Student Number:____________

Maths set: 12ME2-1

SHORE

Year 12
Mathematics - Extension 2
Trial Examination
2010

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- All necessary working should be shown in every question
- A table of standard integrals is attached on the final page

Note: Any time you have remaining should be spent revising your answers.

Total marks - 120

- Attempt Questions 1 - 8
- All questions are of equal value
- Start each question in a new writing booklet
- Write your examination number on the front cover of each booklet to be handed in
- If you do not attempt a question, submit a blank booklet with your examination number and "N/A" on the front cover

DO NOT REMOVE THIS PAPER FROM THE EXAMINATION ROOM

Question 1 **(15 Marks)** Use a Separate Booklet **Marks**

a) Evaluate $\displaystyle\int_{\frac{\pi}{16}}^{\frac{\pi}{12}} sec\,4x\ tan\,4x\, dx$ **2**

b) Find $\displaystyle\int x\, ln\, x\, dx$ **2**

c) Find $\displaystyle\int \frac{9x^3+9x^2+5x+4}{3x+1}\, dx$ **3**

d) i. Find constants a, b and c such that **2**

$$\frac{3x^2-2x-3}{\left(x^2+9\right)\left(x-3\right)} = \frac{ax+b}{x^2+9} + \frac{c}{x-3}$$

ii. Hence find $\displaystyle\int \frac{3x^2-2x-3}{\left(x^2+9\right)\left(x-3\right)}\, dx$. **2**

e) By making the substitution $t = \tan\dfrac{\theta}{2}$, evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{d\theta}{1+sin\,\theta+cos\,\theta}$ **4**

End of Question 1

Question 2 **(15 Marks)** Use a Separate Booklet **Marks**

a) Given $A = 3 + 4i$ and $B = 1 - i$, express the following in the form $x + iy$ where x and y are real numbers.

i. AB **1**

ii. $\frac{A}{iB}$ **2**

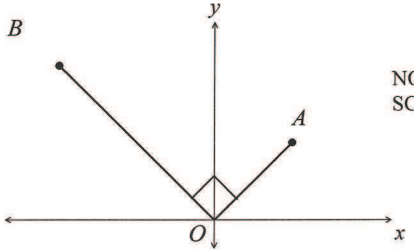
iii. $\sqrt{A}$ (give both possible answers) **3**

b) If $w = \sqrt{3} - i$,

i. Find the exact value of $|w|$ and $arg\, w$. **2**

ii. Find the exact value of w^5 in the form $a + ib$ where a and b are real. **2**

c)



NOT TO SCALE

On the Argand diagram, OA represents the complex number $z_1 = x + iy$, $\angle AOB = \frac{\pi}{2}$ and the length of OB is twice that of OA.

i. Show that OB represents the complex number $-2y + 2ix$. **1**

ii. Given that AOBC is a rectangle, find the complex number represented by OC. **1**

iii. Find the complex number represented by BA. **1**

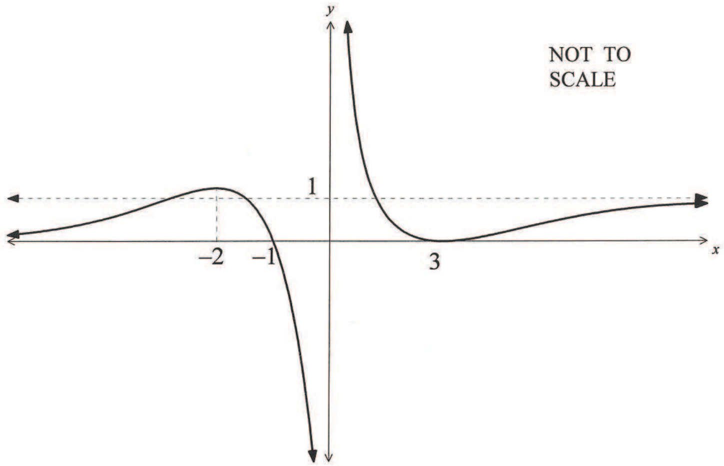
d) Sketch the region on an argand diagram where **2**

$$|z - 1| \le \sqrt{2} \text{ and } 0 \le arg\,(z + i) \le \frac{\pi}{4} \text{ both hold.}$$

End of Question 2

Question 3 **(15 Marks)** Use a Separate Booklet **Marks**

a)



NOT TO SCALE

The diagram shows the graph of the function $y = f(x)$ which has asymptotes, vertically at $x = 0$ and horizontally at $y = 1$ for $x \ge 0$ and at $y = 0$ for $x \le 0$.

Draw separate sketches of the following showing any critical features.

i. $y = \frac{1}{f(x)}$ **3**

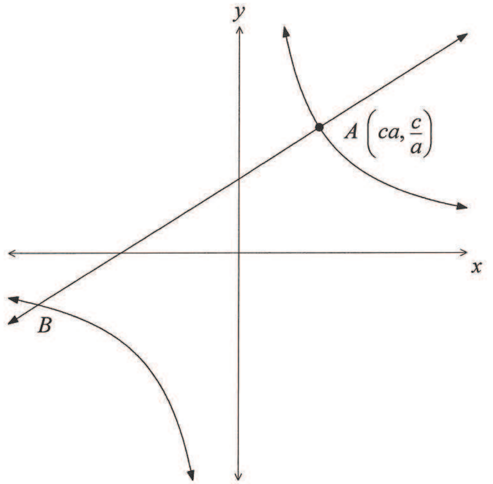
ii. $y = \left[f(x)\right]^2$ **3**

iii. $y = f'(x)$ **2**

Question 3 continues

Question 3 continued **Marks**

b)



