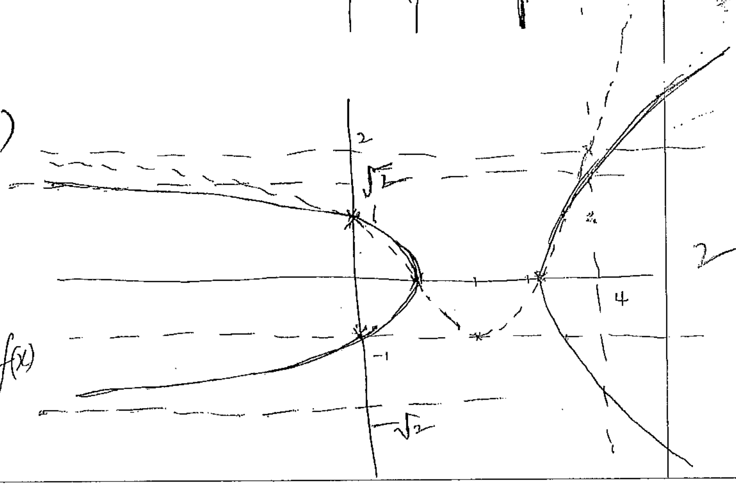
Q	Solutions	Marks	Comments
12c	$Im\left(\frac{1}{z-i}\right)=1$ let $z=x+iy$		
	$\therefore \frac{1}{z-i}=\frac{1}{x-iy-i}$		
	$=\frac{1}{x-(y+1)i}\times\frac{x+(y+1)i}{x+(y+1)i}$		
	$=\frac{x+(y+1)i}{x^2+(y+1)^2}$		
	$=\frac{x}{x^2+(y+1)^2}+\frac{(y+1)}{x^2+(y+1)^2}i$	1	
	Now $Im\left(\frac{1}{z-i}\right)=\frac{y+1}{x^2+(y+1)^2}=1$		
	$\therefore x^2+(y+1)^2=y+1$		
	$x^2+y^2+2y+1=y+1$	1	
	$x^2+y^2+y=0$		
	$x^2+y^2+y+\frac{1}{4}=\frac{1}{4}$		
	$x^2+\left(y+\frac{1}{2}\right)^2=\frac{1}{4}$		
	$\therefore$ Circle centre $\left(0,-\frac{1}{2}\right)$ radius $=\frac{1}{2}$	1	
12d		1ea =2	

St Catherines's School Waverley

Course: Page no. of

Marking Scheme for Task: Academic Year: 2011-12

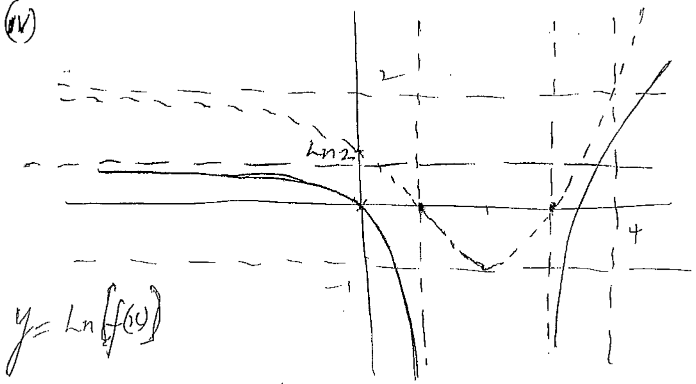
Q	Solutions	Marks	Comments
12e)	$z=\frac{1-t^2+2it}{1+t^2}$		
	$=\frac{1-t^2}{1+t^2}+\frac{2t}{1+t^2}i$		
	let $t=\tan\frac{\theta}{2}$		
	$\therefore z=\cos\theta+i\sin\theta$		
	now $\lvert z\rvert=\sqrt{\cos^2\theta+\sin^2\theta}$		
	$=1$		
	$\therefore z$ lies on unit circle centre origin		
	OR		
	$\lvert z\rvert=\sqrt{\left(\frac{1-t^2}{1+t^2}\right)^2+\left(\frac{2t}{1+t^2}\right)^2}$		
	$=\sqrt{\frac{1-2t^2+t^4}{(1+t^2)^2}+\frac{4t^2}{(1+t^2)^2}}$		
	$=\sqrt{\frac{1+2t^2+t^4}{(1+t^2)^2}}$		
	$=\sqrt{\frac{(1+t^2)^2}{(1+t^2)^2}}$		
	$=\sqrt{1}$	2.	
	$=1$		

