New South Wales Higher School Certificate Mathematics Extension 2 Examination 2003 - Solutions*

By Derek Buchanan

1. (a) Let $u = 1 + e^x \therefore du = e^x\,dx, x = 0 \Rightarrow u = 2, x = 1 \Rightarrow u = 1 + e$
$\therefore \int_0^1 \frac{e^x\,dx}{(1+e^x)^2} = \int_2^{1+e} u^{-2}\,du = [-u^{-1}]_2^{1+e} = -\frac{1}{1+e} + \frac{1}{2} = \frac{e-1}{2(e+1)}$

(b) $\int x^3 \ln x\,dx = \int \ln x \frac{d}{dx}\frac{x^4}{4}\,dx = \frac{x^4}{4}\ln x - \int \frac{x^4}{4}\frac{d}{dx}\ln x\,dx = \frac{x^4}{4}\ln x - \int \frac{x^4}{4}\frac{1}{x}\,dx = \frac{x^4}{4}\ln x - \int \frac{x^3}{4}\,dx$
$= \frac{x^4}{4}\ln x - \frac{x^4}{16} + C$

(c) Letting $u = x - 1$ so $du = dx$, $\int \frac{dx}{\sqrt{x^2-2x+5}} = \int \frac{dx}{\sqrt{(x-1)^2+4}} = \int \frac{du}{\sqrt{u^2+4}} = \ln(u + \sqrt{u^2+4}) + C$
$= \ln(x - 1 + \sqrt{x^2 - 2x + 5}) + C$

(d) (i) $\frac{5x^2-3x+13}{(x-1)(x^2+4)} \equiv \frac{a}{x-1} + \frac{bx-1}{x^2+4} \Rightarrow a(x^2+4) + (x-1)(bx-1) = (a+b)x^2 - (b+1)x + 4a + 1$
$\equiv 5x^2 - 3x + 13 \Rightarrow b + 1 = 3 \therefore b = 2, a + b = 5 \therefore a = 3.$

(ii) $\int \frac{(5x^2-3x+13)\,dx}{(x-1)(x^2+4)} = \int (\frac{3}{x-1} + \frac{2x-1}{x^2+4})\,dx = \int (\frac{3}{x-1} + \frac{2x}{x^2+4} - \frac{1}{x^2+4})\,dx$
$= 3\ln(x-1) + \ln(x^2+4) - \frac{1}{2}\tan^{-1}\frac{x}{2} + C$

(e) Let $x = 3\sin\theta$ so $dx = 3\cos\theta\,d\theta$, $\theta = \sin^{-1}\frac{x}{3}$, $(9 - x^2)^{\frac{3}{2}} = (9(1 - \sin^2\theta))^{\frac{3}{2}} = 27\cos^3\theta$,
$x = 0 \Rightarrow \theta = 0$, $x = \frac{3}{\sqrt{2}} \Rightarrow \theta = \sin^{-1}\frac{3/\sqrt{2}}{3} = \sin^{-1}\frac{1}{\sqrt{2}} = \frac{\pi}{4}$. $\therefore \frac{dx}{(9-x^2)^{\frac{3}{2}}} = \frac{3\cos\theta\,d\theta}{27\cos^3\theta}$
$= \frac{1}{9}\frac{d\theta}{\cos^2\theta} = \frac{1}{9}\sec^2\theta\,d\theta$ & $\therefore \int_0^{3/\sqrt{2}} \frac{dx}{(9-x^2)^{\frac{3}{2}}} = \frac{1}{9}\int_0^{\frac{\pi}{4}} \sec^2\theta\,d\theta = \frac{1}{9}[\tan\theta]_0^{\frac{\pi}{4}} = \frac{1}{9}(1-0) = \frac{1}{9}$

2. (a) (i) $z\overline{w} = (2+i)\overline{(1-i)} = (2+i)(1+i) = 2 - 1 + i + 2i = 1 + 3i.$

(ii) $\frac{4}{z} = \frac{4}{2+i} \cdot \frac{2-i}{2-i} = \frac{8-4i}{4+1} = \frac{8}{5} - \frac{4}{5}i.$

(b) (i) $\alpha = \sqrt{1^2 + 1^2}\text{cis}\frac{3\pi}{4} = \sqrt{2}\text{cis}\frac{3\pi}{4}.$

(ii) $(-1+i)^4 + 4 = (\sqrt{2}\text{cis}\frac{3\pi}{4})^4 + 4 = 4\text{cis}3\pi + 4 = -4 + 4 = 0 \therefore \alpha$ is a root of $z^4 + 4 = 0$.

