St Catherine's School

Year: 12

Subject: Extension II Mathematics

Time Allowed: 3 hours

Date: August 2005

Exam number: 15227508

Directions to candidates:

- All questions are to be attempted.
- All questions are of equal value.
- All necessary **working** must be shown in every question.
- Full marks may not be awarded for careless or badly arranged work.
- Approved calculators and geometrical instruments are required.
- Start each question on a new page.
- Hand in your work in **1 bundle**:
- Attach the question paper

TEACHER'S USE ONLY Total Marks
A 15+15+15+13 = 58
B 14½+8½+14+15=5
TOTAL ~~110~~ 109.

Extension II Trials

Question 1

(a) Integrate $\int \frac{dx}{x\ln x}$ (2m)

(b) (i) Show that $z\bar{z} = |z|^2$ (1m)

(ii) hence or otherwise find and sketch the locus of z:

$(z-2)(\bar{z}-2) = 4$ (3m)

(c) Integrate $\int \frac{2x}{x^2+4x+5}dx$ (4m)

(d) Find the modulus and argument of $z = 1+i$ *and* $w = -1-\sqrt{3}i$ and hence the modulus and argument of, $\frac{z^2}{w^5}$

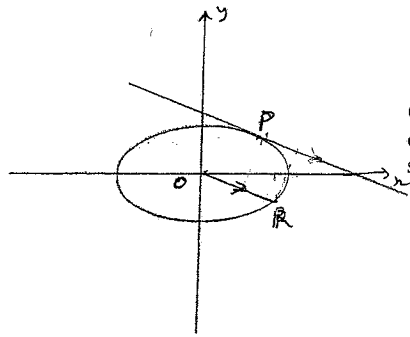
(5m)

Question 2

(a) Show that $x^3 + ax + b = 0$ has a double root if $4a^3 + 27b^2 = 0$ (4m)

(b) (1+i) is a root of the polynomial equation $z^4 - 4z^3 + 10z^2 - 12z + 8 = 0$
Find all the other roots. (4m)

(c) P $(4\cos\theta, 3\sin\theta)$ is a point on the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$.



(i) Show that the gradient of the tangent at P is $\frac{-3\cos\theta}{4\sin\theta}$ (2m)

OR is drawn parallel to the tangent from the origin O meeting the ellipse again at the point R as shown.

(ii) Find the equation of OR. (1m)

(iii) Show that the coordinates of R is given by $(4\sin\theta, -3\cos\theta)$ (2m)

(iv) Find the area of the triangle OPR. (2m)

Question 3

(a) P $(cp, \frac{c}{p})$ and Q $(cq, \frac{c}{q})$ are points on the rectangular hyperbola $xy = c^2$

(i) Show that the equation of the chord PQ is x+pqy=cp+cq (2m)

(ii) Find the locus of the mid point of the chord, given that the chord passes through the point (2, 0) (3m)

(b) Consider the ellipse $\frac{x^2}{9} + \frac{y^2}{16} = 1$

(i) Find the eccentricity (1m)

(ii) Find the length of the major axis (1m)

(iii) Find the coordinates of the foci (1m)

(iv) Find the equation of the directrices (1m)

(v) Sketch the ellipse, showing the above features. (1m)

(c) A particle of mass 5 kilograms attached by a string of length 2 m. to a fixed point describes a horizontal circle with an angular velocity of 3 radians per second.

(i) Copy the diagram and show the forces acting on the particle (1m)

(i) Find the tension in the string (3m)

(ii) Find the distance of the particle below the fixed point. (2m)

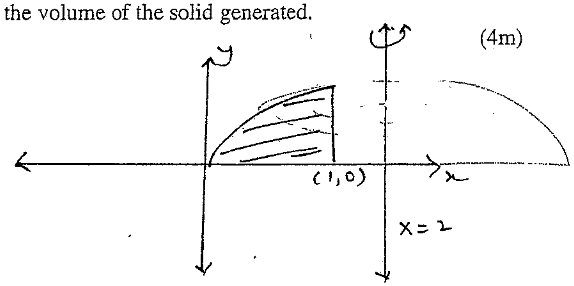
Question 4

(a)(i) A polynomial P(x) is divided by $x^2 - a^2$ and the remainder is px+q. Show that

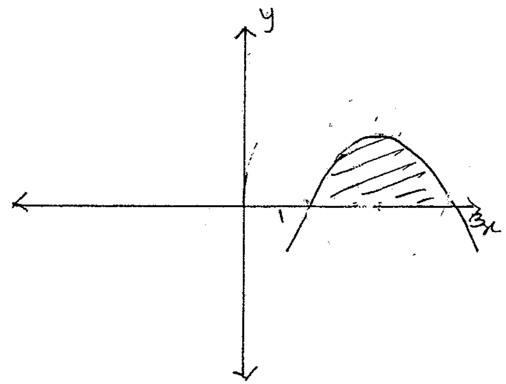
$$p = \frac{1}{2a}(P(a) - P(-a)) \ \ and \ \ q = \frac{1}{2}(P(a) + P(-a))$$ (3m)

(ii) Hence or otherwise find the remainder when the polynomial $x^n - a^n$ is divided by $x^2 - a^2$, for when n is even and when is odd. (3m)

(b) The area bounded by the curve $y = \sqrt{x}$, the x axis and the line x=1 is rotated about the the line x=2. By considering the slices perpendicular to the axis of rotation, find the volume of the solid generated. (4m)



(c) The area bounded by the parabola y = (x-1)(3-x) and the x axis is rotated about the x axis. Use the method of cylindrical shells to find the volume of the solid generated. (5m)



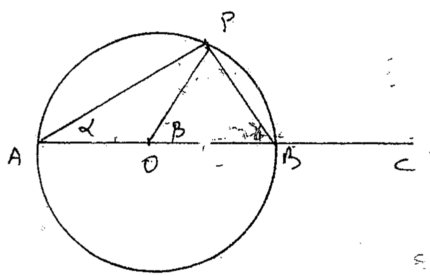
Question 5.