Q	Solutions	Marks	Comments
Q13	(i) $y = \lvert f(x) \rvert$ 	2	
	(ii) $y = \frac{1}{f(x)}$ 	2	
	(iii) $y^2 = f(x)$ 	2	

St Catherines's School Waverley

Course: Page no. of

Marking Scheme for Task: Academic Year: 2011-12

Q	Solutions	Marks	Comments
13a)	(iv) $y = \ln[f(x)]$ 	2	
v)	$x^3 + y - 3xy = 3$		
	$3x^2 + \frac{dy}{dx} - 3y - 3x\frac{dy}{dx} = 0$	1	
	$\frac{dy}{dx}(1-3x) = 3y - 3x^2$		
	$\frac{dy}{dx} = \frac{3y-3x^2}{1-3x}$		
	at $(1, 2)$ $\frac{dy}{dx} = \frac{3}{-2}$	1	
	$y - 2 = -\frac{3}{2}(x-1)$		
	$2y - 4 = -3x + 3$		
	$3x + 2y - 7 = 0$ is eqn of tangent.	1	

Q	Solutions	Marks	Comments
13e	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$		
	(i) $S(ae,0)$ $y = \frac{b}{a}x$ $\therefore m_2 = -\frac{a}{b}$		
	$y - 0 = -\frac{a}{b}(x - ae)$		
	$by = -ax + a^2e$	1	
	$\therefore ax + by - a^2e = 0$		
	(ii) $ax + by - a^2e = 0$ — (1)		
	$y = \frac{b}{a}x$ — (2)	1	
	Sub (2) in (1)		
	$ax + \frac{b^2x}{a} - a^2e = 0$		
	$a^2x + b^2x - a^3e = 0$		
	$x(a^2+b^2) = a^3e$		
	$x = \frac{a^3e}{a^2+b^2}$ — (3)		
	Now $b^2 = a^2(e^2-1)$		
	$b^2 = a^2e^2 - a^2$	1	
	$\therefore a^2 + b^2 = a^2e^2$		
	$\therefore$ from (3) $x = \frac{a^3e}{a^2e^2}$		
	$x = \frac{a}{e}$		
	Which is the corresponding directrix	1	

St Catherines's School Waverley

Course: Page no. of

Marking Scheme for Task: Academic Year: 2011-12

Q	Solutions	Marks	Comments
14a)(iii)	Area $\Delta ABQ = \frac{1}{2} AB.PQ$		
	Now $PQ = \sqrt{\left(cp + \frac{c}{p^3}\right)^2 + \left(\frac{c}{p} + cp^3\right)^2}$		
	$= \sqrt{c^2p^2 + \frac{2c^2}{p^2} + \frac{c^2}{p^6} + \frac{c^2}{p^2} + 2c^2p^2 + c^2p^6}$		
	$= c\sqrt{p^6 + 3p^2 + \frac{3}{p^2} + \frac{1}{p^6}}$	1	
	$= c\sqrt{\left(p^2 + \frac{1}{p^2}\right)^3}$		
	$\therefore$ Area $\Delta ABQ = \frac{1}{2} \times 2c\sqrt{\left(p^2+\frac{1}{p^2}\right)} \times c\sqrt{\left(p^2+\frac{1}{p^2}\right)^3}$	1	
	$= c^2\left(p^2 + \frac{1}{p^2}\right)^2$		
(iv)	$\frac{a}{b} + \frac{b}{a} \geq 2$ for $a>0$ $b>0$ (given)		
	$p^2 + \frac{1}{p^2} \geq 2$	1	
	$\therefore$ Minimum Area $= 4c^2$		
14b)	$P(x) = x^3 - 6x^2 + 9x + c$ double zero.		
	$P'(x) = 3x^2 - 12x + 9 = 0$		
	$x^2 - 4x + 3 = 0$		
	$(x-3)(x-1) = 0$	1	
	$\therefore$ possible double zero for $x=1$, $x=3$		
	If $x=1$ $P(x)=0 \Rightarrow c=-4$		
	$x=3$ $P(x)=0 \Rightarrow c=0$	1	

