St George Girls High School

Trial Higher School Certificate Examination

2012

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen.
- Write your student number on each booklet.
- Board-approved calculators may be used.
- A table of standard integrals is provided at the back of this paper.
- The mark allocated for each question is listed at the side of the question.
- Marks may be deducted for careless or poorly presented work.

Total Marks – 100

Section I – Pages 2 – 4
10 marks

- Attempt Questions 1 – 10.
- Allow about 15 minutes for this section.
- Answer on the sheet provided.

Section II – Pages 5 – 13
90 marks

- Attempt Questions 11 – 16
- Allow about 2 hours 45 minutes for this section.
- Begin each question in a new booklet.
- Show all necessary working in Questions 11 – 16.
- Templates for Q12(a) to be detached and placed in [redacted] answer booklet.

Q12

Students are advised that this is a Trial Examination only and does not necessarily reflect the content or format of the Higher School Certificate Examination.

Section I – (10 marks) **Marks**

Answer this section on the answer sheet provided at the back of this paper.
Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

1. The maximum value of y reached by the ellipse with equation

$$\frac{3(x+3)^2}{5} + \frac{(y-4)^2}{6} = 3$$

is:

A. $-4 + 3\sqrt{2}$

B. $4 + \sqrt{5}$

C. $3\sqrt{2}$

D. $4 + 3\sqrt{2}$

2. The graph of $f(x) = \frac{1}{x^2+mx-n}$, where m and n are real constants, has no vertical asymptotes if

A. $m^2 < 4n$

B. $m^2 > 4n$

C. $m^2 = -4n$

D. $m^2 < -4n$

3. The number of real solutions to $x^4 - x^3 = \text{cosec}^2(x) - \cot^2(x)$ is:

A. 0

B. 1

C. 2

D. 3

4. If $z = \frac{3+4i}{1+2i}$, the imaginary part of z is:

A. -2 B. $-\frac{2}{5}i$ C. $-\frac{2}{5}$ D. $-2i$

Section I (cont'd) **Marks**

5. If $I = \int_0^{\ln 2} \frac{e^x}{e^x + e^{-x}}\, dx$ and $J = \int_0^{\ln 2} \frac{e^{-x}}{e^x + e^{-x}}\, dx$, then the exact value of $I - J$ is:

A. $\ln\left(\frac{5}{2}\right)$ B. $\ln 2$ C. $\ln(5)$ D. $\ln\left(\frac{5}{4}\right)$

6. If $z = \sqrt{3} + i$ then in modulus/argument form $z = 2\text{cis}\frac{\pi}{6}$. If $z^n + (\bar{z})^n$ is to be rational, then the integer $'n'$ can not be:

A. 2
B. 3
C. 5
D. 6

7. Given hyperbola $\mathcal{H}$ with equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ has eccentricity e then the ellipse E with equation $\frac{x^2}{a^2+b^2} + \frac{y^2}{b^2} = 1$ has eccentricity.

A. $-e$ B. $\frac{1}{e}$ C. $\sqrt{e}$ D. e^2

8. What restrictions must be placed on p if α, β, γ are the three, non-zero real roots of the equation $x^3 + px - 1 = 0$?

A. $p > 0$, p is real
B. $p < 0$, p is real
C. $p \geq 0$, p is real
D. $p \leq 0$, p is real

Section I (cont'd) **Marks**

9. Given that $\frac{dy}{dx} = y^2 + 1$, and that $y = 1$ at $x = 0$, then

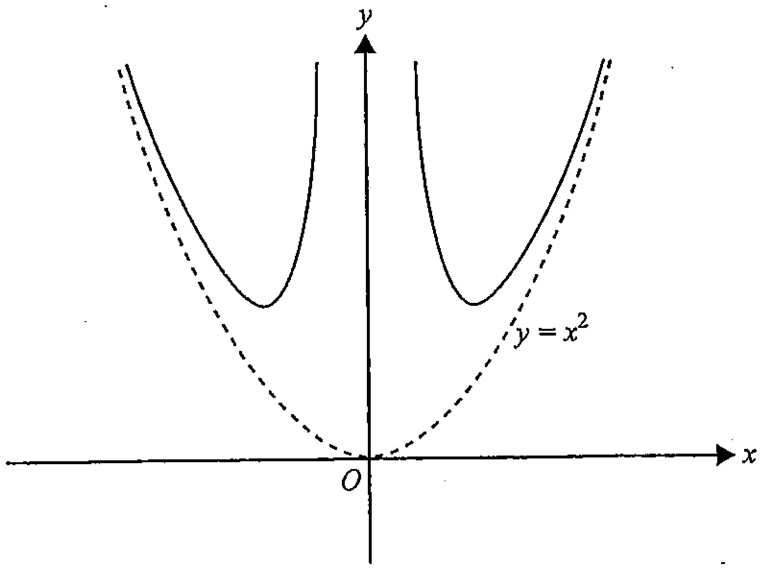
A. $y = \tan\left(x - \frac{\pi}{4}\right)$