NOT TO SCALE

The point $A\left(ca, \frac{c}{a}\right)$, where $a \neq \pm 1$ lies on the hyperbola $xy = c^2$. The normal through A meets the other branch of the curve at B.

i. Show that the equation of the normal through A is **2**

$$y = a^2x + \frac{c}{a}\left(1 - a^4\right)$$

ii. Hence if B has coordinates $\left(cb, \frac{c}{b}\right)$, show that $b = \frac{-1}{a^3}$. **3**

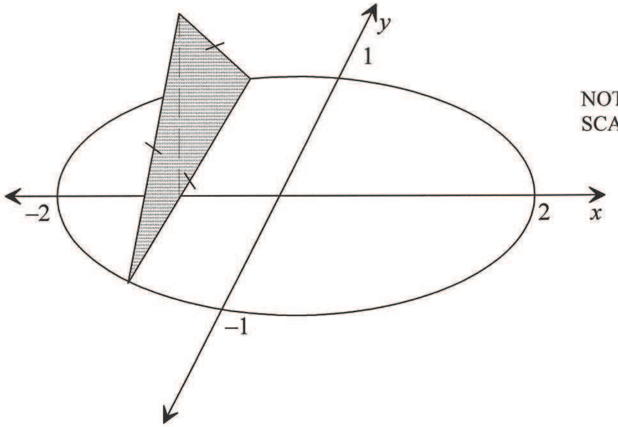
iii. If this hyperbola is rotated clockwise through 45°, show that the equation becomes **2**

$$X^2 - Y^2 = 2c^2.$$

End of Question 3

Question 4 (15 Marks) Use a Separate Booklet **Marks**

a) A solid shape has as its base an ellipse in the XY plane as shown below. Sections taken perpendicular to the X-axis are equilateral triangles. The major and minor axes of the ellipse are 4 metres and 2 metres respectively.



NOT TO SCALE

i. Write down the equation of the ellipse. **1**

ii. Show that the area of the cross-section is given by **2**

$$A = \frac{\sqrt{3}}{4}\left(4 - x^2\right).$$

iii. By using the technique of slicing, find the volume of the solid. **2**

b) The region enclosed by the curve $y = 5x - x^2$, the x axis and the lines $x = 1$ and $x = 3$ is rotated about the y axis. By using the method of cylindrical shells, find the volume of the solid so produced. **4**

Question 4 continues

Question 4 continued **Marks**

c) The roots of the equation $x^3 - 3x^2 + 9 = 0$ are α, β and γ.

i. Determine the polynomial equation with roots α^2, β^2 and γ^2. **2**

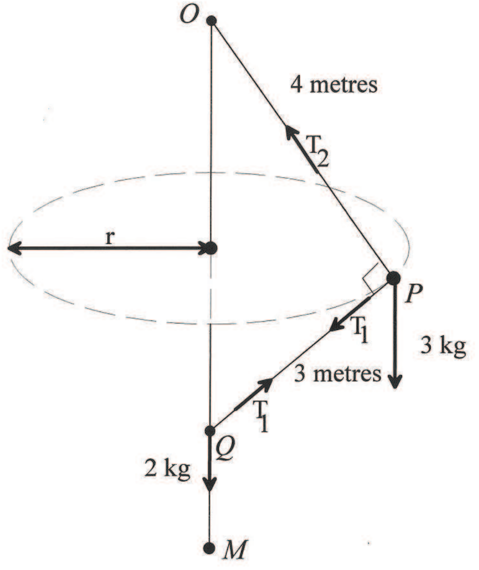
ii. Find the value of $\alpha^2 + \beta^2 + \gamma^2$ and hence evaluate $\alpha^3 + \beta^3 + \gamma^3$. **2**

d) Given that the polynomial $P(x)$ has a double root at $x = \alpha$, show that the polynomial $P'(x)$ will have a single root at $x = \alpha$. **2**

End of Question 4

Question 5 (15 Marks) Use a Separate Booklet **Marks**

a)



NOT TO SCALE

The above sketch shows a smooth vertical rod *OM.* Light inextensible strings *OP* and *QP* are attached to the rod at *O* and a mass of 3kg at *P*. At *Q*, a 2kg mass is free to slide on the rod. *P* is rotating in a horizontal circle about the rod and maintains a right angle at *P*. Let angle POQ be θ.

Note that the distance *OQ* is 5 metres.

i. By considering the forces vertically at P and Q calculate the tension T_1 in PQ and T_2 in OP. (In terms of *g*) **4**

ii. By considering the forces horizontally at P calculate the angular velocity of *P* to maintain this system. Give your answer correct to one decimal place. (Use $g = 10\ ms^{-2}$) **3**