(iii) Coefficients of $z^4 + 4$ are real $\therefore \overline{\alpha} = -1 - i$ is a root of $z^4 + 4 = 0$ & $\therefore (z - \alpha)(z - \overline{\alpha}) = (z^2 - 2\Re(\alpha)z + |\alpha|^2)\big| z^4 + 4$. $\therefore$ a real quadratic factor of $z^4 + 4$ is $z^2 + 2z + 2$.

* The question paper is available at
http://www.boardofstudies.nsw.edu.au/hsc_exams/hsc2003exams/pdf_doc/mathematics_ext2_03.pdf
and there are extensive free resources available for the course at
http://www4.tpgi.com.au/nanahcub/me2.html
such as over 100 trial papers, comprehensive notes and assignments.

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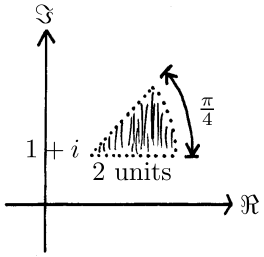
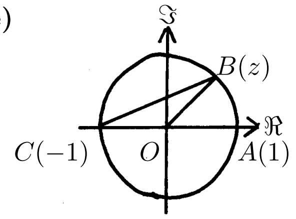
(c)



(d) $(cis\theta)^5 = \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta \Rightarrow$

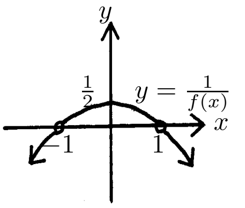
$$\begin{aligned}\cos 5\theta &= \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos(1-\cos^2\theta)^2\\ &= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta(1 - 2\cos^2\theta + \cos^4\theta)\\ &= 11\cos^5\theta - 10\cos^3\theta + 5\cos\theta - 10\cos^3\theta + 5\cos^5\theta\\ &= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta\end{aligned}$$

(e)

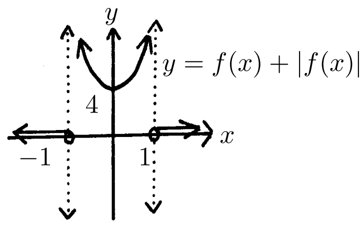


z lies on a unit circle,
$0 \le \arg(z) \le \frac{\pi}{2}$,
$\angle AOB = 2\angle ACB$ ($\angle$ at centre $= 2 \times \angle$ at circumference)
$\therefore 2\arg(z+1) = \arg z$.

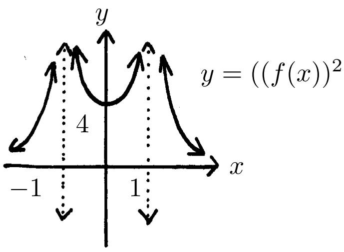
3. (a) (i)



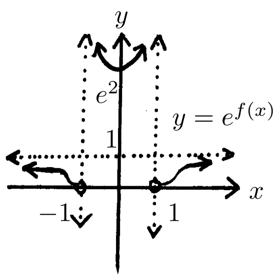
(ii)



(iii)



(iv)



(b) Eccentricity $= \sqrt{1-\frac{4}{9}} = \frac{\sqrt{5}}{3}$, foci are $(\pm\sqrt{9}\cdot\frac{\sqrt{5}}{3}, 0) = (\pm\sqrt{5}, 0)$, directrices are $x = \frac{\sqrt{9}}{\sqrt{5}/3} = \pm\frac{9}{\sqrt{5}}$, i.e., $x = \pm\frac{9\sqrt{5}}{5}$

(c) (i) $y = (x-1)(3-x) = -x^2+4x-3 \Rightarrow x^2-4x+3+y=0 \therefore x = (4\pm\sqrt{16-4(3+y)})/2 = 2\pm\sqrt{1-y} \therefore \ell = 2\sqrt{1-y} \therefore$ outer circle of annulus has radius $(2-2\sqrt{1-y})/2 + 2\sqrt{1-y} = 1+\sqrt{1-y}$ and inner circle of annulus has radius $(2-2\sqrt{1-y})/2 = 1-\sqrt{1-y}$. Hence the area of the annulus is $\pi((1+\sqrt{1-y})^2 - (1-\sqrt{1-y})^2) = \pi(1+1-y+2\sqrt{1-y} - (1+1-y-2\sqrt{1-y})) = 4\pi\sqrt{1-y}$

(ii) Vertex has height $(2-1)(3-2) = 1 \Rightarrow$
$V = \int_0^1 4\pi(1-y)^{\frac{1}{2}}\,dy = 4\pi(-\frac{2}{3})[(1-y)^{\frac{3}{2}}]_0^1 = -\frac{8\pi}{3}(0-1) = \frac{8\pi}{3}$ unit^3.