(a) In a bag there are three counters marked with the digit '3' and five counters marked with the digit '4'. Four counters are drawn out of the bag one at a time with replacement.

(i) Find the probability that three '3's and one '4' will be drawn (2m)

(ii) Find the probability that the sum of the digits on the counters is greater than 15. (2m)

(c) A particle P is moving in a circle of radius a, with uniform speed u. AB is a diameter of the circle and O is the centre. AB is produced to C (fig). Let angle PAC $=\alpha$, angle POC $=\beta$ and $P\hat{B}C = \gamma$. Find, in terms of u, the angular velocity of the particle P about each of the points A, O and B. (4m)



(c) Express $\dfrac{x^2+x-28}{x^2-16}$ as a sum of partial fractions. (3m)

(d) (i) Use De Moivre's theorem to show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ (1m)

(ii) Hence solve $8x^3 - 6x - 1 = 0$ (3m)

Question 6

(a) Let w be a complex root $z^3 = 1$

(i) Show that $1 + w + w^2 = 0$ (2m)

(ii) Find the possible values of w^k, where k is any positive integer. (2m)

(iii) Hence explain why the possible values of $w^k + w^{2k}$ is 2 or -1 (1m)

(iv) Write down the expansions of $(1+w)^n$ *and* $(1+w^2)^n$ (1m)

(v) Deduce that $(1+w)^n + (1+w^2)^n = 3({}^nc_0 + {}^nc_3 + {}^nc_6 + \dots\dots + {}^nc_{3l}) - 2^n$, where l is the largest integer such that $3l \le n$

(use the identity $\sum_{r=0}^{n} {}^nc_r = 2^n$) (3m)

(vi) If n is a multiple of 6, deduce that

$${}^nc_0 + {}^nc_3 + {}^nc_6 + \dots\dots + {}^nc_n = \frac{1}{3}(2^n + 2)$$ (2m)

(b) If $I_n = \int_0^1 \frac{1}{(1+x^2)^n}dx$, n = 1, 2, 3..., show that

$2n\ I_{n+1} = (2n-1)I_n + \frac{1}{2^n}$, for n=1,2,3,...

(4m)

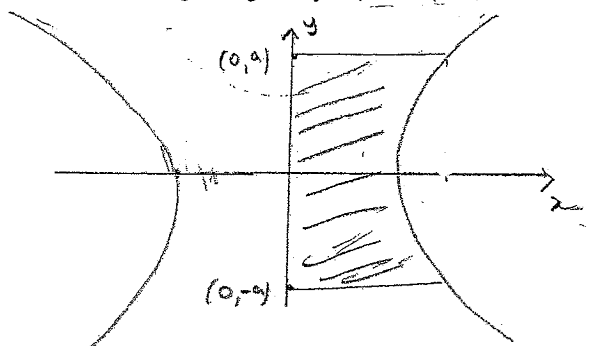
Question 7

(a) (i) Using integration by parts or otherwise show that

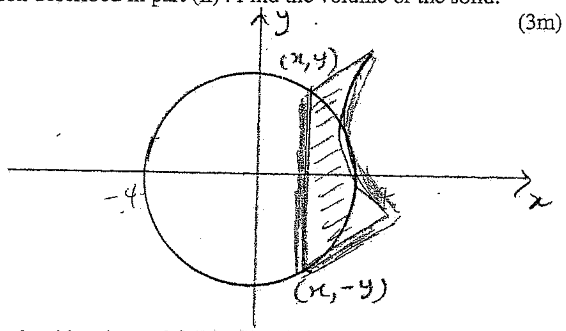
$$\int_0^{\frac{\pi}{4}} \sec^3\theta \, d\theta = \frac{1}{2}(\sqrt{2} + \ln(\sqrt{2}+1))$$

(3m)

(ii) In the given diagram the shaded region R is bounded by a branch of the Hyperbola $x^2 - y^2 = a^2$, the lines y=a and y=-a and the y axis. Show that the area of the region is given by $a^2(\sqrt{2} + \ln(\sqrt{2}+1)$ (4m)



(iii) In the diagram a solid is constructed with base the circle $x^2 + y^2 = 16$. Each cross section perpendicular to the x axis is a plane similar to the region described in part (ii). Find the volume of the solid. (3m)



(b) Assume that the tides rise and fall in Simple Harmonic Motion. At low tide a channel is 9 metres deep and at high tide, it is 12 metres deep. The low tide occurs at 9 am and the high tide at 3pm. A ship needs 11.25 metres of water depth to pass through. Find the earliest time between 9 am and 3 pm, when it is safe for the ship to pass through

(5m)

Question 8

(a) A body of mass 1 kilogram is projected upwards from the ground at 20 metres per second. The particle is under the effect of gravity and the air resistance at any time is equal to $\frac{1}{10}v^2$, where v is the velocity at the time.

(Take the acceleration due to gravity at 10).