St Catherines's School Waverley

Course: Page no. of

Marking Scheme for Task: Academic Year: 2011-12

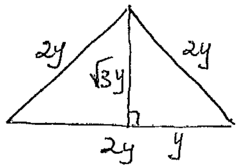
Q	Solutions	Marks	Comments
14a	i) gradient of tangent at P on $xy = c^2$		
	is. $m_1 = -\frac{1}{p^2}$		
	$\therefore$ gradient of normal is $m_2 = p^2$		
	$\therefore$ equation of normal at P:		
	$y - \frac{c}{p} = p^2(x - cp)$		
	$(\times p)$ $py - c = p^3(x - cp)$	1	
	(ii) $py - c = p^3(x - cp)$ —— ①		
	$y = \frac{c^2}{x}$ —— ②		
	Sub ② in ①	1	
	$\frac{pc^2}{x} - c = p^3(x - cp)$		
	$pc^2 - cx = p^3x^2 - cp^4x$		
	$p^3x^2 - cp^4x + cx - pc^2 = 0$		
	$(\div p^3)$ $x^2 - cpx + \frac{c}{p^3}x - \frac{c^2}{p^2} = 0$		
	$x^2 - c\left[p - \frac{1}{p^3}\right]x - \frac{c^2}{p^2} = 0$	1	
	Sum of roots $= c\left[p - \frac{1}{p^3}\right]$		
	root at P is ~~xp~~ cp		
	$\therefore$ x coordinate of Q is $x = -\frac{c}{p^3}$		
	y coordinate of Q is $y = \frac{c^2}{-\frac{c}{p^3}}$	1	
	$\therefore$ Coords of Q $\left(-\frac{c}{p^3}, -cp^3\right)$		