4. (a) (i) $x = r\cos\theta,\ y = r\sin\theta,\ \theta = \omega t,\ \omega$ constant $\Rightarrow \dot\omega = 0,\ \dot x = -r\sin\theta\omega,\ \dot y = r\cos\theta\omega,$
$\ddot x = -r\cos\theta\omega^2 = -\omega^2 x, \ddot y = -r\sin\theta\omega^2 = -\omega^2 y \therefore$ there is an inward radial force of magnitude $m\sqrt{\ddot x^2 + \ddot y^2} = m\sqrt{r^2\omega^4(\cos^2\theta + \sin^2\theta)} = mr\omega^2$ acting on P.

(ii) $Am/r^2 = mr\omega^2 \Rightarrow r = \sqrt[3]{A/\omega^2}$

(b) (i) $y - b\tan\theta = \frac{dy/d\theta}{dx/d\theta}(x - a\sec\theta) = \frac{b\sec^2\theta}{a\sec\theta\tan\theta}(x - a\sec\theta) \therefore \frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = \sec^2\theta - \tan^2\theta = 1.$
$\therefore$ tangent at P is $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1.$

(ii) Tangent at P meets asymptotes where also $y = \pm\frac{bx}{a} \therefore x(\frac{\sec\theta}{a} \pm \frac{\tan\theta}{a}) = 1 \therefore x = \frac{a}{\sec\theta\pm\tan\theta}$ $= \frac{a\cos\theta}{1\pm\sin\theta}$ (B, A respectively) $y_A = \frac{b}{a}\frac{a\cos\theta}{1-\sin\theta},\ y_B = -\frac{b}{a}\frac{a\cos\theta}{1+\sin\theta} \Rightarrow A = (\frac{a\cos\theta}{1-\sin\theta}, \frac{b\cos\theta}{1-\sin\theta}),$
$B = (\frac{a\cos\theta}{1+\sin\theta}, -\frac{b\cos\theta}{1+\sin\theta})$

(iii) Let $T(t, 0)$ be the x-intercept of the tangent at P. Then the height of $\triangle OAT,\ h_1 = \frac{b\cos\theta}{1-\sin\theta}$, and the height of $\triangle OBT,\ h_2 = \frac{b\cos\theta}{1+\sin\theta}$ and the area$(\triangle OAB) = (h_1 + h_2)t/2$. For T, $y = 0$ and $t = a\cos\theta$. $\therefore$ area(△OAB) $= \frac{a\cos\theta}{2}\cdot b\cos\theta(\frac{1}{1-\sin\theta} + \frac{1}{1+\sin\theta}) = \frac{ab}{2}\cos^2\theta\cdot\frac{2}{1-\sin^2\theta} = ab.$

(c) (i) n^n **(ii)** $1 - \frac{n!}{n^n} = \frac{n^{n-1}-(n-1)!}{n^{n-1}}.$

5. (a) (i) $s_1 = \alpha + \beta + \gamma =$ sum of roots $= -\frac{0}{1} = 0.$
$s_2 = \alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha) = 0^2 - 2\frac{p}{1} = -2p$
$s_3 = \alpha^3 + \beta^3 + \gamma^3 = (-p\alpha - q) + (-p\beta - q) + (-p\gamma - q) = -p(\alpha + \beta + \gamma) - 3q = 0 - 3q = -3q$

(ii) For $n > 3$,

$$\begin{aligned}
-ps_{n-2} - qs_{n-3} &= -(\alpha\beta + \beta\gamma + \gamma\alpha)(\alpha^{n-2} + \beta^{n-2} + \gamma^{n-2}) + \alpha\beta\gamma(\alpha^{n-3} + \beta^{n-3} + \gamma^{n-3}) \\
&= -\alpha^{n-1}\beta - \alpha\beta^{n-1} - \alpha\beta\gamma^{n-2} - \alpha^{n-2}\beta\gamma - \beta^{n-1}\gamma - \beta\gamma^{n-1} - \gamma\alpha^{n-1} \\
&\quad - \alpha\gamma\beta^{n-2} - \alpha\gamma^{n-1} + \alpha^{n-2}\beta\gamma + \alpha\gamma\beta^{n-2} + \alpha\beta\gamma^{n-2} \\
&= -\beta\alpha^{n-1} - \alpha\beta^{n-1} - \gamma\beta^{n-1} - \beta\gamma^{n-1} - \gamma\alpha^{n-1} - \alpha\gamma^{n-1} \\
&= -(\alpha + \beta + \gamma)(\alpha^{n-1} + \beta^{n-1} + \gamma^{n-1}) + (\alpha^n + \beta^n + \gamma^n) \\
&= -0(\alpha^{n-1} + \beta^{n-1} + \gamma^{n-1}) + s_n \\
&= s_n
\end{aligned}$$