(i) Explain why the equation of motion while going up is given by

$$\ddot{x} = -(10 + \frac{1}{10}v^2)$$

(1m)

(ii) Taking $\ddot{x} = v\frac{dv}{dx}$ find the greatest height reached (3m)

(iii) Taking $\ddot{x} = \frac{dv}{dt}$, find the time taken to reach this height. (3m)

(b) (i) If $I_{2n} = \int_0^{\frac{\pi}{2}} \cos^{2n}x \, dx$, use integration by parts to show that

$$I_{2n} = \frac{2n-1}{2n} I_{2n-2}$$ (4m)

(ii) Show that $I_4 = \frac{3\pi}{16}$ (1m)

(iii) Show that $I_{2n} = \frac{(2n)!}{(2^n n!)^2}\frac{\pi}{2}$ (3m)

End of Paper

Ext 2 Trials '05

Q.1

1. $\int \frac{dx}{x \ln x}$ if $\ln x = u$, $\frac{1}{x}dx = du$

$= \int \frac{du}{u}$

$= \ln u + c$

$= \ln(\ln x) + c$

2 $z\bar{z} = |z|^2$

Let $z = x + iy$

$\bar{z} = x - iy$

$z\bar{z} = (x+iy)(x-iy)$

$= x^2+y^2$

$|z|^2 = (\sqrt{x^2+y^2})^2$

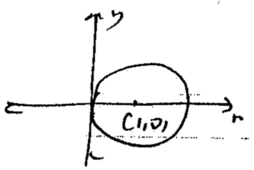
$= x^2+y^2$

$(z-2)(\bar{z}-2) = 4$

$z\bar{z} - 2(z+\bar{z}) + 4 = 4$

$x^2+y^2-2x = 0$

$(x-1)^2+y^2 = 1$



circle centre $(1,0)$

$r = 1$

c) $\int \frac{2x}{x^2+4x+5}\,dx$

$= \int \frac{2x+4-4}{x^2+4x+5}\,dx$

$= \ln|x^2+4x+5| - \int \frac{4}{(x+2)^2+1}\,dx$

$= \ln(x^2+4x+5) - 4\tan^{-1}\frac{x+2}{1} + c$

d) $z = 1+i$

$|z| = \sqrt{2}$; $\operatorname{Arg} z = \frac{\pi}{4}$

$w = -1-\sqrt{3}i$

$|w| = 2$ $\operatorname{Arg} w = -\frac{2\pi}{3}$

$\left|\frac{z^2}{w^5}\right| = \frac{(\sqrt{2})^2}{2^5} = \frac{1}{2^4}$

$\operatorname{Arg}\frac{z^2}{w^5} = 2\operatorname{Arg} z - 5\operatorname{Arg} w \pm 2n\pi$

$= \frac{\pi}{2} + \frac{10\pi}{3}$ $(\pm 2n\pi)$

$= -\frac{\pi}{6}$.

Q.2

$x^3+ax+b=0$

if α is the double root

α is also a root of

$3x^2+a=0$.

Thus

$\alpha^3+a\alpha+b=0$ — (1)

$3\alpha^2+a=0$. — (2)

from (1)

$\alpha(\alpha^2+a) = -b$.

$\alpha^2(\alpha^2+a)^2 = b^2$

$-\frac{a}{3}\left(-\frac{a}{3}+a\right)^2 = b^2$ from (2).

$\frac{-a}{3}\times\frac{4a^2}{9} = b^2$

$-4a^3 = 27b^2$

or $4a^3+27b^2=0$.

(2) $z^4-4z^3+10z^2-12z+8=0$

- roots have real coefficients.

∴ if $(1+i)$ is a root, $(1-i)$ is also a root.

let z and w be the other roots

(Note: if they are complex roots, they are also complex conjugates)

$(z-(1+i))(z-(1-i))$ is a factor.

z^2-2z+2 is a factor.

$$
\begin{array}{r}
z^2 - 2z + 4 \\
z^2-2z+2 \overline{\big)\, z^4-4z^3+10z^2-12z+8} \\
z^4-2z^3+2z^2 \\
\hline
-2z^3+8z^2-12z+8 \\
-2z^3+4z^2-4z \\
\hline
4z^2-8z+8
\end{array}
$$

Thus $(z^2 - 2z + 2)(z^2 - 2z + 4) = 0$

$\therefore \quad z = \underline{1 + i} \qquad z = \dfrac{2 \pm 2\sqrt{3}i}{2}$

$\underline{1 - i}$

ie $\underline{z = 1 + \sqrt{3}i}$

$\underline{z = 1 - \sqrt{3}i}$.

c) $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$

$\dfrac{2x}{16} + \dfrac{2y}{9} y' = 0$

$y' = -\dfrac{x}{16} \div \dfrac{y}{9}$

$= \dfrac{-9x}{16y}$

$y'_{at\,P} = -\dfrac{9}{16} \cdot \dfrac{4\cos\theta}{3\sin\theta}$

$= \dfrac{-3\cos\theta}{4\sin\theta}$.

(ii) Equ. of OR: $y = -\dfrac{3\cos\theta}{4\sin\theta} x$

(iii) $\dfrac{x^2}{16} + \dfrac{\cancel{9}\cos^2\theta}{16\sin^2\theta} \cdot \dfrac{x^2}{\cancel{9}} = 1$.

$x^2\sin^2\theta + x^2\cos^2\theta = 16\sin^2\theta$.

$x = \pm 4\sin\theta$

where R is $x = 4\sin\theta$.