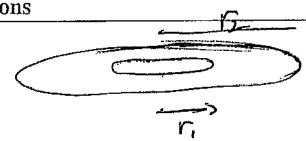
St Catherines's School Waverley

Course: Page no. of

Marking Scheme for Task: Academic Year: 2011-12

Q	Solutions	Marks	Comments
14c	i) $z = \cos\theta + i\sin\theta$		
	$z^n = \cos n\theta + i\sin n\theta$ $\quad z^{-n} = \cos(-n\theta) + i\sin(-n\theta)$		
	$\therefore \frac{1}{z^n} = \cos n\theta - i\sin n\theta$		
	$\therefore z^n + \frac{1}{z^n} = 2\cos n\theta$	1	
	ii) let $z^5 - 1 = 0$		
	$\therefore z^5 = 1$		
	$\cos 5\theta + i\sin 5\theta = 1$		
	$\therefore \cos 5\theta = 1$		
	$5\theta = 0, 2\pi, 4\pi, 6\pi, 8\pi$		
	$\theta = 0, \frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{8\pi}{5}$	1	
	$\therefore$ roots of $z^5 - 1 = 0$ are		
	$z = \text{cis}\,0, \text{cis}\frac{2\pi}{5}, \text{cis}\frac{4\pi}{5}, \text{cis}\frac{6\pi}{5}, \text{cis}\frac{8\pi}{5}$		
	$= \text{cis}\,0, \text{cis}\frac{2\pi}{5}, \text{cis}\frac{4\pi}{5}, \text{cis}\left(-\frac{4\pi}{5}\right), \text{cis}\left(-\frac{2\pi}{5}\right)$		
	$\therefore$ factors of $z^5 - 1$ are		
	$(z-1)\left(z - \text{cis}\frac{2\pi}{5}\right)\left[z - \text{cis}\left(-\frac{2\pi}{5}\right)\right]\left[z - \text{cis}\frac{4\pi}{5}\right]\left[z - \text{cis}\left(-\frac{4\pi}{5}\right)\right]$		
	ie. $(z-1)\left(z^2 - 2\cos\frac{2\pi}{5}z + 1\right)\left(z^2 - 2\cos\frac{4\pi}{5}z + 1\right)$	1	
	(iii) $z^5 - 1 = (z-1)(z^4 + z^3 + z^2 + z + 1)$	1	
	$\therefore z^4 + z^3 + z^2 + z + 1 = \left(z^2 - 2\cos\frac{2\pi}{5}z + 1\right)\left(z^2 - 2\cos\frac{4\pi}{5}z + 1\right)$		
	for $z = 1$: $5 = \left(2 - 2\cos\frac{2\pi}{5}\right)\left(2 - 2\cos\frac{4\pi}{5}\right)$		
	$\frac{5}{4} = \left(1 - \cos\frac{2\pi}{5}\right)\left(1 - \cos\frac{4\pi}{5}\right)$	1	

Q	Solutions	Marks	Comments
Q15	a) (i)		
	area of cross section: 		
	$A = y \times \sqrt{3}y$	1	
	$= \sqrt{3}y^2$		
	(ii) $\frac{x^2}{25} + \frac{y^2}{16} = 1$		
	$\therefore \frac{y^2}{16} = 1 - \frac{x^2}{25}$		
	$y^2 = 16\left(1 - \frac{x^2}{25}\right)$		
	$\therefore A = 16\sqrt{3}\left(1 - \frac{x^2}{25}\right)$	1	
	(iii) $V = \int_{-5}^{5} 16\sqrt{3}\left(1 - \frac{x^2}{25}\right) dx.$	1	
	$= 16\sqrt{3}\int_{-5}^{5}\left(1 - \frac{x^2}{25}\right) dx$		
	$= 16\sqrt{3}\left[x - \frac{x^3}{75}\right]_{-5}^{5}$		
	$= 16\sqrt{3}\left[\left(5 - \frac{125}{75}\right) - \left(-5 + \frac{125}{75}\right)\right]$		
	$= 16\sqrt{3}\left[10 - \frac{250}{75}\right]$	1	
	$= \frac{320\sqrt{3}}{3}$ units^3		

Q	Solutions	Marks	Comments
Q15	b) (i) 		
	$\therefore r_2 = 1 + \sqrt{1-y} + 1$ $\qquad y = 2x - x^2$	1	
	$r_1 = 1 - \sqrt{1-y} + 1$ $\qquad x^2 - 2x + y = 0$		
	$x = \frac{2 \pm \sqrt{4-4y}}{2}$	1	
	$A = \pi\left(r_2^2 - r_1^2\right)$ $\qquad = 1 \pm \sqrt{1-y}$		
	$= \pi\left(r_2 - r_1\right)\left(r_2 + r_1\right)$		
	$= \pi\left(2\sqrt{1-y}\right)(4)$	1	
	$= 8\pi\sqrt{1-y}$ u^2		
	(ii) $V = \int_0^1 8\pi\sqrt{1-y}\, dy$	1	
	$= 8\pi\int_0^1 \sqrt{1-y}\, dy$		
	$= 8\pi\left[\frac{2(1-y)^{3/2}}{-3}\right]_0^1$		
	$= 8\pi\left[+\frac{2}{3}\right]$		
	$= \frac{16\pi}{3}$ units^3	1	