B. $y = \tan\left(x + \frac{\pi}{4}\right)$

C. $x = \log_e \left(\frac{y^2+1}{2}\right)$

D. $y = \frac{1}{3}y^3 + y - \frac{1}{3}$

10.



A possible equation for the graph of the curve shown above is

A. $y = \frac{x^3+a}{x}, \quad a > 0$

B. $y = \frac{x^3+a}{x}, \quad a < 0$

C. $y = \frac{2x^4+a}{x^2}, \quad a > 0$

D. $y = \frac{x^4+a}{x^2}, \quad a < 0$

Section II – Show all working

Question 11 – Start A New Booklet – (15 marks)

Marks

a) Find $\displaystyle\int \frac{dx}{\sqrt{3-4x-4x^2}}$ 2

b) Evaluate $\displaystyle\int_0^{\frac{\pi}{6}} \frac{d\theta}{9-8\cos^2\theta}$ using the substitution $t = \tan\theta$ 3

c) Find $\displaystyle\int \frac{dx}{(x+1)(x^2+4)}$ 3

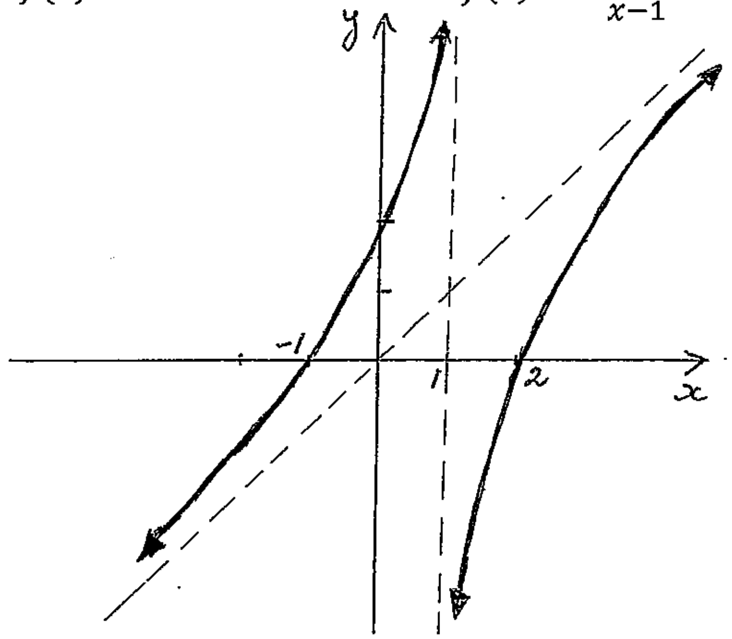
d) Evaluate $\displaystyle\int_0^1 \tan^{-1}x \; dx$ 2

e) If $I_n = \displaystyle\int_0^{\frac{\pi}{2}} \sin^n x \, . dx$ show that $I_n = \frac{n-1}{n} \; . \; I_{n-2}$ 3

Hence evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \sin^5 x \, . dx$ 2

Question 12 – Start A New Booklet – (15 marks) **Marks**

a) The sketch of $y = f(x)$ is shown below where $f(x) = \frac{x^2-x-2}{x-1}$



(i) Show that $y = x$ is an asymptote. 2

(ii) Sketch each of the following on the template provided.