$\therefore \quad y = -\dfrac{3\cos\theta}{\cancel{4\sin\theta}} \cdot \cancel{4\sin\theta} = -3\cos\theta$

Thus $R: (4\sin\theta, -3\cos\theta)$

(iv) $|OR| = \sqrt{16\sin^2\theta + 9\cos^2\theta}$

$\underline{h}$ dist. from $P: (4\cos\theta, 3\sin\theta)$ on

$y = -\dfrac{3\cos\theta}{4\sin\theta} x$

or $3\cos\theta\, x + 4\sin\theta\, y = 0$ is

$d = \left| \dfrac{12\cos^2\theta + 12\sin^2\theta}{\sqrt{16\sin^2\theta + 9\cos^2\theta}} \right| = \dfrac{12}{\sqrt{9\cos^2\theta + 16\sin^2\theta}}$

$\therefore$ Area is $\dfrac{1}{2} \times |OR| \times d$

$= \dfrac{1}{2} \times \cancel{\sqrt{\ }} \times \dfrac{12}{\cancel{\sqrt{\ }}}$

$= 6 \text{ units}^2$.

Q.3 $P: \left(cp, \frac{c}{p}\right)$; $Q\left(cq, \frac{c}{q}\right)$

grad. of PQ: $\dfrac{\frac{c}{q} - \frac{c}{p}}{cq - cp} = \dfrac{p-q}{pq} \times \dfrac{1}{q-p}$.

$= -\dfrac{1}{pq}$.

Equ. of PQ:

$y - \frac{c}{p} = -\frac{1}{pq}(x - cp)$

$pqy - cq = -x + cp$.

$x + pqy = c(p+q)$.

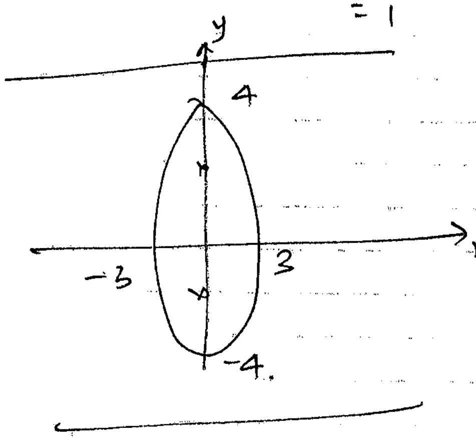
passes through $(2,0)$ $\therefore$ $2 = cp + cq$. — ①

mid. pt. of PQ:

$x = \dfrac{c(p+q)}{2}$ $\quad y = \frac{1}{2}\left(\frac{c}{p} + \frac{c}{q}\right)$.

$= \frac{2}{2}$ from ①

$= 1$ is the locus.



(b). $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$

$9 = 16(1 - e^2)$

$1 - e^2 = \frac{9}{16}$

$e^2 = \frac{7}{16}$ $\quad e = \frac{\sqrt{7}}{4}$ $(e > 0)$

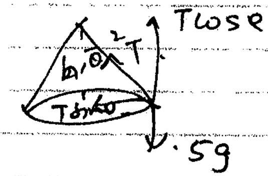
foci: $(0, \pm ae)$

$: (0, \pm\sqrt{7})$ —.

$|$major axis$|$ = 8.

directrices: $y = \pm 4 \times \frac{4}{\sqrt{7}}$

$y = \pm \frac{16}{\sqrt{7}}$.

c) 

$\omega = 3$ rad/s.

$T\cos\theta = 5g$ — ①

$T\sin\theta = 5 \times 2\sin\theta \times 9$

$T = 90$ N.

sub in ①

$90\cos\theta = 5g$

$\cos\theta = \frac{5g}{90}$. also, $\cos\theta = \frac{h}{2}$.

$\therefore \frac{h}{2} = \frac{5g}{90}$

$h = \frac{10g}{90} = \frac{g}{9}$ m.

Q.4 $P(x) = (x^2 - a^2).Q(x) + px + q$.

$P(a) = ap + q$ — ①

$P(-a) = -ap + q$ — ②

$\dfrac{P(a) + P(-a)}{2} = q$. & $\dfrac{P(a) - P(-a)}{2a} = p$.

The rem: $P(x): x^n - a^n$.

$P(a) = a^n - a^n$

$= 0$

$p(-a) = (-a)^n - a^n$

$= (-1)^n a^n - a^n.$

$= a^n - a^n$ when n is even.

$= 0$

$(-1)^n a^n - a^n,$

$= -a^n - a^n$ when n is odd.

$= -2a^n$

$\therefore$ The rem. is $px + q$, where

when n is even.

$p = \frac{1}{2a}\left(P(a) - P(-a)\right)$

$= 0.$

$q = 0$ ... The rem is zero

when is odd.

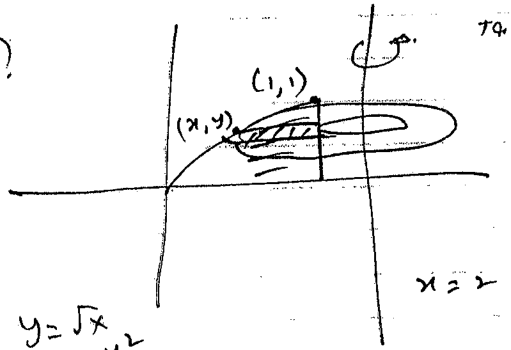
$p = \frac{1}{2a}\left(0 - -2a^n\right)$

$= a^{n-1}.$

$q = \frac{1}{2}\left(-2a^n\right) = -a^n.$

$\therefore$ The rem is $a^{n-1}x - a^n.$

b)



take a slice at (x,y)