Question 5 continues

Question 5 continued

Marks

b) By taking logarithms of both sides and then differentiating implicitly, verify the rule for differentiating the quotient $y = \frac{u(x)}{v(x)}$ is given by **3**

$$\frac{dy}{dx} = \frac{v(x)u'(x) - u(x)v'(x)}{(v(x))^2}$$

c) i. Show that the recurrence (reduction) formula for **3**

$$I_n = \int tan^{\,n}\, x dx \qquad \text{is} \qquad I_n = \frac{1}{n-1}\tan^{\,n-1} x - I_{n-2}\,.$$

ii. Hence evaluate $\int_0^{\frac{\pi}{4}} tan^{\,3} x\, dx$ **2**

End of Question 5

Question 6 (15 Marks) Use a Separate Booklet **Marks**

a) A solid of unit mass is dropped from above the ground.
Air resistance is proportional to the speed (v) of the mass.
(acceleration under gravity = g)

i. Write the equation for the acceleration of the mass. (Use k as the constant of proportionality) **1**

ii. Show that the velocity (v) of the solid after t seconds is given by **3**

$$v = \frac{g}{k}\left(1 - e^{-kt}\right)$$

iii. Show that the displacement (x) of the solid at velocity v is given by **3**

$$x = \frac{g}{k^2}\left[\ln\frac{g}{g-kv} - \frac{kv}{g}\right].$$

iv. Using either part (ii) or (iii) deduce the terminal velocity of the mass. **1**

b) Given $z = cos\ \theta + isin\ \theta$, and using De Moivres' Theorem

i. Find an expression for $cos\ 4\theta$ in terms of powers of $cos\ \theta$. **3**

ii. Determine the roots of the equation $z^4 = -1$. **2**

iii. Using the fact that $z^n + \frac{1}{z^n} = 2\cos n\theta$, find an expression for $\cos^4\theta$ in terms of $cos\ 2\theta$ and $cos\ 4\theta$. **2**

End of Question 6

Question 7 **(15 Marks)** Use a Separate Booklet **Marks**

a) When a polynomial $P(x)$ is divided by $(x-1)$ the remainder is 3 and when divided by $(x-2)$ the remainder is 5. Find the remainder when the polynomial is divided by $(x-1)(x-2)$. **2**

b) i. Prove that $\cos[(k-1)\theta] - 2\cos\theta\cos k\theta = -\cos[(k+1)\theta]$. **1**

ii. Hence, using mathematical induction, prove that if n is a positive integer then **4**

$$1 + \cos\theta + \cos 2\theta + \ldots\ldots\ldots\ldots + \cos(n-1)\theta = \frac{1 - \cos\theta - \cos n\theta + \cos[(n-1)\theta]}{2 - 2\cos\theta}$$

c) A mass of 20 kg hangs from the end of a rope under gravity and is hauled up vertically from rest by winding up the rope. The pulling force on the rope starts at 250 N and decreases uniformly by 10 N for every metre wound up. In other words, the pulling force is *250-10x* Newtons where x m is the height above the initial starting point. **3**

Find the velocity of the mass when 10 metres have been wound up.

(Neglect the weight of the rope and take $g = 10ms^{-2}$)

d) i. Show that $\dfrac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta} = \sin\theta + i\cos\theta$ where $\theta \neq \dfrac{3\pi}{2} + 2k\pi$; k integer. **3**

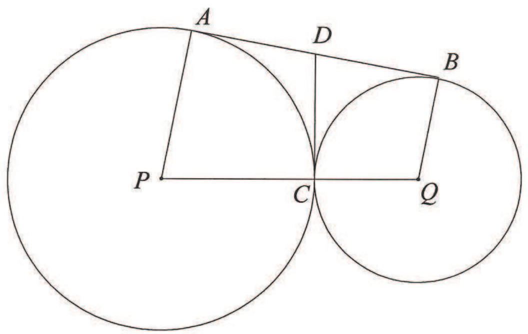
ii. Hence prove that $\left(\dfrac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}\right)^n = \cos(\dfrac{n\pi}{2} - n\theta) + i\sin(\dfrac{n\pi}{2} - n\theta)$ **2**

where n is a poistive integer and where $\theta \neq \dfrac{3\pi}{2} + 2k\pi$; k integer.

End of Question 7

Question 8 **(15 Marks)** Use a Separate Booklet **Marks**

a)



NOT TO SCALE

In the diagram PCQ is a straight line joining the centres of the circles P and Q. AB and DC are common tangents.

Copy or trace the diagram into your writing booklet.

i. Explain why CDBQ is a cyclic quadrilateral. **2**

ii. Show that ΔADC ||| ΔBQC. **3**

iii. Show that PD || CB. **3**

b) Given $2\cos A \sin B = \sin(A+B) - \sin(A-B)$

If $P = 1 + 2\cos\theta + 2\cos 2\theta + 2\cos 3\theta$

i. Prove that $P\sin\dfrac{\theta}{2} = \sin\dfrac{7\theta}{2}$. **2**

ii. Hence show that $1 + 2\cos\dfrac{2\pi}{7} + 2\cos\dfrac{4\pi}{7} + 2\cos\dfrac{6\pi}{7} = 0$ **2**

iii. By writing P in terms of $\cos\theta$, prove that $\cos\dfrac{2\pi}{7}$ is a root of the polynomial equation **3**

$$8x^3 + 4x^2 - 4x - 1 = 0$$

End of Examination

Solutions Maths Ext 2 Trial HSC 2010 Shore

1 (a) $I = \left[\frac{1}{4}\sec 4x\right]_{\pi/16}^{\pi/12}$

$= \frac{1}{4}\left(\sec\frac{\pi}{3} - \sec\frac{\pi}{4}\right)$

$= \frac{1}{4}\left(2 - \sqrt{2}\right)$

OR $\frac{2-\sqrt{2}}{4}$ (2)

(b) $I = \int \ln x \, d\left(\frac{x^2}{2}\right)$

$= \frac{x^2}{2}\ln x - \int \frac{x^2}{2} d(\ln x)$

$= \frac{1}{2}x^2\ln x - \int \frac{x}{2} dx$

$= \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + c$

OR $\frac{1}{4}x^2(2\ln x - 1) + c$ (2)