$\therefore s_n = -ps_{n-2} - qs_{n-3}.$

(iii) $\therefore \frac{\alpha^5+\beta^5+\gamma^5}{5} = \frac{s_5}{5} = \frac{-ps_3-qs_2}{5} = \frac{-p(-3q)-q(-2p)}{5} = pq = \frac{-2p}{2}\cdot\frac{-3q}{3} = \frac{s_2}{2}\cdot\frac{s_3}{3}$
$= (\frac{\alpha^2+\beta^2+\gamma^2}{2})(\frac{\alpha^3+\beta^3+\gamma^3}{3})$

(b) (i) $\ddot x = -k\dot x \Rightarrow \frac{d\dot x}{dt} = -k\dot x \therefore \frac{d\dot x}{\dot x} = -k\,dt \therefore \int\frac{d\dot x}{\dot x} = \int -k\,dt \therefore \ln\dot x = -kt + c_1$ for a constant c_1.
$\therefore \dot x = Ae^{-kt}$ for a constant $A = e^{c_1}$. $t = 0 \Rightarrow \dot x = A = u\cos\alpha$. $\therefore \dot x = ue^{-kt}\cos\alpha.$

(ii) $\dot y = \frac{1}{k}((ku\sin\alpha + g)e^{-kt} - g)...(*) \Rightarrow t = 0 \Rightarrow \dot y = \frac{1}{k}(ku\sin\alpha + g - g) = u\sin\alpha$ & $\therefore$ $(*)$ satisfies the initial condition. Moreover, $\ddot y = -k\cdot\frac{1}{k}((ku\sin\alpha) + g)e^{-kt} = -((ku\sin\alpha) + g)e^{-kt}$ $= -k\cdot\frac{1}{k}((ku\sin\alpha) + g)e^{-kt} - g) - g = -k\dot y - g$ & $\therefore$ $(*)$ also satisfies $\ddot y = -k\dot y - g$.

(iii) Particle reaches maximum height when $\dot y = 0$ & $\therefore$ $\frac{1}{k}((ku\sin\alpha + g)e^{-kt} - g) = 0$
$\therefore$ $e^{-kt} = \frac{g}{ku\sin\alpha+g}$ $\therefore$ $e^{kt} = \frac{ku\sin\alpha+g}{g}$. Hence $t = \frac{1}{k}\ln\frac{ku\sin\alpha+g}{g}$

(iv) $\frac{dx}{dt} = ue^{-kt}\cos\alpha \Rightarrow x = \int ue^{-kt}\cos\alpha\,dt = -\frac{1}{k}ue^{-kt}\cos\alpha + c_2$ for a constant c_2. $t = 0 \Rightarrow$ $x = 0 \Rightarrow c_2 = \frac{u\cos\alpha}{k}$ & $\therefore$ $x = -\frac{1}{k}ue^{-kt}\cos\alpha + \frac{u\cos\alpha}{k}$. Hence the limiting value of the horizontal displacement of the particle is $\lim_{t\to\infty} x = \frac{u\cos\alpha}{k}$.

6. (a) (i) $\cos(a+b)x + \cos(a-b)x = \cos ax \cos bx - \sin ax \sin bx + \cos ax \cos -bx - \sin ax \sin -bx$
$= \cos ax \cos bx - \sin ax \sin bx + \cos ax \cos bx + \sin ax \sin bx = 2\cos ax \cos bx$

(ii) $\int \cos 3x \cos 2x\, dx = \frac{1}{2}\int(\cos(3+2)x + \cos(3-2)x)\, dx = \frac{1}{2}\int(\cos 5x + \cos x)\, dx$
$= \frac{1}{10}\sin 5x + \frac{1}{2}\sin x + C.$

(b) (i) $s_3 = s_2 + 2s_1 = 2 + 2(1) = 4$ & $s_4 = s_3 + 3s_2 = 4 + 3(2) = 10.$

(ii) $x \geq 0 \Rightarrow 2\sqrt{x}\, x \geq 0 \Rightarrow (\sqrt{x}+x)^2 = x + 2\sqrt{x}\, x + x^2 \geq x + x^2 = x(x+1)\ \therefore\ \sqrt{x}+x \geq \sqrt{x(x+1)}.$