$\Delta V = \pi\left((2-x)^2 - 1^2\right)\Delta y$

$= \pi\left(4 - 4x + x^2 - 1\right)$

$= \pi\left(3 - 4x + x^2\right)\Delta y$

$= \pi\left(3 - 4y^2 + y^4\right)\Delta y.$

$\therefore V = \pi\int_0^1 \left(3 - 4y^2 + y^4\right) dy$

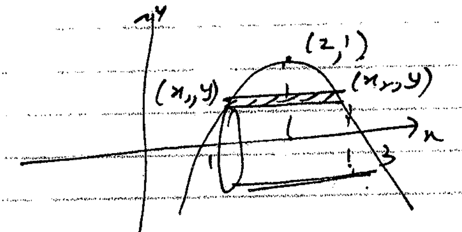
$= \pi\left(3y - \frac{4}{3}y^3 + \frac{y^5}{5}\right)_0^1$

$= \pi\left(3 - \frac{4}{3} + \frac{1}{5}\right) u^3.$

$= \pi\left(\frac{45 - 20 + 3}{15}\right) u^3.$

$= \frac{28\pi u^3}{15}.$

c)



consider a cyl. shell at a dist. y from the axis of rotation.

$\Delta V = 2\pi\, y\, (x_2 - x_1)\, \Delta y$

((x_1,y) & (x_2,y) are as shown).

$y = (x-1)(3-x)$

$= -x^2 + 4x - 3.$

x_1, x_2 are roots of

$x^2 - 4x + 3 + y = 0$

$\therefore x_1 + x_2 = 4$

$x_1 x_2 = 3 + y.$

$(x_1 - x_2)^2 = 4^2 - 4(3+y)$

$= 16 - 12 - 4y$

$= 4 - 4y$

$\therefore x_1 - x_2 = \sqrt{4 - 4y}$

$x_1 - x_2 = 2\sqrt{1-y}$

$\therefore \Delta V = 2\pi y\left(2\sqrt{1-y}\right)$

$= 4\pi\, y\sqrt{1-y}\ \Delta y$

$\therefore V = 4\pi\int_0^1 y\sqrt{1-y}\, dy$

Let $1 - y = u.$

$-dy = du.$

$y = 0;\ u = 1$

$y = 1;\ u = 0$

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$$\therefore V = 4\pi \int_1^0 (1-u)\sqrt{u}\,(-du)$$

$$= 4\pi \int_0^1 (\sqrt{u} - u^{3/2})\,du.$$

$$= 4\pi \left[\tfrac{2}{3} u^{\frac{3}{2}} - \tfrac{2}{5} u^{\frac{5}{2}} \right]_0^1$$

$$= 4\pi \left(\tfrac{2}{3} - \tfrac{2}{5} \right)$$

$$= 8\pi \times \tfrac{2}{15} = \tfrac{16\pi}{15} \; u^3.$$

Q.5

$P(3) = \frac{3}{8} = p$

$P(4) = \frac{5}{8} = q.$

4. Counters)

Consider $(p+q)^4 = {}^4C_0\, p^4 + {}^4C_1\, p^3 q + {}^4C_2\, p^2 q^2 + {}^4C_3\, p q^3 + {}^4C_4\, q^4.$

P(3 3's & one 4)

$$= {}^4C_1 \cdot \left(\tfrac{3}{8}\right)^3 \left(\tfrac{5}{8}\right)$$

$$= 4 \times \frac{27 \times 5}{8^4}.$$

(ii). For sum > 15, we need to draw 4 4's.

$P(4\ 4\text{'s}) = {}^4C_0 \left(\tfrac{5}{8}\right)^4.$

$= \left(\tfrac{5}{8}\right)^4.$

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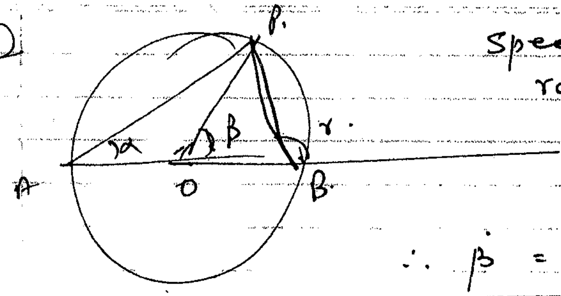
(c)



Speed : u.

radius : a.

$u = a\omega$

$\therefore \omega = \frac{u}{a}.$

$\therefore \dot{\beta} = \frac{u}{a}.$

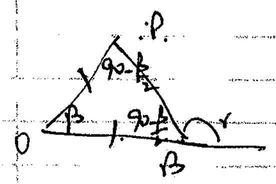
β is an ext. angle of the isos ΔOAP

$\therefore \beta = 2\alpha$

$\therefore \alpha = \frac{\beta}{2}$

$\dot{\alpha} = \frac{1}{2}\dot{\beta} = \frac{1}{2}\frac{u}{a}.$

γ is an ext. $\angle$ of the isos. ΔOPB.



