Student Number:____________________

St. Catherine's School
Waverley

August 2008

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Extension II Mathematics

Time allowed: *3 Hours + 5 mins Reading Time*

INSTRUCTIONS

- Write your STUDENT NUMBER on each page
- All questions are of equal value
- Marks for each part of a question are indicated
- All questions should be attempted on the separate paper provided
- All necessary working should be shown
- Start each question on a NEW page
- Approved scientific calculators and drawing templates may be used
- Standard integrals are printed at the end of the paper

QUESTION 1 (15 marks) — **Marks**

a)

(i) Show that the equation of the tangent at a point P $(4\cos\theta, 3\sin\theta)$ to the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ is $3x\cos\theta+4y\sin\theta=12$ — 3

(ii) Find the eccentricity, the coordinates of the foci and the equations of the directrices of the hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$ — 2

(iii) If the tangent to the ellipse at P (in part (i)) goes through a focus of the hyperbola (in part (ii)), show that P must lie on a the corresponding directrix of the hyperbola. — 2

(iv) Show that the gradient of the tangent at P is either 1 or -1. — 2

b) Consider the Hyperbola $x^2-y^2=16$

(i) Show that the eccentricity of the Hyperbola is $\sqrt{2}$ — 1

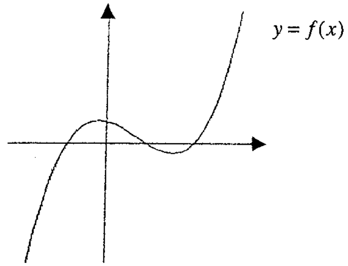
(ii) State the equation of the asymptotes — 1

(iii) This hyperbola is rotated anticlockwise through 45^0 to assume the equation $xy=c^2$, explain why $c^2=8$ — 2

(iv) Find the coordinates of the foci to $xy=c^2$ — 2

QUESTION 2 (15 marks) Start a *new* page. **Marks**

a) The graph shown is $y = f(x)$,
where $f(x) = (x-1)(x+1)(x-2)$



Sketch the graph of each of the following graphs on separate Number Planes:

(approx $\frac{1}{3}$ page each)

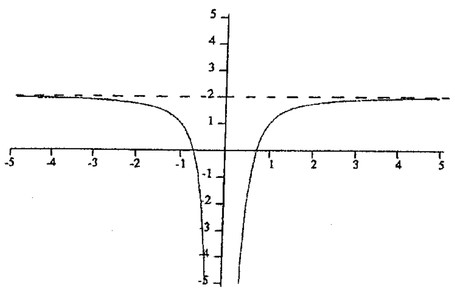
(i) $y = |f(x)|$ — 1

(ii) $y = f(|x|)$ — 2

(iii) $y^2 = f(x)$ — 2

(iv) $y = f(x-1)$ — 1

b) The graph shown is of $y = f(x)$, where $f(x) = 2 - \frac{1}{x^2}$



(i) Sketch the graph of $y = (f(x))^2$ — 2

(ii) Graph $y = x$ on the same Number Plane as $y = f(x)$ and state the values of x for which $2 - \frac{1}{x^2} > x$ — 3

c) Use the graph of $u = \cos x$ *and* $y = e^u$ to sketch the graph $y = e^{\cos x}$ clearly labelling key points. — 4

QUESTION 3 (15 marks) Start a *new* page. **Marks**

a) Integrate $\int \frac{2x+3}{x^2+2x+5}dx$ — 4

b) Integrate $\int \frac{e^x}{\sqrt{1-e^{2x}}}dx$ — 3

c) (i) Use the substitution $x = a - t$ where a is a constant, to prove that — 2

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx$$

(ii) Hence or otherwise show that $\int_0^1 x(1-x)^{99}\,dx = \frac{1}{10100}$ — 2

d) (i) Show that $(1-\sqrt{x})^{n-1}\sqrt{x} = (1-\sqrt{x})^{n-1} - (1-\sqrt{x})^n$ — 1

(ii) If $I_n = \int_0^1 (1-\sqrt{x})^n dx$, for $n \geq 0$, show that $I_n = \frac{n}{n+2} I_{n-1}$, — 3

QUESTION 4 (15 marks) Start a *new* page. **Marks**

a) Factorise $x^4 + x^2 + 1$ over the set of real numbers **1**

b) The equation $x^3 - 5x^2 + 5 = 0$ has roots $\alpha,\ \beta\ and\ \gamma$.

(i) Find the cubic equation with integer coefficients whose roots are **2**

$\alpha - 1, \beta - 1$ and $\gamma - 1$

(ii) Find the cubic equation with integer coefficients whose roots are **2**

α^2, β^2 and γ^2

(iii) Find the value of $\alpha^3 + \beta^3 + \gamma^3$ **1**

c) (i) Prove that the equation $ax^2 + bx + c = 0$ has a double root if $b^2 - 4ac = 0$ **2**

(ii) Prove that the equation $ax^3 + bx^2 + cx + d = 0$ has a triple root at $x = h$ **3**

if $\frac{b}{3a} = \frac{c}{b} = \frac{3d}{c} = -h$

d) The real number x is a solution of $x^2 - x - 1 = 0$.