(c)

$$\begin{array}{r|l} & 3x^2 + 2x + 1 \\ 3x+1 & 9x^3 + 9x^2 + 5x + 4 \\ & 9x^3 + 3x^2 \\ & \quad 6x^2 + 5x \\ & \quad 6x^2 + 2x \\ & \qquad 3x + 4 \\ & \qquad 3x + 1 \\ & \qquad\quad 3 \end{array}$$

$I = \int 3x^2 + 2x + 1 + \frac{3}{3x+1} dx$

$= x^3 + x^2 + x + \ln|3x+1| + c$ (3)

(d)(i) If $x \doteqdot 3$, $\frac{27-6-3}{18} \doteqdot c$

$\therefore c = 1$

Also $3x^2 - 2x - 3 = (ax+b)(x-3) + c(x^2+9)$

Equating coeff x^2: $a + c = 3$

Equating const: $-3 = -3b + 9c$

$\therefore a = 2$ and $b = 4$ (2)

(ii) $\therefore I = \int \frac{2x+4}{x^2+9} + \frac{1}{x-3} dx$

$= \int \frac{2x}{x^2+9} + \frac{4}{x^2+3^2} + \frac{1}{x-3} dx$

$= \ln(x^2+9) + \frac{4}{3}\tan^{-1}\frac{x}{3} + \ln|x-3| + c$

OR $\ln|(x^2+9)(x-3)| + \frac{4}{3}\tan^{-1}\frac{x}{3} + c$ (2)

(e) $t = \tan\frac{\theta}{2} \Rightarrow \theta = 2\tan^{-1}t \Rightarrow \frac{d\theta}{dt} = \frac{2}{1+t^2}$

$\therefore d\theta = \frac{2dt}{1+t^2}$: $\theta = 0, t = 0; \theta = \frac{\pi}{2}, t = 1$

Also $\tan\theta = \frac{2t}{1-t^2}$, $\sin\theta = \frac{2t}{1+t^2}$, $\cos\theta = \frac{1-t^2}{1+t^2}$

$\therefore I = \int_0^1 \frac{2dt}{1 + \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}} \times \frac{1}{1+t^2}$

$\therefore I = \int_0^1 \frac{2dt}{1+t^2+2t+1-t^2}$

$= \int_0^1 \frac{dt}{1+t}$

$= \left[\ln|1+t|\right]_0^1$

$= \ln 2 - \ln 1$

$= \ln 2$ (4)

2(a)(i) $AB = (3+4i)(1-i)$

$= 3 + 4 - 3i + 4i$

$= 7 + i$ (1)

(ii) $\frac{A}{iB} = \frac{3+4i}{i+1} \times \frac{i-1}{i-1}$

$= \frac{-3-4-4i+3i}{-1-1}$

$= \frac{7}{2} + \frac{i}{2}$ (2)

(iii) $\sqrt{A} = \sqrt{3+4i} = x + iy$ say

$\therefore 3 + 4i = x^2 - y^2 + 2xyi$

Equating: $x^2 - y^2 = 3$ & $xy = 2$

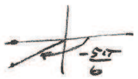
$\therefore x = \pm 2$, $y = \pm 1$ (3)

$\therefore \sqrt{3+4i} = 2 + i$ or $-2 - i$

(b)(i)

$|w| = 2$, $\arg w = \frac{-\pi}{6}$ (2)

(ii) $w^5 = \left[2\,\text{cis}\left(\frac{-\pi}{6}\right)\right]^5$



$= 2^5\,\text{cis}\,\frac{-5\pi}{6}$

$= 32\left(-\frac{\sqrt{3}}{2} - \frac{i}{2}\right)$

$= -16\sqrt{3} - 16i$ (2)

(c)(i) OB represents $2i(x+iy) = 2ix - 2y$

$= -2y + 2ix$ (1)

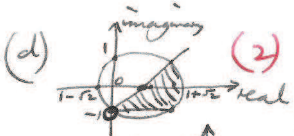
(ii) OC represents $(x+iy) + (-2y+2ix) = (x-2y) + (2x+y)i$ (1)

(iii) $\overrightarrow{BA} = \overrightarrow{BO} + \overrightarrow{OA}$

$= (2y - 2ix) + (x + iy)$

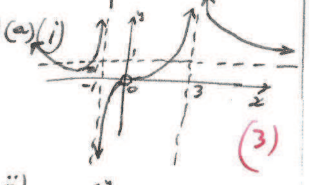
$= (x+2y) + i(-2x+y)$ (1)

2(d)



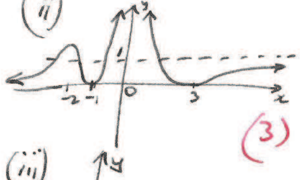
(2)

3(a)(i)



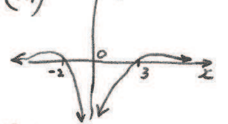
(3)

(ii)



(3)

(iii)



(2)

(b) Now $x = ca$ and $y = \frac{c}{a}$

(i) $\therefore \frac{dx}{da} = c$ & $\frac{dy}{da} = \frac{-c}{a^2}$

$\therefore \frac{dy}{dx} = \frac{\frac{dy}{da}}{\frac{dx}{da}}$

$= \frac{-c/a^2}{c}$

$= \frac{-1}{a^2}$

Normal grad a^2

$\therefore \frac{y - c/a}{x - ca} = a^2$

$\therefore y - c/a = a^2x - a^3c$

$\therefore y = a^2x + \frac{c}{a} - a^3c$

$= a^2x + \frac{c}{a}(1 - a^4)$ (2)