$\gamma = 90 + \frac{\beta}{2}.$

$\therefore \dot{\gamma} = \frac{1}{2}\dot{\beta} = \frac{1}{2}\frac{u}{a}.$

(c)

$$\begin{array}{r|l} & 1 \\ x^2-16) & x^2 + x - 28 \\ & x^2 \quad\;\; - 16 \\ \hline & x - 12 \end{array}$$

$$\therefore \frac{x^2+x-28}{x^2-16} = 1 + \frac{x-12}{x^2-16}.$$

Consider $\frac{x-12}{(x-4)(x+4)} = \frac{A}{x-4} + \frac{B}{x+4}$

$\therefore x - 12 \equiv A(x+4) + B(x-4)$

Let $x = 4$. $\quad -8 = 8A \quad \underline{A = -1}.$

Let $x=-4$; $-16 = -8B$
$B = 2$

$$\therefore \frac{x^2+x-28}{x^2-16} = 1 - \frac{1}{x-4} + \frac{2}{x+4}$$

1). Consider,
$(\cos\theta + i\sin\theta)^3$
$= \cos 3\theta + i\sin 3\theta$ by DeMoivre Thm.

also $(\cos\theta + i\sin\theta)^3$

$= {}^3C_0 \cos^3\theta + 3\cos^2\theta(i\sin\theta) + 3\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3$

equating real parts
$\therefore \cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$
$= \cos^3\theta - 3\cos\theta(1-\cos^2\theta)$
$= 4\cos^3\theta - 3\cos\theta$

if $\cos 3\theta = \frac{1}{2}$; $4\cos^3\theta - 3\cos\theta = \frac{1}{2}$
or $8\cos^3\theta - 6\cos\theta - 1 = 0$

hence if α, β, γ are the roots of the above eqn.
$\cos\alpha$, $\cos\beta$, & $\cos\gamma$ would be the solution to
$8x^3 - 6x - 1 = 0$

α, β, γ are the roots to $\cos 3\theta = \frac{1}{2}$
$3\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}$.

$\therefore$ The roots of $8x^3 - 6x - 1 = 0$ are

$\cos\frac{\pi}{9}$, $\cos\frac{5\pi}{9}$ & $\cos\frac{7\pi}{9}$.

Q.6

$z^3 = 1$.
$z^3 - 1 = 0$
$(z-1)(z^2+z+1) = 0$
$z = 1$; $z = \frac{-1 \pm \sqrt{3}i}{2}$

if $w = \frac{-1+\sqrt{3}i}{2}$, $w^2 = \left(\frac{-1+\sqrt{3}i}{2}\right)^2$
$= \frac{-1-\sqrt{3}i}{2}$.

Thus $1, w, w^2$ are the roots $z^3 - 1 = 0$
$\therefore 1 + w + w^2 = 0$.

if $k = 3m$; $w^k = w^{3m} = (w^3)^m = 1^m = 1$

if $k = 3m+1$; $w^k = w^{3m+1} = w^{3m}.w = w$

if $k = 3m+2$; $w^k = w^{3m+2} = w^{3m}.w^2 = w^2$.

$\therefore$ The possible values are $1, w, w^2$

iii) $w^k + w^{2k} = 1+1$
$= 2$ if $k = 3m$

$w^k + w^{2k} = w + w^2$
$= -1$ if $k = 3m+1$

$w^k + w^{2k} = w^2 + w^4$
$= w^2 + w$
$= -1$ if $k = 3m+2$.

v) $(1+\omega)^n = {}^nC_0 + {}^nC_1\,\omega + {}^nC_2\,\omega^2 + \dots + {}^nC_n\,\omega^n.$

$(1+\omega^2)^n = {}^nC_0 + {}^nC_1\,\omega^2 + {}^nC_2\,\omega^4 + \dots + {}^nC_n\,\omega^{2n}.$

vi) $(1+\omega)^n + (1+\omega^2)^n$

$= 2\,{}^nC_0 + {}^nC_1(\omega+\omega^2) + {}^nC_2(\omega^2+\omega^4) + {}^nC_3(\omega^3+\omega^6)$

$+ \dots + {}^nC_n(\omega^n+\omega^{2n}),$

$= 2\,{}^nC_0 - {}^nC_1 - {}^nC_2 + 2\,{}^nC_3 - {}^nC_4 - {}^nC_5 + 2\,{}^nC_6 \dots$

$= 2({}^nC_0 + {}^nC_3 + {}^nC_6 + \dots + {}^nC_{3k}) - ({}^nC_1 + {}^nC_2 + {}^nC_4 + \dots)$

$= 3({}^nC_0 + {}^nC_3 + {}^nC_6 + \dots + {}^nC_{3k}) - ({}^nC_0 + {}^nC_1 + {}^nC_2 + \dots)$

$= 3({}^nC_0 + {}^nC_3 + {}^nC_6 + \dots + {}^nC_{3k}) - \sum_{r=0}^{n} {}^nC_r$

$= 3({}^nC_0 + {}^nC_3 + \dots + {}^nC_{3k}) - 2^n.$

vii) if n is a multiple of 6;

$(1+\omega)^n = (-\omega^2)^{64} = \omega^{124} = (\omega^3)^{44} = 1^{44} = 1$

$(1+\omega^2)^n = (-\omega)^{64} = \omega^{64} = (\omega^3)^{24} = 1^{24} = 1.$

$\therefore (1+\omega)^n + (1+\omega^2)^n = 2.$

Thus $2 = 3({}^nC_0 + {}^nC_3 + \dots + {}^nC_n) - 2^n$

$3({}^nC_0 + {}^nC_3 + \dots + {}^nC_n) = 2 + 2^n.$

or ${}^nC_0 + {}^nC_3 + \dots + {}^nC_n = \frac{1}{3}(2^n + 2)$

b) $I_n = \int_0^1 \frac{1}{(1+x^2)^n}\,dx.$

Let $u = (1+x^2)^{-n}$ $\quad v' = 1$

$v = x$

$u' = -n(1+x^2)^{-n-1}(2x)$

$= -\frac{2nx}{(1+x^2)^{n+1}}$

$\int u v' = uv - \int u'v.$

$\therefore I_n = \left(\frac{x}{(1+x^2)^n}\right)_0^1 + 2n\int_0^1 \frac{x^2}{(1+x^2)^{n+1}}\,dx$

