Sydney Girls High School

2011
TRIAL HIGHER SCHOOL
CERTIFICATE EXAMINATION

Mathematics

Extension 2

General Instructions

- Reading Time - 5 minutes
- Working time - 3 hours
- Attempt ALL questions
- ALL questions are of equal value
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Standard integrals are supplied
- Board-approved calculators may be used.
- Diagrams are not to scale
- Each question attempted should be started on a new sheet. Write on one side of the paper only.

This is a trial paper ONLY. It does not necessarily reflect the format or the contents of the 2011 HSC Examination Paper in this subject.

Name: Nancy Xie

Teacher: Mr. Kaliha

Marks

Question 1 (15 marks)

(a) Find $\int_0^1 x(5x^2-2)^4\,dx$ 3

(b) Find $\int \cot x\,dx$ 2

(c) Find $\int \frac{1}{x(x^2-1)}\,dx$ 3

(d) Using the substitution $t=\tan\frac{x}{2}$, or otherwise, evaluate $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{dx}{1-\cos x}$ 4

(e) Evaluate $\int_e^{e^2} \log_e x\,dx$ 3

Marks

Question 2 (15 marks) Start a new page

(a) Let $z = 3 + 2i$

(i) Find $\overline{z}$ 1

(ii) Find $\frac{1}{z}$ in the form $x + iy$ 2

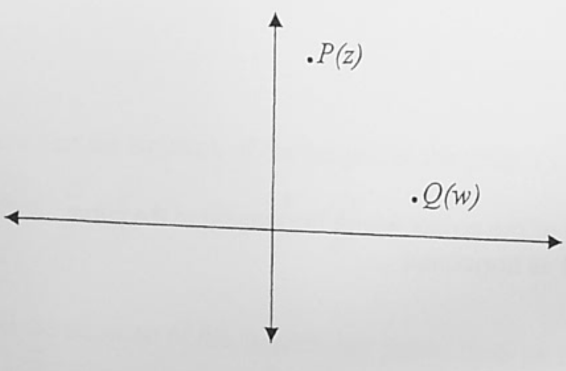
(iii) Find z^{-2} in the form $x + iy$ 1

(b) (i) Express $1-\sqrt{3}i$ in modulus-argument form. 2

(ii) Find $(1-\sqrt{3}i)^5$ in the form $x + iy$ 2

(c) Sketch the region in the complex plane where the inequalities $z \leq 2$ and $\left|\arg z\right| \leq \frac{\pi}{4}$ hold simultaneously. 2

(d) The points P and Q on the Argand diagram represent the complex numbers z and w respectively



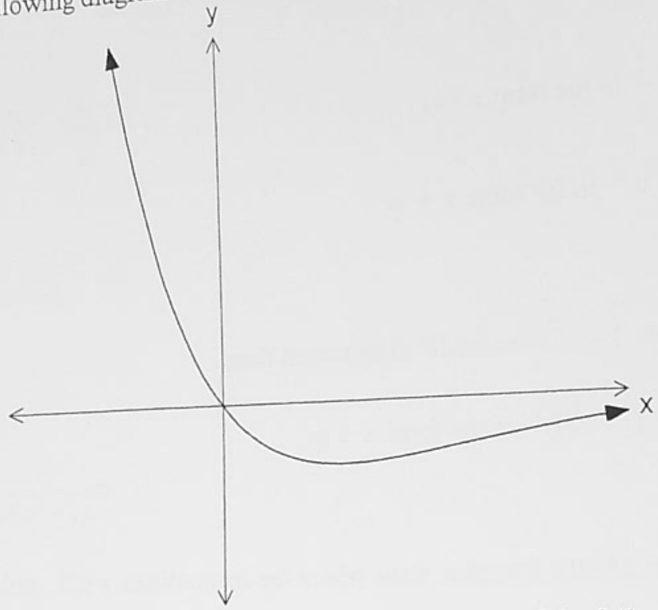
Copy the diagram and mark on it the following points:

(i) The point A representing $-z$ 1
(ii) The point B representing $2w$ 1
(iii) The point S representing $\overline{z}$ 1
(iv) The point T representing iw 1
(v) The point U representing $z + w$ 1

Marks

Question 3 (15 marks) Start a new page

(a) The following diagram shows the graph of $y = f(x)$



Draw separate one-third page sketches of the graphs of the following:

(i) $y = \left| f(x) \right|$ 1

(ii) $y = \dfrac{1}{f(x)}$ 2

(iii) $y = e^{f(x)}$ 2

(iv) $y = f\left(|x|\right)$ 2

(b) Find the coordinates of the points where the tangent to the curve $x^2 + xy + y^2 = 12$ is horizontal 3