(ii) Subs in $\left(cb, \frac{c}{b}\right)$

$\therefore \frac{c}{b} = a^2 \cdot cb + \frac{c}{a}(1 - a^4)$

$\left(\times\frac{ab}{c}\right)$: $a = a^3b^2 + b - ba^4$

$\therefore 0 = a^3b^2 + b - ba^4 - a$

$= b(a^3b + 1) - a(ba^3 + 1)$

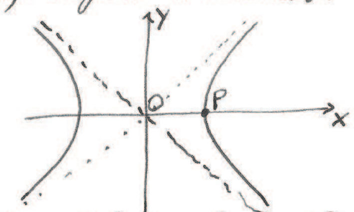
$= (a^3b + 1)(b - a)$

But $a \neq \pm 1 \therefore a \neq b$ by symmetry

$\therefore a^3b = -1$

$\therefore b = \frac{-1}{a^3}$ (3)

(iii) Through 45° it becomes:



This is of the form $\frac{X^2}{a^2} - \frac{Y^2}{b^2} = 1$

But it is rectangular $\therefore a = b$

$\therefore$ form $X^2 - Y^2 = k^2$ for some k.

On original hyperbola (c, c) is $\sqrt{2}c$ from the origin

$\therefore OP = \sqrt{2}c \therefore P(\sqrt{2}c, 0)$

Subs in $(\sqrt{2}c)^2 - 0^2 = k^2 \Longrightarrow k = \sqrt{2}c$

$\therefore$ new eqn $X^2 - Y^2 = 2c^2$ (2)

4(a)(i) $\frac{x^2}{4} + y^2 = 1$ (1)

(ii) $y^2 = 1 - \frac{x^2}{4}$ ([illegible])

$\therefore 2y = 2\sqrt{1 - \frac{k^2}{4}}$, each side of the equilateral Δ.

Area $= \frac{1}{2}ab\sin C$

$= \cancel{\frac{1}{2}} \cdot \cancel{2}\sqrt{1-\frac{k^2}{4}} \cdot 2\sqrt{1-\frac{k^2}{4}}\sin 60°$

$= \sqrt{3}\left(1 - \frac{k^2}{4}\right)$

$= \frac{\sqrt{3}}{4}(4 - k^2)$ (2)

(iii) $\Delta V = \frac{\sqrt{3}}{4}(4 - k^2)\Delta k$

$\therefore V = \lim_{\Delta k \to 0} \sum_{-2}^{2} \frac{\sqrt{3}}{4}(4 - k^2)\Delta k$

$= \int_{-2}^{2} \frac{\sqrt{3}}{4}(4 - k^2) dk$

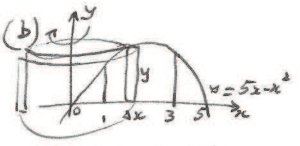
$= 2\int_0^2 \frac{\sqrt{3}}{4}(4 - k^2) dk$

$= 2 \cdot \frac{\sqrt{3}}{4}\left[4k - \frac{k^3}{3}\right]_0^2$

$= \frac{\sqrt{3}}{2} \cdot \left(8 - \frac{8}{3}\right)$

$= \frac{8\sqrt{3}}{3} u^3$ (2)

4(b)



Ignoring 2nd order differences

$\delta V \doteqdot 2\pi x \cdot y \cdot \Delta x$

$\therefore V = \lim_{\Delta x \to 0} \sum_{1}^{3} 2\pi x y \, dx$

$= \int_1^3 2\pi x . (5x - x^2) dx$

$= \int_1^3 2\pi (5x^2 - x^3) dx$

$= 2\pi \left[\frac{5x^3}{3} - \frac{x^4}{4} \right]_1^3$

$= 2\pi \left(45 - \frac{81}{4} - \frac{5}{3} + \frac{1}{4} \right)$

$= \frac{140\pi}{3} u^3$ (4)

(c) $x = \alpha, \beta, \gamma$, let $y = \alpha^2, \beta^2, \gamma^2$

(i) $\therefore y = x^2 \Rightarrow x = \sqrt{y}$ say

$\therefore (\sqrt{y})^3 - 3(\sqrt{y})^2 + 9 = 0$

$\therefore y\sqrt{y} - 3y + 9 = 0$

$\therefore y\sqrt{y} = 3y - 9$

$\therefore y^2 \cdot y = (3y - 9)^2$

$\therefore y^3 = 9y^2 - 54y + 81$

$\therefore y^3 - 9y^2 + 54y - 81 = 0$ (2)

(ii) Using sum of the roots

$\alpha^2 + \beta^2 + \gamma^2 = 9$

Now $\alpha^3 - 3\alpha^2 + 9 = 0$

$\beta^3 - 3\beta^2 + 9 = 0$

$\gamma^3 - 3\gamma^2 + 9 = 0$

$\therefore \alpha^3 + \beta^3 + \gamma^3 - 3(\alpha^2 + \beta^2 + \gamma^2) + 27 = 0$

$\therefore \alpha^3 + \beta^3 + \gamma^3 = 27 - 27$

$= 0$ (2)

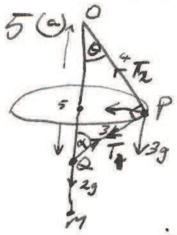
(d) Let $P(x) = (x - \alpha)^2 . Q(x)$ where $Q(\alpha) \neq 0$

$\therefore P'(x) = Q(x) . 2(x - \alpha)' + (x - \alpha)^2 . Q'(x)$

$= (x - \alpha) [2 Q(x) + (x - \alpha) . Q'(x)]$

$\therefore P'(\alpha) = 0 \Rightarrow P'(x)$ has a single root as $Q(\alpha) \neq 0$ with $(x - \alpha)$ not being a factor. (2)