(iii) r.t.p. : $s_n \geq \sqrt{n!}\ \forall n \in \mathbb{Z}^+$
Pf. : $s_1 = 1 \geq \sqrt{1!} = 1, s_2 = 2 \geq \sqrt{2!} = \sqrt{2} \because 4 > 2 \therefore$ true for $n = 1, 2$.
If true for $n = k, k+1$ then $s_k \geq \sqrt{k!}$ & $s_{k+1} \geq \sqrt{(k+1)!}$ whereupon $s_{k+2} = s_{k+1} + (k+1)s_k \geq \sqrt{(k+1)!} + (k+1)\sqrt{k!} = \sqrt{k+1}\sqrt{k!} + (k+1)\sqrt{k!} = \sqrt{k!}(\sqrt{k+1} + k + 1) \geq \sqrt{k!}\sqrt{(k+1)((k+1)+1)} = \sqrt{(k+2)!}$ from **(ii)** with $x = k+1 > 0$.
So if true for $n = 1, 2, \therefore$ for 3. True for 2,3$\Rightarrow$ true for 4, etc. $\therefore$ by induction true $\forall n \in \mathbb{Z}^+$. $\square$

(c) (i) $x \geq 0, y \geq 0 \Rightarrow (\frac{x+y}{2})^2 - xy = \frac{(x-y)^2}{4} \geq 0$ & $\therefore\ \frac{x+y}{2} \geq \sqrt{xy}$

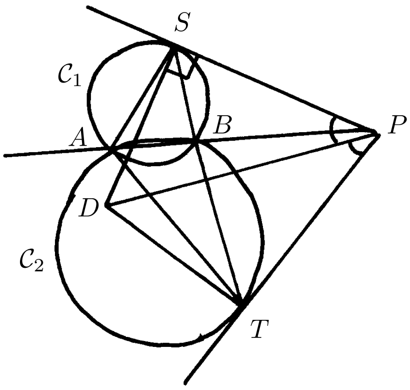
(ii) $a^4+b^4+c^4 = \frac{1}{2}(a^4+b^4) + \frac{1}{2}(b^4+c^4) + \frac{1}{2}(c^4+a^4) \geq \sqrt{a^4b^4} + \sqrt{b^4c^4} + \sqrt{c^4a^4} = a^2b^2 + a^2c^2 + b^2c^2$ from **(i)**.

(iii) $(a-b)^2 = a^2 - 2ab + b^2 \geq 0 \Rightarrow a^2 + b^2 \geq 2ab$
$(b-c)^2 = b^2 - 2bc + c^2 \geq 0 \Rightarrow b^2 + c^2 \geq 2bc$
$(c-a)^2 = c^2 - 2ca + a^2 \geq 0 \Rightarrow c^2 + a^2 \geq 2ca$
Adding, $a^2+b^2+b^2+c^2+c^2+a^2 = 2(a^2+b^2+c^2) \geq 2(ab+bc+ca) \Rightarrow a^2+b^2+c^2 \geq ab+bc+ca$
Replacing a by ab, b by bc, c by ca,
$$(ab)^2 + (bc)^2 + (ca)^2 \geq (ab)(bc) + (bc)(ca) + (ca)(ab)$$
$$\therefore a^2b^2 + b^2c^2 + c^2a^2 \geq ab^2c + bc^2a + ca^2b$$
$\therefore a^2b^2 + a^2c^2 + b^2c^2 \geq a^2bc + b^2ac + c^2ab$

(iv) If $a+b+c = d$, then
$$\begin{aligned} a^4+b^4+c^4 &\geq a^2b^2 + a^2c^2 + b^2c^2 \text{ from (ii)} \\ &\geq a^2bc + b^2ac + c^2ab \text{ from (iii)} \\ &= abc(a+b+c) \\ &= abcd \end{aligned}$$
$\therefore a^4+b^4+c^4 \geq abcd$

7. (a) $x(3-x^2) = x(\sqrt{3}-x)(\sqrt{3}+x),\ 0 \leq x \leq \sqrt{3} \Rightarrow$
$V = \int_0^{\sqrt{3}} 2\pi xy\, dx = \int_0^{\sqrt{3}} 2\pi x x(3-x^2)\, dx = 2\pi\int_0^{\sqrt{3}}(3x^2 - x^4)\, dx = 2\pi[x^3 - \frac{x^5}{5}]_0^{\sqrt{3}} = 2\pi(3\sqrt{3} - \frac{9\sqrt{3}}{5})$
$= \frac{12\pi\sqrt{3}}{5}$ unit^3.