(α) $y = |f(x)|$ 2

(β) $y = f(1-x)$ 2

(γ) $y^2 = f(x)$ 2

b) Consider the curve C: $x^2 + xy + y^2 = 9$

(i) Find $\frac{dy}{dx}$ 1

(ii) Find all stationary points and points where $\frac{dy}{dx}$ is not defined. 4

(iii) Sketch C clearly showing the above features and intercepts on the x, y axes. 2

Question 13 – Start A New Booklet – (15 marks)

Marks

a) If $z = (1+i)^{-1}$.

(i) Express $\bar{z}$ in modulus-argument form. **2**

(ii) If $(\bar{z})^9 = a + ib$ where a and b are real numbers, find the values of a and b **2**

b) Sketch each of the following on separate Argand diagrams.

(i) $|z - 2 + 3i| = |z + 2 - 3i|$ **2**

(ii) $\arg(z + 3 - i) = \frac{3\pi}{4}$ **2**

c) (i) On an Argand diagram sketch $\left|z - \sqrt{2} - \sqrt{z}\, i\right| = 1$ **2**

(ii) Find the minimum values of $|z|$ and $\arg z$ **3**

d)

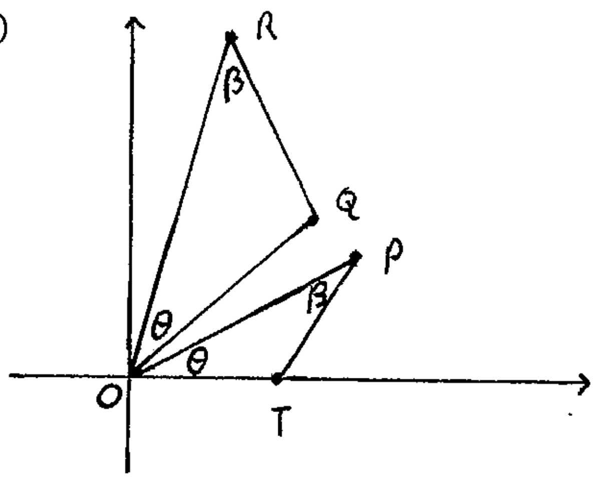


Fig I

The points T, P and Q in the complex plane correspond to the complex numbers 1, $\sqrt{3} + i$ and $2 + 2i$ respectively. **2**

Triangles OTP and OQR are similar with corresponding angles as shown in Fig I. Find the complex number represented by R (in modulus argument form).

Question 14 – Start A New Booklet – (15 marks)

Marks

a) The polynomial equation $x^3 - 6x^2 + 3x - 2 = 0$ has roots α, β, γ. 2

Evaluate $\alpha^3 + \beta^3 + \gamma^3$

b) Prove that if a polynomial $P(x)$ has a zero of multiplicity 'm' then the derived polynomial $P'(x)$ has that same zero with multiplicity '$m - 1$' 1

c) Given that $-2 - i$ is a zero of $P(x) = x^4 + 6x^3 + 14x^2 + 14x + 5$, find all zeros of $P(x)$ 3

d) (i) Prove that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ by use of de Moivre's theorem. 2

(ii) Find the general solution of $\cos 3\theta = \frac{1}{2}$ 1

(iii) Solve for $x : 8x^3 - 6x - 1 = 0$ 3

(iv) Find a polynomial of least degree which has zeros

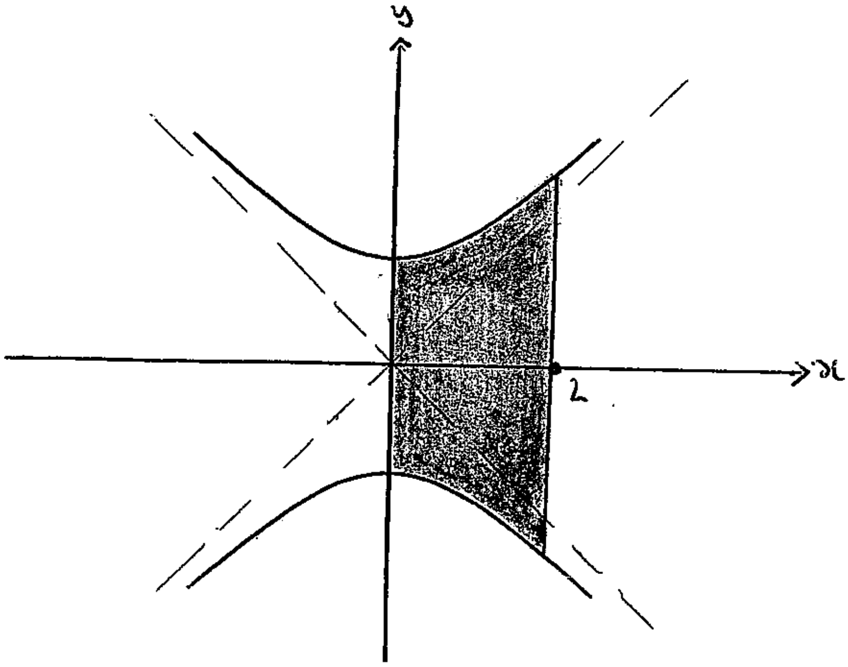
$\sec^2\frac{\pi}{9}, \sec^2\frac{5\pi}{9}, \sec^2\frac{7\pi}{9}$ 2