Consider the series $1 + x + x^2 + x^3 + \dots\dots\dots + x^{2n-1}$

(i) Write down S, the sum of this series **1**

(ii) Use the binomial theorem to show that $S = \sum_{r=1}^{n} {}^nC_r\ x^{r+1}$ **3**

QUESTION 5 (15 marks) Start a *new* page. **Marks**

a) For a sequence of numbers $a_1 = 2$; $a_2 = 3$ and $a_n = 3\ a_{n-1} - 2\ a_{n-2}$, **4**
for all integers $n \geq 3$, prove by mathematical induction that

$a_n = 2^{n-1} + 1$ for all $n \geq 1$

b) The point P (x_1, y_1) lies on the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$

(i) Show that the equation of the tangent of the tangent at P is **3**

$\sqrt{y_1}\ x + \sqrt{x_1}\ y = \sqrt{a}\sqrt{x_1 y_1}$

(ii) The tangent meets the coordinate axes at S and T, show that $OS + OT = a$ **2**

c) (i) Show that $x > \tan^{-1} x$ *for* $x > 0$ **3**

(ii) By evaluating $\int_0^1 x\ dx$ *and* $\int_0^1 \tan^{-1}x\ dx$, show that $2 > \pi - \ln 4$ **3**

QUESTION 6 (15 marks) Start a *new* page. **Marks**

a) The region between the curve $y = e^x$, the y-axis and the line $y = e$ is rotated about the line $y = e$. Use the method of slicing to find the volume of the solid generated. **4**

b) (i) The ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ is rotated about the line $x = 4$. Use the method of cylindrical shells to show that the volume V of the solid generated is given by **3**

$$V=\frac{8\pi}{3}\int_{-3}^{3}(4-x)\sqrt{9-x^2}\,dx$$

(ii) Hence find the Volume **3**

c) If $f(x)=\cos^{-1}(\sin x)$

(i) Show $f'(x)=\pm 1$ **2**

(ii) Hence or otherwise sketch the graph of $y = f(x)$ for $-2\pi \le x \le 2\pi$ **3**

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QUESTION 7 (15 marks) Start a *new* page. **Marks**

a) The If $\frac{a}{c}=\frac{a-b}{b-c}$, then b is called the Harmonic Mean of a and c.

(i) Show that $b=\frac{2ac}{a+c}$ **1**

(ii) Prove that the reciprocals of a,b and c are in Arithmetic progression. **2**

b) (i) Show that the condition for the line y = mx+c to be a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $c^2=a^2m^2+b^2$ **3**

(ii) Hence show that the pair of tangents from the point (3,4) to the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ are at right angles to each other. **3**

c) Use De Moivre's theorem to show that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$ **1**

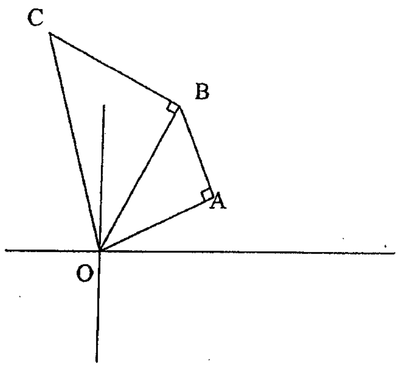
(i) Deduce that $8x^3-6x+1=0$ has solutions x= $\sin\theta$, where $\sin 3\theta=\frac{1}{2}$ **2**

(ii) Find the roots of $8x^3-6x+1=0$ in terms of $\sin\theta$ **2**

(iii) Hence evaluate $\sin\frac{\pi}{18}\sin\frac{5\pi}{18}\sin\frac{13\pi}{18}$ **1**

QUESTION 8 (15 marks) Start a *new* page. **Marks**

a)



OAB is an isosceles right angled triangle, right angled at A.

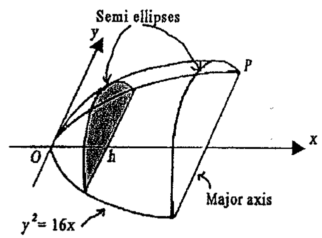
Also OBC is an isosceles right angled triangle, right angled at B.

If the point A and C represent complex numbers α and β respectively, show that

(i) $OC = 2 \times OA$ **2**

(ii) $4\alpha^2 + \beta^2 = 0$ **3**

b)



The base of a solid P is the region in the xy plane enclosed by the parabola $y^2 = 16x$ and the line $x = 6$, and each cross-section perpendicular to the x axis is a semi-ellipse with the minor axis one half of the major axis and the major axis is on the parabola.

i) Show that the area of the semi-ellipse at $x = h$ is $4\pi h$ **2**

(You may assume the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to be πab).

ii) Find the volume of the solid P. **2**

c) Let $p = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$. The complex number $\alpha = p + p^2 + p^4$ is a root of the quadratic equation $x^2 + ax + b = 0$, where a and b are real.

Noting that p is a complex root of $z^7 = 1$

(i) Prove that $1 + p + p^2 + \ldots\ldots + p^6 = 0$ **1**

(ii) The second root of the quadratic equation $x^2 + ax + b = 0$ is β. **3**

Show that $\beta = p^3 + p^5 + p^6$.

(iii) Hence find the values of the coefficients a and b **2**