(b)



(i) $\angle PSB = \angle PAS$ ($\angle$s in alt. seg.)
$\angle SPB = \angle APS$ (common)
$\therefore \triangle ASP ||| \triangle SBP$ (AAA)

(ii) $\therefore \frac{PS}{PA} = \frac{PB}{PS}$ (sides proportional in similar $\triangle$s)
$\therefore PS^2 = AP.BP$.
Likewise, $\triangle ATP ||| \triangle TBP$ & $PT^2 = AP.BP$
$\therefore PT = PS$.

(iii) DP common to $\triangle DSP, \triangle DTP$ and $TP = SP$ from **(ii)**
$\angle SPD = \angle TPD$ (DP bisects $\angle SPT$)
$\therefore \triangle DSP \equiv \triangle DTP$ (SAS)
$\therefore \angle DSP = \angle DTP = 90°$ (corresponding $\angle$s). But PT is a tangent.
$\therefore DT$ passes through the centre of $\mathcal{C}_2$ (tangent$\perp$radius)

(c)† **(i)** $P_n \sin \frac{\alpha}{2^n} = (\prod_{j=1}^{n} \cos \frac{\alpha}{2^j}) \sin \frac{\alpha}{2^n} = (\prod_{j=1}^{n-1} \cos \frac{\alpha}{2^j}) \cos \frac{\alpha}{2^n} \sin \frac{\alpha}{2^n} = P_{n-1} \cos \frac{\alpha}{2^n} \sin \frac{\alpha}{2^n}$
$= \frac{1}{2} P_{n-1} (2 \cos \frac{\alpha}{2^n} \sin \frac{\alpha}{2^n}) = \frac{1}{2} P_{n-1} \sin(2 \frac{\alpha}{2^n}) = \frac{1}{2} P_{n-1} \sin \frac{\alpha}{2^{n-1}}$.

(ii) <u>r.t.p.</u> : $P_n = \frac{\sin \alpha}{2^n \sin \frac{\alpha}{2^n}} \forall n \in \mathbb{Z}^+$.
<u>Pf.</u> : $\frac{\sin \alpha}{2^1 \sin \frac{\alpha}{2^1}} = \frac{2 \sin \frac{\alpha}{2} \cos \frac{\alpha}{2}}{2 \sin \frac{\alpha}{2}} = \cos \frac{\alpha}{2} = P_1 \therefore$ true for $n = 1$.
If true for $n = k, P_k = \frac{\sin \alpha}{2^k \sin \frac{\alpha}{2^k}}$ & $\therefore$
$P_{k+1} = \frac{\frac{1}{2} P_k \sin \frac{\alpha}{2^k}}{\sin \frac{\alpha}{2^{k+1}}} = \frac{1}{2} \frac{\sin \alpha}{2^k \sin \frac{\alpha}{2^k}} \frac{\sin \frac{\alpha}{2^k}}{\sin \frac{\alpha}{2^{k+1}}} = \frac{1}{2} \frac{\sin \alpha}{2^k \sin \frac{\alpha}{2^{k+1}}} = \frac{\sin \alpha}{2^{k+1} \sin \frac{\alpha}{2^{k+1}}}$ & $\therefore$ if true for $n = k$, then true for $n = k + 1$.
It is true for $n = 1$ & $\therefore$ true for $n = 2$ & $\therefore$ also true for $n = 3, 4, 5, ...$, i.e., by induction $\forall n \in \mathbb{Z}^+$, $P_n = \frac{\sin \alpha}{2^n \sin \frac{\alpha}{2^n}}$. $\square$

(iii) $\sin x < x$ for $x > 0$. Let $x = \frac{\alpha}{2^n}$. $\therefore \frac{\sin \alpha}{\prod_{j=1}^{n} \cos \frac{\alpha}{2^j}} = \frac{\sin \alpha}{\sin \alpha / 2^n \sin \frac{\alpha}{2^n}} = 2^n \sin \frac{\alpha}{2^n} < 2^n \frac{\alpha}{2^n} = \alpha$ from **(ii)**, i.e., $\frac{\sin \alpha}{\prod_{j=1}^{n} \cos \frac{\alpha}{2^j}} < \alpha$.

8. (a) (i) Residues mod 3 are 0, 1, 2 $\Rightarrow$ $1 + \omega^k + \omega^{2k} = 1 + 1 + 1$ or $1 + \omega + \omega^2 = 0$ or $1 + \omega^2 + \omega = 0$. $\therefore$ 2 possible values of $1 + \omega^k + \omega^{2k}$ are 0, 3.