Q	Solutions	Marks	Comments
Q15	c)(i) $a = m_1 h + c_1$		
	when $h=0$ $a=4 \Rightarrow c_1 = 4$		
	$\therefore a = m_1 h + 4$		
	when $h=10$ $a=0 \Rightarrow m_1 = -\frac{2}{5}$ $\therefore a = -\frac{2}{5}h + 4$	1	
	$b = m_2 h + c_2$		
	when $h=0$ $b=0 \Rightarrow c_2 = 0$		
	$\therefore b = m_2 h$		
	when $h=10$ $b=2 \Rightarrow m_2 = \frac{1}{5}$ $\therefore b = \frac{1}{5}h$	1	
	$x = m_3 h + c_3$		
	when $h=0$ $x=4 \Rightarrow c_3 = 4$		
	$\therefore x = m_3 h + 4$		
	when $h=10$ $x=2 \Rightarrow m_3 = -\frac{1}{5}$ $\therefore x = -\frac{1}{5}h + 4$	1	
	Now $A = \frac{-\frac{1}{5}h+4}{2}\left[-\frac{2}{5}h+4+\frac{1}{5}h\right]$		
	$= \frac{-h+20}{10}\left[4-\frac{1}{5}h\right]$		
	$= \frac{-h+20}{10}\left[\frac{20-h}{5}\right]$		
	$= \frac{20-h}{10}\cdot\frac{20-h}{5}$		
	$= \frac{400-40h+h^2}{50}$		
	$= 8 - \frac{4h}{5} + \frac{h^2}{50}$	1	

St Catherines's School Waverley

Course:

Page no. of

Marking Scheme for Task:

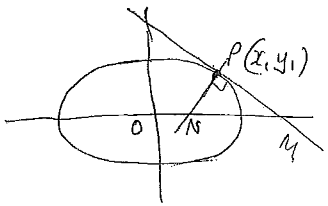
Academic Year: 2011-12

Q	Solutions	Marks	Comments
15c	ii) $V = \int_0^{10}\left(8 - \frac{4h}{5} + \frac{h^2}{50}\right)dh$	1	
	$= \left[8h - \frac{4h^2}{10} + \frac{h^3}{150}\right]_0^{10}$		
	$= \left[80 - \frac{400}{10} + \frac{1000}{150}\right]$		
	$= 80 - 40 + \frac{20}{3}$		
	$= \left[40 + \frac{20}{3}\right]$		
	$= \frac{140}{3}$ units^3	1	

St Catherines's School Waverley

Course: Page no. of

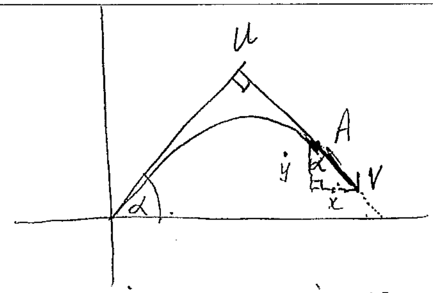
Marking Scheme for Task: Academic Year: 2011-12

Q	Solutions	Marks	Comments
Q16	a) ax^3+bx^2+cx+d. let roots be $\alpha, 2\alpha, 3\alpha$		
	Sum of roots $= \boxed{6\alpha = -\frac{b}{a}}$ — ①		
	Sum prod 2x $= 2\alpha^2+3\alpha^2+6\alpha^2 = \boxed{11\alpha^2 = \frac{c}{a}}$ — ②	1	
	product of roots $= \boxed{6\alpha^3 = -\frac{d}{a}}$ — ③		
	①×② $66\alpha^3 = -\frac{bc}{a^2}$	1	
	③×11 $66\alpha^3 = -\frac{11d}{a}$		
	$\therefore \frac{bc}{a^2} = \frac{11d}{a}$		
	$\therefore bc = 11ad$.	1	
	b) tangent at P. 		
	$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$		
	let $y=0$ $x=\frac{a^2}{x_1}$		
	$\therefore OM = \frac{a^2}{x_1}$	1	
	Normal at P: $\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2-b^2$		
	let $y=0$ $x=(a^2-b^2)\frac{x_1}{a^2}$	1	
	$\therefore ON = (a^2-b^2)\cdot\frac{x_1}{a^2}$		
	Now $OM.ON = a^2-b^2\cdot\frac{x_1}{a^2}\cdot\frac{a^2}{x_1}$	2	
	$= a^2-b^2$		
	$= a^2e^2$ (Note $b^2=a^2(1-e^2)$, $b^2=a^2-a^2e^2$)		

