SYDNEY SECONDARY COLLEGE
BLACKWATTLE BAY CAMPUS

2011
TRIAL EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 120

- Attempt Questions 1 – 8
- Questions are of equal value

Total marks – 120
Attempt Questions 1-8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Show that $\frac{1}{1+e^x} = \frac{e^{-x}}{1+e^{-x}}$ **1**

(ii) Hence, or otherwise, find $\int \frac{1}{1+e^x}\,dx$ **2**

(b) Using the substitution $u = \cos x$, or otherwise, **3**

evaluate $\int_0^{\frac{\pi}{2}} cos^4x\, sin^3x\, dx$

(c) By completing the square, find $\int \frac{1}{9x^2+6x+5}\,dx$ **3**

(d) (i) Express $\frac{4}{x^2(x+4)}$ in the form $\frac{Ax+B}{x^2} + \frac{C}{x+4}$ **2**

(ii) Hence find $\int \frac{4}{x^2(x\ +4)}\,dx$ **2**

(e) Find $\int sin^{-1}x\ dx$ **2**

Question 2 (15 marks) Use a SEPARATE writing booklet.

(a) Let $z = 3 - 4i$ and $w = 4 - 3i$. Find in the form $a + bi$ where a and b are real,

(i) $z + iw$ **1**

(ii) $z^2 + w^2$ **1**

(b) Find real numbers x, y such that

$3(x + 2i - 4yi) = (x - yi)(2 + 5i) + 17 + 26i$ **2**

(c) Express $4 - 4i$ in modulus-argument form. **2**

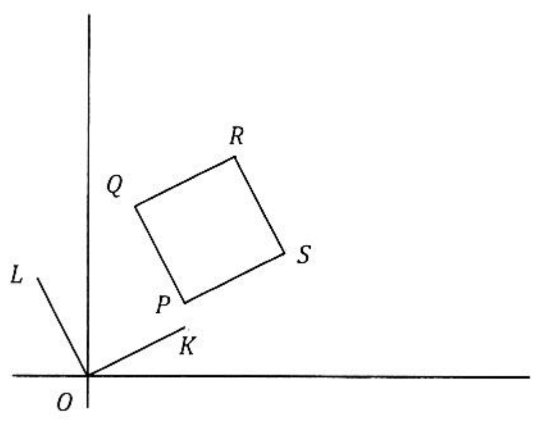
(d) Solve $z^2 + (1 + i)z + 2i = 0$ **2**

(e) If $z = rcis\theta$ simplify $arg(iz^3)$ **2**

(f) If the complex number $w = \frac{z-1}{z-2i}$ is real, **2**

find the locus of z given that $z = x + yi$.

(g)



In the Argand diagram, $PQRS$ is a square, with OK and OL parallel and equal to PS and PQ respectively.
The vertices P and S correspond to the complex numbers z_1 and z_2 respectively.

(i) Explain why the point K corresponds to $z_2 - z_1$ **1**

(ii) What complex number corresponds to the point L ? **1**

(iii) What complex number corresponds to the vertex Q ? **1**

Question 3 (15 marks) Use a SEPARATE writing booklet.

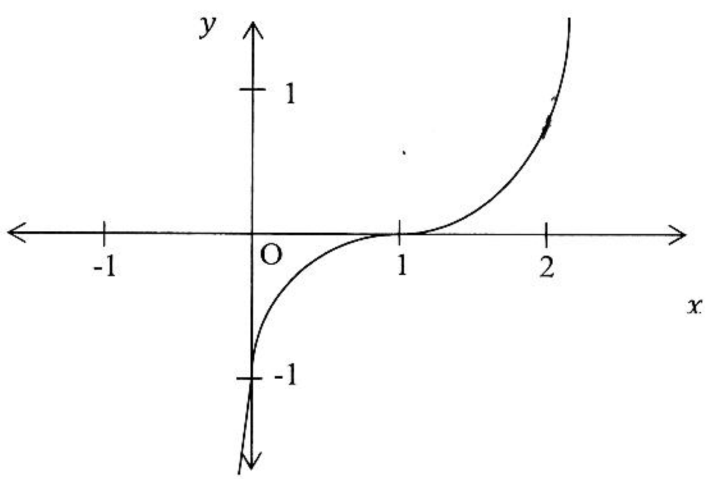
(a) Consider the function $f(x) = 1 - x^{\frac{1}{3}}$

(i) Use the first derivative to determine whether the curve is increasing or decreasing. **2**