5(a)



Horizontally at P :

$m r \omega^2 = T_1 \sin\alpha + T_2 \sin\theta$

$\therefore 3 r \omega^2 = T_1 \sin\alpha + T_2 \sin\theta$

Now $\frac{r}{4} = \sin\theta$. But $\sin\theta = \frac{3}{5}$

$\therefore r = \frac{12}{5}$. $\sin\alpha = \frac{12/5}{3}$

$\therefore 3\left(\frac{12}{5}\right)\omega^2 = T_1 \cdot \frac{4}{5} + T_2 \cdot \frac{3}{5}$

$\therefore 36\omega^2 = 4T_1 + 3T_2$ ①

Vertically at Q: $0 = T_1 \cos\alpha - 2g$

$= T_1 \times \frac{3}{5} - 2g$

$\therefore T_1 = \frac{10g}{3}$ N

Vertically at P: $0 = T_2 \cos\theta - T_1 \cos\alpha - 3g$

$= T_2 \times \frac{4}{5} - \frac{10g}{3} \times \frac{3}{5} - 3g$

$\therefore T_2 \times \frac{4}{5} = +5g$

$\therefore T_2 = \frac{25g}{4}$ N (4)

Subs into ① $36\omega^2 = 4 \times \frac{10g}{3} + 3 \times \frac{25g}{4}$

$\therefore \omega^2 = \frac{1925}{216}$ where $g = 10$

$\therefore \omega \doteqdot 3{\cdot}0 \text{ rad.s}^{-1}$ (3)

(b) $y = \frac{u}{v}$

$\therefore \ln y = \ln u - \ln v$

$\therefore \frac{1}{y} \times \frac{dy}{dx} = \frac{1}{u} \times \frac{du}{dx} - \frac{1}{v} \times \frac{dv}{dx}$

$\therefore \frac{v}{u} \times \frac{dy}{dx} = \frac{1}{u} \times \frac{du}{dx} - \frac{1}{v} \times \frac{dv}{dx}$

$\therefore \frac{dy}{dx} = \frac{1}{u} \times \frac{du}{dx} \times \frac{u}{v} - \frac{1}{v} \times \frac{dv}{dx} \times \frac{u}{v}$

$= \frac{1}{v} \times \frac{du}{dx} - \frac{u}{v^2} \times \frac{dv}{dx}$

$= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ (3)

5(c)(i) $I_n = \int \tan^{n-2} x . \tan^2 x \, dx$

$= \int \tan^{n-2} x . (\sec^2 x - 1) dx$

$= \int \tan^{n-2} x . \sec^2 x \, dx - I_{n-2}$

Let $u = \tan x$, $du = \sec^2 x \, dx$

$\therefore I_n = \int u^{n-2} du - I_{n-2}$

$= \frac{u^{n-1}}{n-1} - I_{n-2}$

$= \frac{\tan^{n-1} x}{n-1} - I_{n-2}$ (3)

$\therefore I_3 = \left[\frac{\tan^2 x}{3-1} \right]_0^{\pi/4} - \int_0^{\pi/4} \tan x \, dx$

$= \frac{1}{2} - \int_0^{\pi/4} \frac{\sin x}{\cos x} dx$

Let $v = \cos x$, $dv = -\sin x \, dx$

$\therefore I_3 = \frac{1}{2} + \int_1^{1/\sqrt{2}} \frac{1}{v} dv$

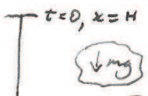
$= \frac{1}{2} + [\ln v]_1^{1/\sqrt{2}}$

$= \frac{1}{2} + \ln \frac{1}{\sqrt{2}} - \ln 1$

$= \frac{1}{2} - \frac{1}{2} \ln 2$

$= \frac{1}{2}(1 - \ln 2)$ (2)

6.
(i)



Here $m = 1$ & $F = ma$

$\therefore \cancel{ma} = g - kv$ (1)

(ii) $\therefore \frac{dv}{dt} = g - kv$

$\therefore \frac{dv}{g - kv} = dt$

$\therefore \int_0^V \frac{-kv}{g - kv} = \int_0^T -k \, dt$

$\therefore \left[\ln|g - kv| \right]_0^V = \left[-kt \right]_0^T$

$\therefore \ln|g - kV| - \ln g = -kT$

$\therefore kT = \ln\left|\frac{g}{g - kV}\right|$

$\therefore -kT = \ln\left|\frac{g - kV}{g}\right|$

$\therefore \frac{g - kV}{g} = e^{-kT}$

$\therefore g - kV = g e^{-kT}$

$\therefore kV = g(1 - e^{-kT})$

$\therefore V = \frac{g}{k}(1 - e^{-kT})$ (3)

(iii) $v \frac{dv}{dx} = g - kv$

$\therefore \frac{v \, dv}{g - kv} = dx$

$$g - kv \overline{\big) \, v \quad \frac{g}{k}} \quad -\frac{1}{k}$$
$$-kv \; -\frac{g}{k}$$
$$\frac{g}{k}$$

$\therefore \left(-\frac{1}{k} + \frac{g/k}{g - kv} \right) dv = dx$

$\therefore \int_0^V \left(-\frac{1}{k} - \frac{g}{k^2} \cdot \frac{-k}{g - kv} \right) dv = \int_0^X dx$

$\therefore \left[\frac{-v}{k} - \frac{g}{k^2} \ln|g - kv| \right]_0^V = [x]_0^X$