(c) The zeros of $2x^3 - 3x^2 + 4x - 1$ are α, β and γ

Find a cubic polynomial with integer coefficients whose zeros are

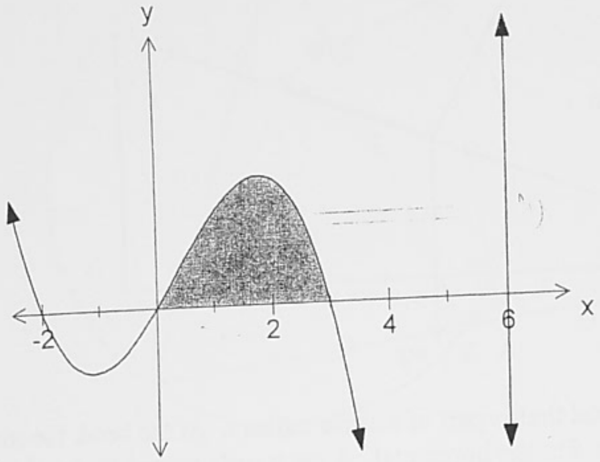
(i) $2\alpha,\ 2\beta$ and 2γ 2

(ii) $\alpha^2,\ \beta^2$ and γ^2 3

Marks

Question 4 (15 marks) Start a new page

(a) The region shaded in the diagram is bounded by the *x-axis* and the curve $y = 6x + x^2 - x^3$ 4



The shaded region is rotated about the line $x = 6$.
Find the volume generated.

(b) (i) Show that the equation of the tangent at the point $(x_1 y_1)$ on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $\frac{x_1 x}{a^2}+\frac{y_1 y}{b^2}=1$ 2

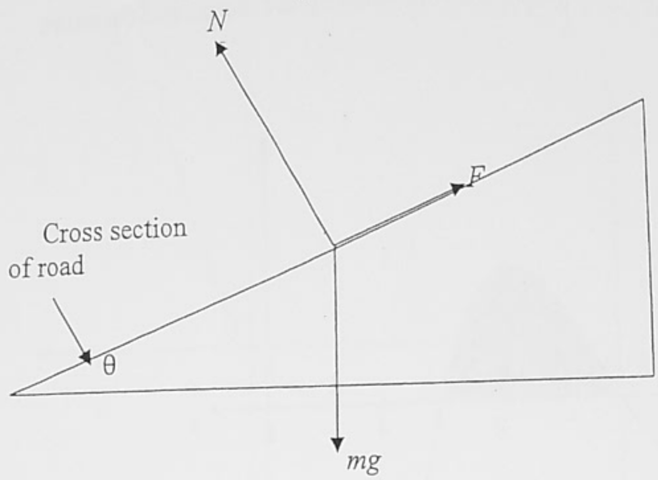
(ii) Find the equation of the tangent that passes through the point $(1,\frac{3\sqrt{3}}{2})$ on the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$ 1

(iii) Find the equation of the tangent parallel to the one in (ii) 2

(iv) Find the equation of the chord joining the points of contact of the tangents in (ii) and (iii) 1

Question 4 (continued)

(c)



A road contains a bend that is part of a circle radius r. At the bend, the road is banked at an angle θ to the horizontal. A car travels around the bend at constant speed v. Assume that the car is represented by a point of mass m, and that the forces acting on the car are the gravitational force mg, a sideways friction force F (acting up the road as drawn) and a normal reaction N to the road.

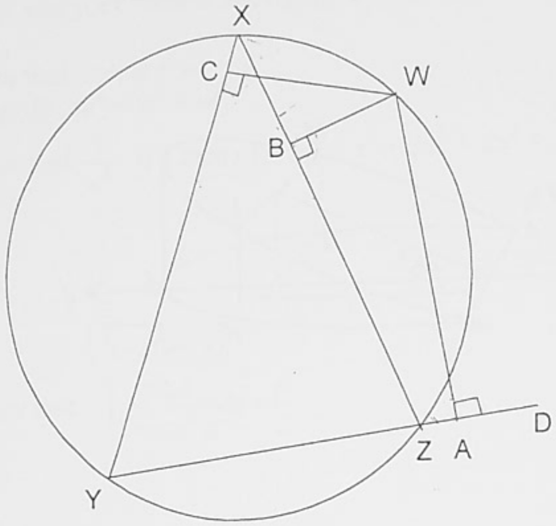
(i) By resolving the horizontal and vertical components of force, find expressions for $N\cos\theta$ and $N\sin\theta$ 3

(ii) Show that $N = \dfrac{m(v^2 + gr\cot\theta)}{r}\sin\theta$ 2

Marks

Question 5 (15 marks) Start a new page

(a)



In the diagram W, X, Y and Z are concyclic, and the points A, B, C are the perpendiculars from W to YZ produced, ZX and XY respectively.