(v) Hence evaluate $\sec^2\frac{\pi}{9} + \sec^2\frac{5\pi}{9} + \sec^2\frac{7\pi}{9}$ 1

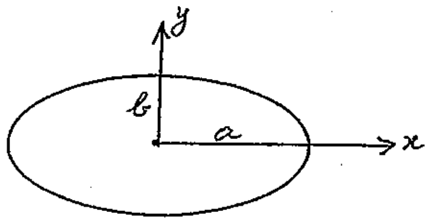
Question 15 – Start A New Booklet – (15 marks)

Marks

a) Using the method of cylindrical shells, find the volume generated by revolving the area bounded by the lines $\begin{cases} x = 2 \\ x = 0 \end{cases}$ and the two branches of the hyperbola $\frac{y^2}{9} - \frac{x^2}{4} = 1$ about the y-axis (as shown in the diagram) 3



b) (i)

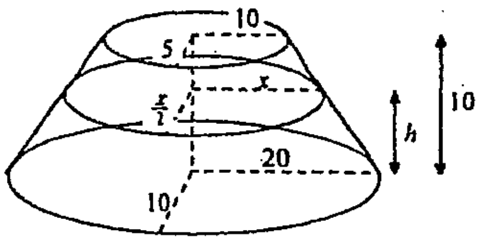


The ellipse shown has equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Prove that the area enclosed by this ellipse is πab 3

Question 15 (cont'd) **Marks**

b) (ii)



A solid of height 10 m stands on horizontal ground.

- The base of the solid is an ellipse with semi-axes of 20 m and 10 m.
- The top of the solid is an ellipse with semi-axes of 10 m and 5 m.

Horizontal cross-sections taken parallel to the base and at height h metres above the base are ellipses with semi-axes x metres and $\frac{x}{2}$ metres.

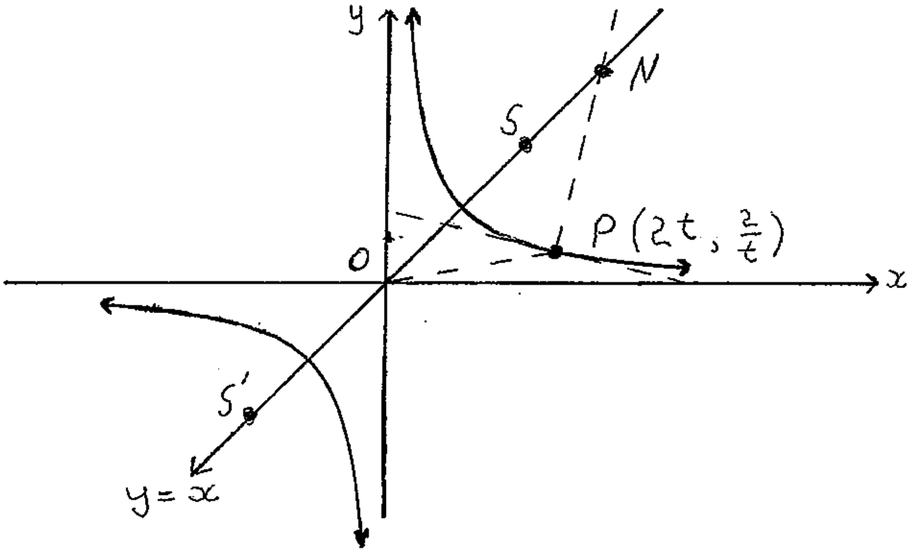
The centres of these elliptical cross-sections and the base lie on a vertical straight line, and the extremities of their semi-axes lie on sloping straight lines as shown in the diagram.