$\therefore \frac{-V}{k} - \frac{g}{k^2} \ln|g - kV| + \frac{g}{k^2} \ln g = X - 0$

$\therefore X = \frac{g}{k^2} \left[-\frac{kV}{g} + \ln\left|\frac{g}{g - kV}\right| \right]$ (3)

(iv) From part (ii) $\lim_{T \to \infty} (e^{-kT}) = 0$

$\therefore$ Terminal velocity $W = \frac{g}{k}$ (1)

(b)(i) $(\text{cis } 4\theta) = (\text{cis } \theta)^4$ by de Moivres' Rm

$\therefore \cos 4\theta + i \sin 4\theta = \cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4i\cos\theta\sin^3\theta + \sin^4\theta$

Equating $\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$

$= \cos^4\theta - 6\cos^2\theta(1 - \cos^2\theta) + (1 - \cos^2\theta)^2$

$= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta$

$= 8\cos^4\theta - 8\cos^2\theta + 1$ Ⓐ (3)

(ii) $\text{cis } 4\theta = \text{cis } \pi$ or $\text{cis } 3\pi$ or $\text{cis } 5\pi$ or $\text{cis } 7\pi$

$\therefore 4\theta = \pi, 3\pi, 5\pi, 7\pi$

$\therefore \theta = \frac{\pi}{4}$ or $\frac{3\pi}{4}$ or $\frac{5\pi}{4}$ or $\frac{7\pi}{4}$

$\therefore z = \text{cis } \frac{\pi}{4}$ or $\text{cis } \frac{3\pi}{4}$ or $\text{cis } \frac{5\pi}{4}$ or $\text{cis } \frac{7\pi}{4}$ (2)

(iii) Let $n = 1$ $\therefore z + \frac{1}{z} = 2\cos\theta$

$\therefore \left(z + \frac{1}{z}\right)^4 = (2\cos\theta)^4$

$\therefore \cos^4\theta = \frac{1}{16}\left(z^4 + 4z^2 + 6 + 4\frac{1}{z^2} + \frac{1}{z^4}\right)$

$= \frac{1}{16}(2\cos 4\theta + 4 \times 2\cos 2\theta + 6)$

$= \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$ (2)

OR $8\cos^4\theta = \cos 4\theta + 8\cos^2\theta + 1$ from Ⓐ

$= \cos 4\theta + 4(2\cos^2\theta - 1) + 3$

$= \cos 4\theta + 4(\cos 2\theta) + 3$

$\therefore \cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$

7 (b) (i) $LHS = \cos(k\theta - \theta) - 2\cos\theta\cos k\theta$

$= \cos k\theta\cos\theta + \sin k\theta\sin\theta - 2\cos\theta\cos k\theta$

$= \sin k\theta\sin\theta - \cos\theta\cos k\theta$

$RHS = -\cos(k\theta + \theta)$

$= -\cos k\theta\cos\theta + \sin k\theta\sin\theta$

$\therefore LHS = RHS$ (1)

(ii) Step 1 Let $n = 1$

$LHS = 1$, $RHS = \dfrac{1 - \cos\theta - \cos\theta + \cos 0}{2 - 2\cos\theta}$

$= \dfrac{2 - 2\cos\theta}{2 - 2\cos\theta}$

$= 1$

Step 2 $\therefore n = 1$ sats proposition.

Suppose prop. true for $n = k$, integer

i.e. supp. $1 + \ldots + \cos(k-1)\theta = \dfrac{1 - \cos\theta - \cos k\theta + \cos(k-1)\theta}{2 - 2\cos\theta}$

Step 3 Now $1 + \ldots + \cos k\theta = \dfrac{1 - \cos\theta - \cos k\theta + \cos(k-1)\theta}{2 - 2\cos\theta} + \cos k\theta$

$= \dfrac{1 - \cos\theta - \cos k\theta + \cos(k-1)\theta + 2\cos k\theta - 2\cos k\theta\cos\theta}{2 - 2\cos\theta}$

$= \dfrac{1 - \cos\theta + \cos k\theta - \cos(k+1)\theta}{2 - 2\cos\theta}$ (using (i))

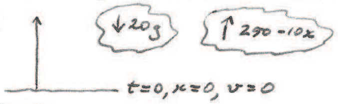
$= \dfrac{1 - \cos\theta + \cos((k+1)-1)\theta - \cos(k+1)\theta}{2 - 2\cos\theta}$

$\therefore$ prop true when $n = k+1$

Step 4 If true for $n = 1$ then true for $n = 1+1 = 2$ etc

As true for $n = 1$, $\therefore$ true for all pos integers n (4)

(c)



$F = ma$

$\therefore 20a = 250 - 10x - 20g$

$\therefore a = \frac{50}{20} - \frac{1}{2}x$

$\therefore v.\frac{dv}{dx} = \frac{5}{2} - \frac{1}{2}x$

$\therefore \int_0^v v.dv = \int_0^{10} \frac{1}{2}(5 - x)\,dx$

$\therefore \left[\frac{1}{2}v^2\right]_0^v = \left[\frac{1}{2}5x - \frac{x^2}{4}\right]_0^{10}$

$\therefore \frac{1}{2}v^2 = 25 - 25$

$\therefore v = 0$, stationary (3)

7

(a) Let $P(x) = (x-1)(x-2)Q(x) + ax + b$

Let $x = 1$ $\therefore P(1) = a + b$

But $P(x) = (x-1)Q_1(x) + 3 \Rightarrow P(1) = 3$

$\therefore a + b = 3$ ①

Similarly, let $x = 2$, $5 = 2a + b$ ②

② − ① : $\therefore a = 2$, $b = 1$

$\therefore$ remainder $2x + 1$ (2)