(i) Show that $\angle WBA = \angle WZA$ 2

(ii) Show that $\angle WBC + \angle WXC = 180^0$ 2

(iii) Deduce that the points A, B and C lie in the same straight line. 2

(b) For each integer $n \geq 0$, let $I_n = \int_0^1 x^n e^x \, dx$

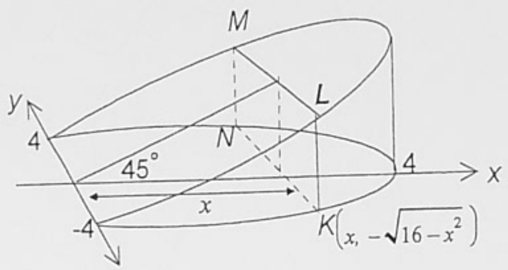
(i) Show that for $n \geq 1, I_n = e - nI_{n-1}$ 2

(ii) Hence, or otherwise, calculate I_4 2

Marks

Question 5 (continued)

(c) The base of a right cylinder is the circle in the xy-plane with centre 0 and radius 4. A wedge is obtained by cutting this cylinder with the plane through the y-axis at 45^0 to the xy-plane, as shown in the diagram.



A rectangular slice KLMN is taken perpendicular to the base of the wedge at a distance x from the y-axis.

(i) Show that the area of KLMN is given by $x\sqrt{64-4x^2}$ 2

(ii) Find the volume of the wedge. 3

Marks

Question 6 (15 marks) Start a new page

(a) Let w be the complex number satisfying $w^3 = 1$ and $Im(w) > 0$

(i) Show that $1 + w + w^2 = 0$ 2

(ii) Simplify $w^4 + w^6 + w^8$ 2

(iii) Show that $\frac{1}{w^2}$ is a zero of $P(x) = x^4 + 3x^3 + 2x^2 + x - 1$ 2

(b) (i) Show that $\int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx = \frac{\pi}{2}$ 2

(ii) By making the substitution $x = \pi - u$, 3

find $\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$

(c) (i) Show that the equation of the tangent at the point $P(ct, \frac{c}{t})$ on the hyperbola $xy=c^2$ is $x+t^2y=2ct$ 2

(ii) Find the equation of the locus of the mid point PG if G is the x intercept of the tangent in (i) 2

Marks

Question 7 (15 marks) Start a new page

(a) The curves $y=\sin x$ and $y=\cot x$ intersect at a point A whose x-coordinate is a

(i) Show that $\frac{d}{dx}(\cot x)=-\text{cosec}^2 x$ 1

(ii) Show that the curves intersect at right angles at A 3

(iii) Show that $\text{cosec}^2 a=\frac{1+\sqrt{5}}{2}$ 2

(b) The force of attraction between the earth and a communications satellite in circular orbit around it is given by $F=\frac{mgR^2}{x^2}$ where x is the distance of the satellite from the earth's centre, m is the mass of the satellite, g is gravity and R is the radius of the earth. A 300kg satellite is orbiting the earth at 3000m above the surface of the earth.
If $R = 6400$km and $g = 10ms^{-2}$ find

(i) The velocity of the satellite correct to one significant figure 2
(ii) The period of the satellite correct to the nearest minute 2
(iii) F 1

(c) (i) Differentiate $\sin^{-1} x-\sqrt{1-x^2}$ 2
(ii) Hence show that
$\int_0^a \sqrt{\frac{1+x}{1-x}}dx=\sin^{-1} a+1-\sqrt{1-a^2}$ for $0<a<1$ 2

Marks

Question 8 (15 marks) Start a new page

(a) (i) Show that $2\cos A\sin B = \sin(A+B) - \sin(A-B)$ 1

(ii) Show that $\sin\frac{\theta}{2}(1+2\cos\theta+2\cos 2\theta+2\cos 3\theta) = \sin\frac{7\theta}{2}$. 2

(iii) Show that if $\theta = \frac{2\pi}{7}$, then $1+2\cos\theta+2\cos 2\theta+2\cos 3\theta = 0$. 2

(iv) By writing $1+2\cos\theta+2\cos 2\theta+2\cos 3\theta$ in terms of $\cos\theta$ prove that $\cos\frac{2\pi}{7}$ is a solution of $8x^3+4x^2-4x-1=0$ 2

(b) Consider the function $f(x) = e^x(1-\frac{x}{8})^8$

(i) Find the turning points of the graph of $y = f(x)$. 2

(ii) Sketch the curve $y=f(x)$ and label the turning points and any asymptotes . 2

(iii) From your graph deduce that $e^x \le (1-\frac{x}{8})^{-8}$ for $x < 8$. 2

(iv) Using (iii), show that $\left(\frac{9}{8}\right)^8 \le e \le \left(\frac{8}{7}\right)^8$ 2

--- End of Exam ---