(ii) For what values of x is the curve concave up ? **2**

(iii) Sketch the curve showing the main features. **1**

(b) The diagram shows the graph of $f(x) = (x-1)^3$



Draw separate $\frac{1}{3}$ page sketches of :

(i) $y = |f(x)|$ **1**

(ii) $y = \sqrt{f(x)}$ **2**

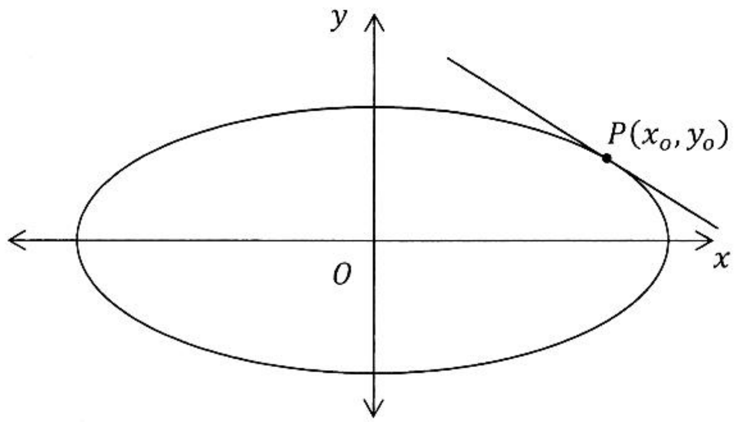
(iii) $y = \dfrac{1}{f(x)}$ **2**

(iv) $y = \dfrac{1}{f(x)} + x$ **2**

(c) Find the equation of the tangent to the curve $x^2y + 2x - 2xy = 0$ at the point $(1, 2)$. **3**

Question 4 (15 marks) Use a SEPARATE writing booklet.

(a) The diagram shows the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ with the tangent at $P(x_o, y_o)$.



(i) Show that the slope of the tangent to the curve at P is $-\dfrac{x_o b^2}{y_o a^2}$. **2**

(ii) Find the equation of the normal at P. **2**

(iii) Find the locus of P given that the normal passes through the centre of the ellipse. **2**

(iv) Give a geometric description of the locus. **1**

(b) (i) Show that the equation of the tangent to the hyperbola $xy = 1$ at the point $P\left(t, \frac{1}{t}\right)$ is given by $x + t^2 y = 2t$. **2**

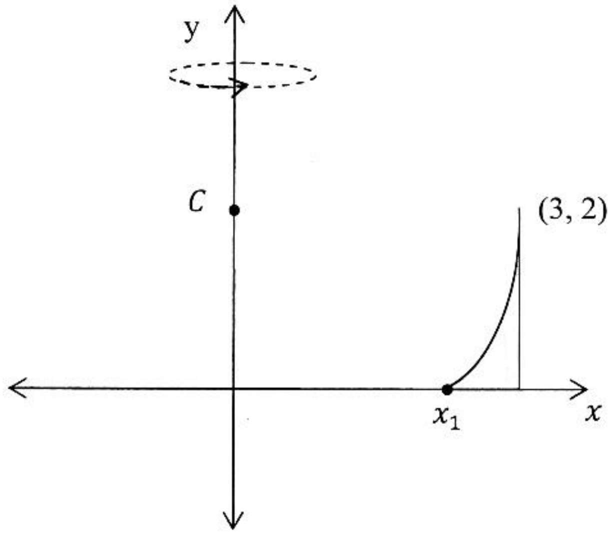
(ii) A line is drawn through the origin, perpendicular to this tangent and meets it at T. Show that the coordinates of the point T are **3**

$$x = \frac{2t}{1+t^4} \quad \text{and} \quad y = \frac{2t^3}{1+t^4}$$

(iii) As P moves on the hyperbola, show that the locus of T is given by $4xy = (x^2 + y^2)^2$. **3**

Question 5 (15 marks) Use a SEPARATE writing booklet.

(a) In the diagram below, the shaded region is bounded by the x axis, the line $x = 3$ and the circle with centre $C(0,2)$ and radius 3.