$= \frac{1}{2^n} + 2n\left[\int_0^1 \frac{x^2+1-1}{(x^2+1)^{n+1}}\,dx\right]$

$I_n = \frac{1}{2^n} + 2n\left[\int_0^1 \frac{1}{(x^2+1)^n}\,dx - \int \frac{1}{(x^2+1)^{n+1}}\,dx\right]$

$\therefore I_n = +2n\left[I_n - I_{n+1}\right] + \frac{1}{2^n}.$

$(1-2n)\,I_n$

$(1+2n)\,I_n = \cancel{\#}\,2n\,I_{n+1} + \frac{1}{2^n}. \longrightarrow$ or $2n\,I_{n+1} = 2n$

$\therefore I_n = \frac{2n}{2n+1}\,I_{n+1} + \frac{1}{2n+1}\,\frac{1}{2^n}.$

Q7.

a) $\int_0^{\frac{\pi}{4}} \sec^3\theta \, d\theta$

Let $u = \sec\theta$ $\qquad v' = \sec^2\theta$

$u' = \sec\theta\tan\theta$ $\qquad v = \tan\theta$

$$\int uv' = uv - \int u'v$$

$$\int_0^{\frac{\pi}{4}} \sec^3\theta \, d\theta = \left(\sec\theta\tan\theta\right)_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \sec\theta \underbrace{\tan^2\theta}_{(\sec^2\theta - 1)} d\theta$$

$$= \sqrt{2} - \left(\int_0^{\frac{\pi}{4}} \sec^3\theta - \int_0^{\pi/4} \sec\theta \, d\theta\right)$$

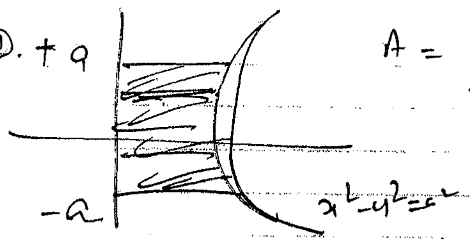
$$I = \sqrt{2} - I + \left(\ln|\sec\theta + \tan\theta|\right)_0^{\frac{\pi}{4}}$$

$$2I = \sqrt{2} + \left(\ln(\sqrt{2}+1) - \ln(1+0)\right)$$

$$= \sqrt{2} + \ln(\sqrt{2}+1)$$

$$I = \frac{1}{2}\left(\sqrt{2} + \ln|\sqrt{2}+1|\right).$$

(ii)



$$A = \int_{-a}^{a} x \, dy.$$

$$x^2 - y^2 = a^2.$$

$$x^2 = a^2 + y^2$$

$$x = \sqrt{a^2+y^2}.$$

$$A = \int_{-a}^{a} \sqrt{a^2+y^2} \, dy$$

$$= 2\int_0^{a} \sqrt{a^2+y^2} \, dy \quad (\sqrt{a^2+y^2} \text{ is an even fn.})$$

Let $y = a\tan\theta$

$dy = a\sec^2\theta \, d\theta$

$a^2 + y^2 = a^2 + a^2\tan^2\theta$

$= a^2\sec^2\theta.$

$y = 0; \quad \theta = 0$

$y = a; \quad \theta = \frac{\pi}{4}$

$$\therefore A = 2\int_0^{\frac{\pi}{4}} a\sec\theta \cdot a\sec^2\theta \, d\theta$$

$$= 2a^2 \int_0^{\frac{\pi}{4}} \sec^3\theta \, d\theta.$$

$$= 2a^2 \cdot \left[\frac{1}{2}\left(\sqrt{2} + \ln(\sqrt{2}+1)\right)\right]$$

$$= a^2\left[\sqrt{2} + \ln(\sqrt{2}+1)\right].$$

(iii) $\Delta V = y^2 \cdot \left(\sqrt{2} + \ln(\sqrt{2}+1)\right) \Delta x.$

$$V = \left(\sqrt{2} + \ln(\sqrt{2}+1)\right) \int_{-4}^{4} y^2 \, dx$$

$$V = \sqrt{2} + \ln(\sqrt{2}+1) \int_{-4}^{4} (16 - x^2)\, dx$$

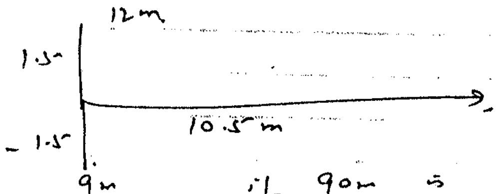
$$= 2\left(\sqrt{2} + \ln(\sqrt{2}+1)\right) \int_0^4 (16 - x^2)\, dx$$

$$= 2\left(\sqrt{2} + \ln(\sqrt{2}+1)\right) \left(16x - \frac{x^3}{3}\right)_0^4$$

$$= 2\left(\sqrt{2} + \ln(\sqrt{2}+1)\right) \left(64 - \frac{64}{3}\right)$$

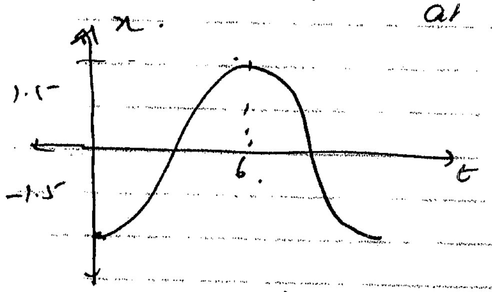
$$= \frac{256}{3} \left(\sqrt{2} + \ln(\sqrt{2}+1)\right) u^3 .$$

b).



if 9am is 0 hrs.
3 pm : 6 hrs :

Origin: (9 am, 10.5m) ; place the axes at this origin.