St Catherines's School Waverley

Course: Page no. of

Marking Scheme for Task: Academic Year: 2011-12

Q	Solutions	Marks	Comments
16c			
	(i) $\dot{x} = U\cos\alpha$ $\dot{y} = U\sin\alpha - gt$		
	At A. $\dot{y} = U\sin\alpha - gt$ $\dot{x} = U\cos\alpha$	1	
	also $\dot{x} = V\sin\alpha$		
	$\therefore V\sin\alpha = U\cos\alpha$		
	$\therefore V = U\cot\alpha$	1	
	(ii) At A $\dot{y} = U\sin\alpha - gt$ and $\dot{y} = -V\cos\alpha$ (downward)		
	$\therefore U\sin\alpha - gt = -V\cos\alpha$		
	$gt = U\sin\alpha + V\cos\alpha$		
	$t = \frac{U\sin\alpha + V\cos\alpha}{g}$	1	
	but from (i) $V = U\cot\alpha$		
	$\therefore t = \frac{U\sin\alpha + \frac{U\cos^2\alpha}{\sin\alpha}}{g}$		
	$= \frac{U\sin^2\alpha + U\cos^2\alpha}{\sin\alpha\, g}$		
	$= \frac{U}{g\sin\alpha}$.	1	

Q	Solutions	Marks	Comments
Q16	d) $u_1 = 1 \quad U_2 = -12 \quad U_n = U_{n-1} + 6U_{n-2} \quad n \geq 3$		
	Prove $U_n = -6\left[(-2)^{n-2} + 3^{n-2}\right]$ for all $n \geq 1$		
	(1) for $n=1$ $\quad U_1 = -6\left[(-2)^{-1} + (3)^{-1}\right]$ $= -6\left[-\frac{1}{2} + \frac{1}{3}\right]$ $= 1$ true	1	
	for $n=2$ $\quad U_2 = -6\left[(-2)^0 + 3^0\right]$ $= -12$ true.		
	(2) assume true for $n=k$ and $n=k-1$ $\quad 3 \leq k \leq \infty$		
	ie assume $U_k = -6\left[(-2)^{k-2} + 3^{k-2}\right]$ $\quad U_{k-1} = -6\left[(-2)^{k-3} + 3^{k-3}\right]$	1	
	(3) ATP true for $n=k+1$		
	ie. ATP. $U_{k+1} = -6\left[(-2)^{k-1} + 3^{k-1}\right]$		
	LHS $U_{k+1} = U_k + 6U_{k-1}$		
	$= -6\left[(-2)^{k-2} + 3^{k-2}\right] - 6^2\left[(-2)^{k-3} + 3^{k-3}\right]$		
	$= -3(-2)^{k-1} + 2.3^{k-1} - 3^2(-2)^{k-1} - 2^2.3^{k-1}$		
	$= -6(-2)^{k-1} - 6.3^{k-1}$	2	
	$= -6\left[(-2)^{k-1} - 3^{k-1}\right]$		
	$=$ RHS		
	(4) true for $n=k+1$ if true for $n=k$ since true for $n=1,2$ then by mathematical induction true for all $n \geq 1$.		