(d) (i) $LHS = \dfrac{1 + \sin\theta + i\cos\theta}{1 + \sin\theta - i\cos\theta} \times \dfrac{1 + \sin\theta + i\cos\theta}{1 + \sin\theta + i\cos\theta}$

$= \dfrac{1 + 2\sin\theta + \sin^2\theta - \cos^2\theta + 2i\cos\theta + 2i\sin\theta\cos\theta}{1 + 2\sin\theta + \sin^2\theta + \cos^2\theta}$

$= \dfrac{2\sin\theta + 2\sin^2\theta + (1 + \sin\theta)2i\cos\theta}{2(1 + \sin\theta)}$

$= \sin\theta + i\cos\theta = RHS$ (3)

(ii) $LHS = (\sin\theta + i\cos\theta)^n$

$= \left[\cos\left(\frac{\pi}{2} - \theta\right) + i\sin\left(\frac{\pi}{2} - \theta\right)\right]^n$

$= \cos\left(\frac{n\pi}{2} - n\theta\right) + i\sin\left(\frac{\pi n}{2} - n\theta\right)$

using de Moirre's theorem

$= RHS$ (2)

8. (a) (i) Let $\angle ADC = \alpha$

In PADC, $\angle PAD = \angle DCP = 90°$ (angle bet. tangent + radius is 90°)

$\therefore$ PADC is a cyclic quad. (opp. angles add to 180°)

In CDPQ, $\angle DCQ = \angle DBQ = 90°$ (similarly)

$\therefore$ CDPQ is a cyclic quad (similarly)

(ii) $\angle ADC = \angle BQC = \alpha$ (ext. angle of cyclic quad DBQC equal to remote int $\angle$) (2)

AD = DC (equal tangents) & BQ = QC (equal radii)

$\therefore \Delta$s ADC & BQC are isosceles (2 pairs of equal sides)

Also, as $\angle ADC = \angle BQC = \alpha$,

$\therefore \Delta ADC ||| \Delta BQC$ (2 equal angles & sides about the angles in the same ratio) (3)

(iii) $\angle QCB = \frac{180 - \alpha}{2}$ using angle sum & 2 equal angles

Also $\angle APC = 180 - \alpha$ (opp angles supp in cyclic quad)

$\therefore \angle DPC = \frac{180 - \alpha}{2}$ (kite ADCP has diag. bisect $\angle$)

$\therefore PD \parallel CB$ (corresp. $\angle$s equal) (3)

OR using ADCP as a cyclic quad, one can show that $\angle DPC = \angle CAD = \angle QCB$

$\therefore PD \parallel CB$ (corresp. $\angle$s equal)

8 (b) (i) $P\sin\frac{\theta}{2} = \sin\frac{\theta}{2} + 2\cos\theta.\sin\frac{\theta}{2} + 2\cos 2\theta.\sin\frac{\theta}{2} + 2\cos 3\theta\sin\frac{\theta}{2}$

$= \sin\frac{\theta}{2} + \sin\left(\theta + \frac{\theta}{2}\right) - \sin\left(\theta - \frac{\theta}{2}\right) + \sin\left(2\theta + \frac{\theta}{2}\right) - \sin\left(2\theta - \frac{\theta}{2}\right) + \sin\left(3\theta + \frac{\theta}{2}\right) - \sin\left(3\theta - \frac{\theta}{2}\right)$

$= \cancel{\sin\frac{\theta}{2}} + \cancel{\sin\frac{3\theta}{2}} - \cancel{\sin\frac{\theta}{2}} + \cancel{\sin\frac{5\theta}{2}} - \cancel{\sin\frac{3\theta}{2}} + \sin\frac{7\theta}{2} - \cancel{\sin\frac{5\theta}{2}}$

$= \sin\frac{7\theta}{2}$ (2)

(ii) Let $\theta = \frac{2\pi}{7}$ $\therefore$ $P\sin\frac{2\pi}{14} = \sin\pi$

$\therefore P = 0$

$\therefore 1 + 2\cos\theta + 2\cos 2\theta + 2\cos 3\theta = 0$ (2)

(iii) $\cos 2\theta = 2\cos^2\theta - 1$ & $\cos 3\theta = \cos(2\theta + \theta)$

$= \cos 2\theta\cos\theta - \sin 2\theta\sin\theta$

$= (2\cos^2\theta - 1)\cos\theta - 2\sin\theta\cos\theta.\sin\theta$

$= 2\cos^3\theta - \cos\theta - 2\cos\theta\sin^2\theta$

$= 2\cos^3\theta - \cos\theta - 2\cos\theta(1 - \cos^2\theta)$

$= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta$

$= 4\cos^3\theta - 3\cos\theta$

$\therefore P = 1 + 2\cos\theta + 2(2\cos^2\theta - 1) + 2(4\cos^3\theta - 3\cos\theta)$

$= 1 + 2\cos\theta + 4\cos^2\theta - 2 + 8\cos^3\theta - 6\cos\theta$

$= -1 - 4\cos\theta + 4\cos^2\theta + 8\cos^3\theta$

$= 8x^3 + 4x^2 - 4x - 1$ where $x = \cos\theta$

But $P = 0$ when $\theta = \frac{2\pi}{7}$ $\therefore$ $x = \cos\frac{2\pi}{7}$ & $8x^3 + 4x^2 - 4x - 1 = 0$

(3)