The motion can be described by the equation.

$$x = -1.5 \cos nt .$$

where $\frac{2\pi}{n}$ is the period

$$\therefore \frac{2\pi}{n} = 12$$

$$2\pi = 12n$$

$$n = \frac{2\pi}{12} = \frac{\pi}{6}$$

$$\therefore x = -1.5 \cos \frac{\pi}{6} t .$$

At 11.25 m depth ; $x = 11.25 - 10.5$
$= 0.75$.

$$\therefore 0.75 = -1.5 \cos nt .$$

$$-\frac{1}{2} = \cos \frac{\pi}{6} t$$

$$\therefore \frac{\pi}{6} t = \frac{2}{3}\pi .$$

$$t = 4 .$$

$\therefore$ 4 hrs from 9am ie at 1pm is the earliest time when the ship can pass through.

Q.8

Note: Resistance opposes motion.

↓mg ↓R. if going up is considered positive;

$$F = ma$$

$$-mg - R = ma$$

$$mg - \frac{1}{10} v^2 = ma$$

m = 1kg

$\therefore a = -g - \frac{1}{10}v^2.$

$= -(10 + \frac{1}{10}v^2)$

$v\frac{dv}{dx} = -\frac{100+v^2}{10}.$

$\int \frac{10v\,dv}{100+v^2} = -\int dx$

$\therefore \frac{1}{2}\ln(100+v^2) = -\frac{1}{10}x + c$

at $x=0;\ v=20.$

$\therefore \frac{1}{2}\ln 500 = c.$

$\therefore \frac{1}{10}x = \frac{1}{2}\ln 500 - \frac{1}{2}\ln(100+v^2)$

$x = 5\ln 500 - 5\ln(100+v^2)$

$= 5\ln\frac{500}{100+v^2}.$

At the greatest ht; H; $v=0$

$\therefore H = 5\ln 5.$

(a) $\frac{dv}{dt} = -\frac{100+v^2}{10}$

$\int \frac{dv}{100+v^2} = -\frac{1}{10}\int dt$

$\frac{1}{10}\tan^{-1}\frac{v}{10} = -\frac{1}{10}t + c$

at $t=0;\ v=20$

$\therefore \frac{1}{10}\tan^{-1}2 = \cancel{\tfrac{1}{10}}\,c$

$\therefore \frac{1}{10}t = \frac{1}{10}\tan^{-1}2 - \frac{1}{10}\tan^{-1}\frac{v}{10}$

$t = \tan^{-1}2 - \tan^{-1}\frac{v}{10}$

at the greatest ht; $v=0$

$\therefore t = \tan^{-1}2 - \tan^{-1}0$

$= \tan^{-1}2$

(b) $I_{2n} = \int_0^{\frac{\pi}{2}} \cos^{2n}x\,dx$

Let $u = \cos^{2n-1}x$ $\qquad v' = \cos x$

$\qquad v = \sin x$

$u' = (2n-1)\cos^{2n-2}x\,(-\sin x)$

$\int uv' = uv - \int u'v$

I_{2n}

$= \left(\sin x\cos^{2n-1}x\right)_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}}(2n-1)\sin^2 x\,.\cos^{2n-2}x\,dx$

$= (2n-1)\int_0^{\frac{\pi}{2}}\cos^{2n-2}x\,(1-\cos^2 x)\,dx.$

$$I_{2n} = (2n-1)\left(I_{2n-2} - I_{2n}\right)$$

$$(1+2n-1)\, I_{2n} = (2n-1)\, I_{2n-2}$$

$$2n\, I_{2n} = (2n-1)\, I_{2n-2}$$

$$I_{2n} = \frac{2n-1}{2n}\, I_{2n-2}.$$

$$I_4 = \frac{3}{4}\, I_2$$

$$= \frac{3}{4}\cdot\frac{1}{2}\cdot I_0$$

$$= \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2}$$

$$= \frac{3}{16}\pi.$$

$$I_0 = \int_0^{\pi/2} (\cos x)^0\, dx$$

$$= \int_0^{\pi/2} 1\, dx$$

$$= \frac{\pi}{2}$$

(iii) $$I_{2n} = \frac{2n-1}{2n}\, I_{2n-2}$$

$$= \frac{2n-1}{2n}\cdot\frac{2n-3}{2n-2}\cdot I_{2n-4}$$

$$= \left(\frac{2n-1}{2n}\right)\frac{(2n-3)}{2n-2}\cdot\frac{2n-5}{2n-4}\cdot\frac{2n-7}{2n-6}\cdots I_0.$$

$$= \frac{(2n-1)}{2n}\cdot\frac{(2n-3)}{2(n-1)}\cdot\frac{(2n-5)}{2(n-2)}\cdot\frac{(2n-7)}{2(n-4)}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

$$= \frac{(2n-1)\cdot(2n-3)\cdots 3\cdot 1}{2^n\; n!}\cdot\frac{2n(2n-2)(2n-4)\cdots\cancel{2}}{2n(2n-2)(2n-4)\cdots 2}\cdot\frac{\pi}{2}$$

$$= \frac{2n\,(2n-1)\,(2n-2)\cdots 3\cdot 2\cdot 1}{2^n\cdot n!\quad 2^n\cdot n!}\cdot\frac{\pi}{2}$$

$$= \frac{(2n)!}{(2^n\cdot n!)^2}\cdot\frac{\pi}{2}.$$