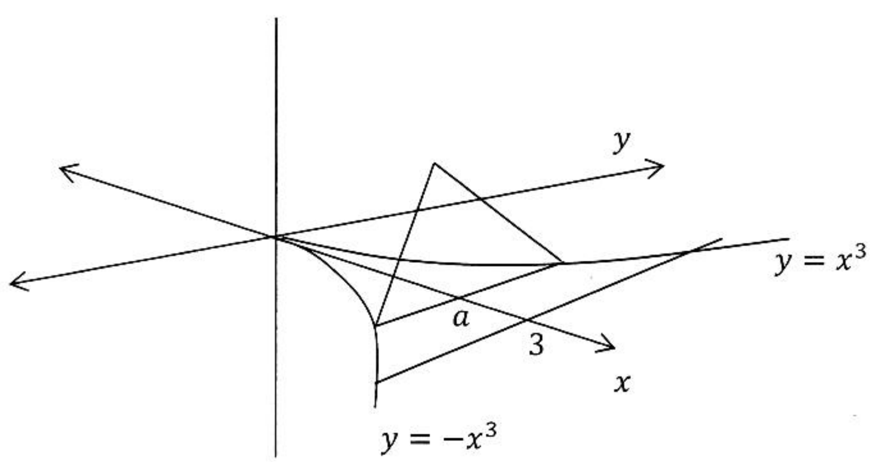


(i) Write down the equation of the circle with centre $(0,2)$ and radius 3 units. **1**

(ii) Find the value of x_1 , that is, the point of intersection of the circle with the positive x axis. **1**

(iii) Using the method of cylindrical shells find the volume of the solid formed when the region is rotated about the y axis. **2**

(b) The base of a solid is the region in the xy plane enclosed by the curves $y = x^3$, $y = -x^3$ and the line $x = 3$. Each cross-section perpendicular to the x axis is an equilateral triangle.



(i) Show that the area of the triangular cross-section at $x = a$, is $a^6\sqrt{3}$ **2**

(ii) Hence find the volume of the solid. **2**

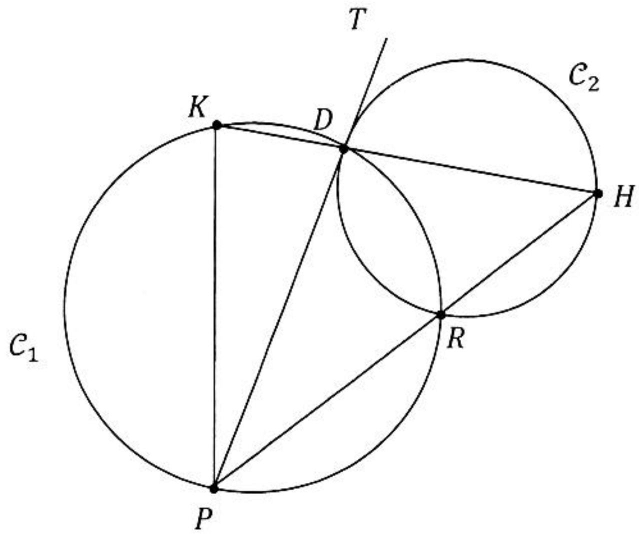
Question 5 continues on page 6

Question 5 (continued)

(c) Two circles C_1 and C_2 intersect at D and R as shown in the diagram.

The tangent TD to C_2 at D meets C_1 at P. The line PR meets C_2 at H.

The line HD meets C_1 at K.



(i) Copy or trace the diagram . **1**

(ii) Prove that ΔDPK is isosceles. **3**

(d) Solve $\sqrt{7x-3} - \sqrt{2x+1} = 2$ **3**

End of Question 5

Question 6 (15 marks) Use a SEPARATE writing booklet.

(a) A particle with a mass of one unit moves in a straight line against a resistance which is equal to $v + v^2$, where v is the velocity of the particle. Initially the particle is at the origin with velocity v_1 where $v_1 > 0$.