(α) Prove that $x = 20 - h$ **2**

(β) Find the volume of the solid correct to the nearest cubic metre. **3**

Question 15 (cont'd) Marks

c) The diagram shows the hyperbola $xy = 4$



(i) What are the coordinates of the foci S and S' ? 1

(ii) The point $P(2t, \frac{2}{t})$ lies on the curve, where $t \neq 0$. The normal at P intersects the straight line $y = x$ at N. O is the origin.

Given the equation of the normal at P is $y = t^2x + \frac{2}{t} - 8$

(α) Find the coordinates of N 1

(β) Show that the triangle OPN is isosceles 2

Question 16 – Start A New Booklet – (15 marks) **Marks**

a) A parachutist of mass M is initially located travelling downward in a straight line with a speed of v_0. [let $x = 0$ at $t = 0$]

If the resistance on the parachute is proportional to the speed and the gravitational force is g.

(i) Show that the speed, v, can be given as

$$v = \frac{g}{k} - \left(\frac{g}{k} - v_0\right) e^{-kt}$$ 3

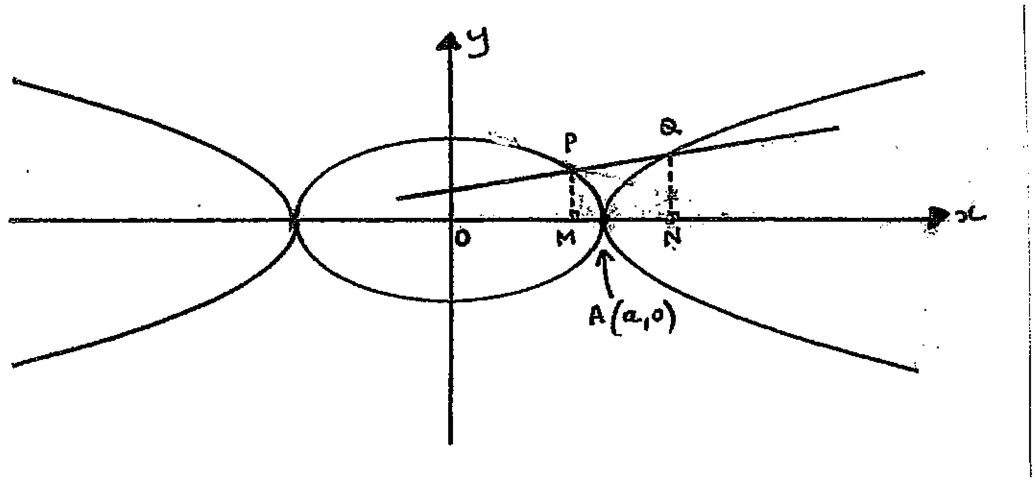
(k) is constant of proportionality.

(ii) Find the parachutist's "terminal" velocity. 1

Questions 16 b) continued on next page

Question 16 (cont'd)

b) $P(a\cos\theta,\ b\sin\theta)$ and $Q(a\sec\theta,\ b\tan\theta)$ lie on the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ and the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$, respectively as shown.



M and N are the feet of the perpendicular from P and Q respectively to the x-axis. $0<\theta<\frac{\pi}{2}$, and QP meets the x-axis at K. A is the point $(a,0)$.

(i) Given $\Delta KPM \| \Delta KQN$, show that $\dfrac{KM}{KN}=\cos\theta$ 1

(ii) Hence, show that K has coordinates $(-a,0)$ 2

(iii) Show that the tangent to the ellipse at P has equation $\dfrac{x\cos\theta}{a}+\dfrac{y\sin\theta}{b}=1$, and deduce it passes through N 3

(iv) Given that the tangent to the hyperbola at Q has equation $\dfrac{x\sec\theta}{a}-\dfrac{y\tan\theta}{b}=1$, show that the tangent passes through M. 2

If T is the point of intersection of PN and QM, show that AT is perpendicula to the x-axis.

c) Using mathematical induction prove that 3

$$\sum_{r=1}^{n} r^3 < n^2(n+1)^2$$

End of Paper