(ii) $(1 + \omega)^n = \sum_{j=0}^{n} \binom{n}{j} \omega^j$, $(1 + \omega^2)^n = \sum_{j=0}^{n} \binom{n}{j} \omega^{2j}$.

(iii) $\ell = \lfloor \frac{n}{3} \rfloor$ $\therefore \sum_{j=0}^{n} \binom{n}{j} (1 + \omega^j + \omega^{2j}) = \sum_{j=0}^{\ell} \binom{n}{3j} (3) + 0$ from **(i)** & $\therefore \because \sum_{j=0}^{n} \binom{n}{j} = 2^n$, $\sum_{j=0}^{\ell} \binom{n}{3j} = \frac{1}{3} (\sum_{j=0}^{n} \binom{n}{j} + (1 + \omega)^n + (1 + \omega^2)^n) = \frac{1}{3} (2^n + (1 + \omega)^n + (1 + \omega^2)^n)$.

(iv) Let $n = 6m$, $m \in \mathbb{Z}^+ \cup \{0\}$. Then $\ell = \frac{n}{3}$ & $\sum_{j=0}^{n/3} \binom{n}{3j} = \frac{1}{3} (2^n + (1 + \omega)^n + (1 + \omega^2)^n)$
$= \frac{1}{3} (2^n + (-\omega^2)^{6m} + (-\omega)^{6m}) = \frac{1}{3} (2^n + 1 + 1) = \frac{1}{3} (2^n + 2)$

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†**Note:** This question can be extended to get Vieta's formula for π as follows. $\lim_{n \to \infty} P_n = \lim_{n \to \infty} \frac{\sin \alpha}{\alpha \sin \frac{\alpha}{2^n} / \frac{\alpha}{2^n}}$
$= \frac{\sin \alpha}{\alpha} \lim_{n \to \infty} \frac{\alpha / 2^n}{\sin \frac{\alpha}{2^n}} = \frac{\sin \alpha}{\alpha} \lim_{t \to 0} \frac{t}{\sin t} = \frac{\sin \alpha}{\alpha}$ where $t = \frac{\alpha}{2^n}$. $\therefore \prod_{j=1}^{\infty} \cos \frac{\alpha}{2^j} = \frac{\sin \alpha}{\alpha}$. Let $\alpha = \frac{\pi}{2}$. $\therefore \prod_{j=1}^{\infty} \cos \frac{\pi}{2^{j+1}} = \frac{1}{\pi/2} = \frac{2}{\pi}$.
Now $\cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$, so $2 \cos \frac{\pi}{4} = \sqrt{2}$. $\cos \theta = \sqrt{\frac{1}{2}(1 + \cos 2\theta)} = \frac{1}{2} \sqrt{2 + 2 \cos 2\theta} \Rightarrow \cos \frac{\pi}{8} = \frac{1}{2} \sqrt{2 + \sqrt{2}}$,
$\cos \frac{\pi}{16} = \frac{1}{2} \sqrt{2 + \sqrt{2 + \sqrt{2}}}$, etc. $\therefore \frac{2}{\pi} = (\frac{1}{2} \sqrt{2}) \left(\frac{1}{2} \sqrt{2 + \sqrt{2}} \right) \left(\frac{1}{2} \sqrt{2 + \sqrt{2 + \sqrt{2}}} \right) \cdots$ (Vieta's formula), or
$\pi = 2 \Big/ (\frac{1}{2} \sqrt{2}) \left(\frac{1}{2} \sqrt{2 + \sqrt{2}} \right) \left(\frac{1}{2} \sqrt{2 + \sqrt{2 + \sqrt{2}}} \right) \cdots$

(b)‡ (i)

$$\begin{aligned}
I_n &= \tfrac{q^{2n}}{n!}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\tfrac{\pi^2}{4}-x^2)^n\cos x\,dx \\
&= \tfrac{q^{2n}}{n!}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\tfrac{\pi^2}{4}-x^2)^n\tfrac{d}{dx}\sin x\,dx \\
&= \tfrac{q^{2n}}{n!}[(\tfrac{\pi^2}{4}-x^2)^n\sin x]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \tfrac{q^{2n}}{n!}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} n(\tfrac{\pi^2}{4}-x^2)^{n-1}(-2x)\sin x\,dx \\
&= \tfrac{2q^{2n}}{(n-1)!}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x(\tfrac{\pi^2}{4}-x^2)^{n-1}\tfrac{d}{dx}-\cos x\,dx \\
&= \tfrac{2q^{2n}}{(n-1)!}[-x(\tfrac{\pi^2}{4}-x^2)^{n-1}\cos x]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\
&\quad - \tfrac{2q^{2n}}{(n-1)!}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}((n-1)(-2x^2)(\tfrac{\pi^2}{4}-x^2)^{n-2}(-\cos x)+(\tfrac{\pi^2}{4}-x^2)^{n-1}(-\cos x))\,dx \\
&= \tfrac{2q^{2n}}{(n-1)!}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\tfrac{\pi^2}{4}-x^2)^{n-1}\cos x\,dx - \tfrac{4q^{2n}}{(n-2)!}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^2(\tfrac{\pi^2}{4}-x^2)^{n-2}\cos x\,dx
\end{aligned}$$