(i) Show that the displacement of the particle is given by **2**

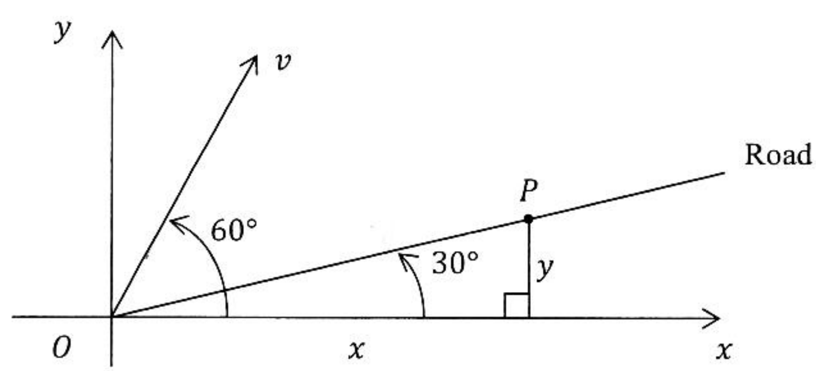
$$x = -log_e(1 + v) + c$$

(ii) Show that $t = log_e \left(\frac{v_1(1+v)}{v(1+v_1)}\right)$ **3**

(iii) Find v as a function of t. **3**

(iv) Find the limiting value of v as $t \to \infty$ **1**

(b) A particle is projected up towards an inclined road. The angle of projection is $60°$ and the road makes an angle of $30°$ with the horizontal. The speed of projection is $40\ m/s$. Let P be the point of landing of the particle and assume that $x = vt\cos\theta$, $y = -\frac{1}{2}gt^2 + vt\sin\theta$ and $g = 10\ m/s^2$.



(i) Show that $y = \dfrac{x}{\sqrt{3}}$ and $t = \dfrac{x}{20}$ **2**

(ii) Find the distance OP **4**

Question 7 (15 marks) Use a SEPARATE writing booklet.

(a) Find the real roots of $x^3 - 2x^2 - 15x + 36 = 0$ given that this equation has a double root. **3**

(b) If $(x - 5)$ is a factor of $P(x) = x^3 - ax^2 - 2ax - 5$, and a is an integer

(i) Find a **1**

(ii) Factorise $P(x)$ over the complex numbers. **2**

(c) The roots of the polynomial $P(x) = 2x^3 - 4x^2 - 3x - 1$ are α, β and γ. Find the values of :

(i) $(\alpha - 1)(\beta - 1)(\gamma - 1)$ **2**

(ii) $(\beta + \gamma - \alpha)(\gamma + \alpha - \beta)(\alpha + \beta - \gamma)$ **2**

(d) A sequence is defined to be a function, say $f(n)$, whose domain is the set of positive integers n $(n = 1, 2, 3, ...)$.

Consider the sequence $1, 1, 1, ...$ where $f(1) = 1$, $f(2) = 1$ and $f(3) = 1$.

Let $f(4) = \alpha$ and $f(n) = an^3 + bn^2 + cn + d$ where α, a, b, c, d are constants

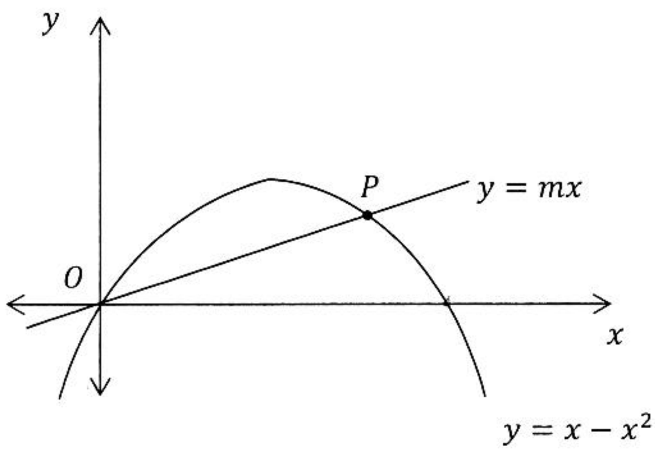
(i) By solving simultaneously, find expressions for a, b, c and d in terms of α. **4**

(ii) Express $f(n)$ in terms of α and n and hence show that $f(4)$, that is, the next number in the sequence, is α , where α is any number. **1**

Question 8 (15 marks) Use a SEPARATE writing booklet.

(a) The area bounded by the curve $y = x - x^2$ and the x axis, from $x = 0$ to $x = 1$ is to be divided equally by the line $y = mx$.

The line $y = mx$ and the curve $y = x - x^2$ intersect at $(0, 0)$ and $P(x, y)$.



(i) Find the coordinates of P in terms of m. **2**

(ii) Calculate the value of m in surd form. **4**

(b) Consider the graph of $y = (c + 1)x^2 + cx + c - 2$

(i) For what values of c is this graph a parabola ? **1**

(ii) Find the values of c such that the whole parabola is below the x axis. **3**

(c) (i) Solve $|x - 1| + |x + 2| > 3$ **3**

(ii) Solve $\frac{x}{x^2-1} \geq 0$ **2**

End of paper

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \, \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$