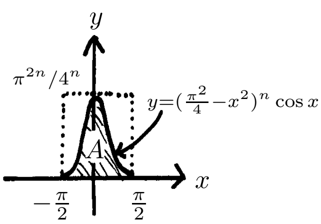
(ii) $\frac{2q^{2n}}{(n-1)!}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\frac{\pi^2}{4}-x^2)^{n-1}\cos x\,dx = 2q^2I_{n-1}$ &

$$\begin{aligned}
-\tfrac{4q^{2n}}{(n-2)!}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^2(\tfrac{\pi^2}{4}-x^2)^{n-2}\cos x\,dx &= -\tfrac{4q^{2n}}{(n-2)!}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\tfrac{\pi^2}{4}-(\tfrac{\pi^2}{4}-x^2))(\tfrac{\pi^2}{4}-x^2)^{n-2}\cos x\,dx \\
&= -\tfrac{4q^{2n}}{(n-2)!}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\tfrac{\pi^2}{4}(\tfrac{\pi^2}{4}-x^2)^{n-2}\cos x\,dx \\
&\quad + \tfrac{4q^{2n}}{(n-2)!}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\tfrac{\pi^2}{4}-x^2)^{n-1}\cos x\,dx \\
&= -4q^4\tfrac{q^{2n-4}}{(n-2)!}\tfrac{\pi^2}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\tfrac{\pi^2}{4}-x^2)^{n-2}\cos x\,dx \\
&\quad + 4q^2\tfrac{q^{2n-2}(n-1)}{(n-1)!}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\tfrac{\pi^2}{4}-x^2)^{n-1}\cos x\,dx \\
&= -4q^4\frac{p^2}{4q^2}I_{n-2}+4q^2(n-1)I_{n-1}
\end{aligned}$$

$\therefore I_n = 4q^2(n-1)I_{n-1}+2q^2I_{n-1}-p^2q^2I_{n-2} = (4n-2)q^2I_{n-1}-p^2q^2I_{n-2}$ for $n\geq 2$.

(iii) $n,p,q\in\mathbb{Z}$. If I_{k-1} & $I_{k-2}\in\mathbb{Z}$ then so too is $I_k=(4k-2)q^2I_{k-1}-p^2q^2I_{k-2}$ $\because (4k-2)q^2,\ p^2q^2\in\mathbb{Z}$. We are given that $I_0=2$ and also $I_1=4q^2$ (not $4q$ as in the question paper) and so $I_0,I_1\in\mathbb{Z}$. $\therefore$ by induction, $I_n\in\mathbb{Z}\ \forall n\in\mathbb{Z}^+\cup\{0\}$

(iv)



$-\frac{\pi}{2}<x<\frac{\pi}{2}\Rightarrow\cos x>0$ & $\frac{\pi^2}{4}-x^2>0\Rightarrow I_n>0$. From the diagram, it is clear that $I_n=\frac{q^{2n}}{n!}A<\frac{q^{2n}}{n!}\pi\frac{\pi^{2n}}{4^n}=\frac{q^{2n}}{n!}\frac{p}{q}(\frac{p}{q})^{2n}\frac{1}{2^{2n}}$ and thus $0<I_n<\frac{p}{q}(\frac{p}{2})^{2n}\frac{1}{n!}$

(v) For n sufficiently large, $\frac{p}{q}(\frac{p}{2})^{2n}\frac{1}{n!}<1\Rightarrow 0<I_n<1$ from **(iv)**. But $I_n\in\mathbb{Z}$ - contradiction. $\therefore$ the assumption that $\exists p,q\in\mathbb{Z}:\pi=\frac{p}{q}$ is false. Hence reductio ad absurdum, $\pi\notin\mathbb{Q}$. □

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‡The fact that π is irrational also follows from the fact that π is transcendental. You can read about this in the book "Galois Theory" by Ian Stewart.