S.G.H.S - 1998 - 4U TRIALS

Question One i) Find the following integrals

[2] a) $\int \frac{2}{x^2-2x+4}.dx$

[2] b) $\int \frac{dx}{(4+x^2)^{\frac{3}{2}}}$

[2] c) $\int \tan^3 x.dx$

[3] ii) Evaluate $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{d\theta}{\sin\theta - \cos\theta + 1}$

[3] iii) If $I_n = \int_0^{\frac{\pi}{2}} \sin^n x.dx$ a) show that $I_n = \frac{n-1}{n}.I_{n-2}$

[3] b) hence evaluate $\int_0^{\frac{\pi}{2}} \sin^7 x.dx$

Question Two:

[2] i) If $z = 1 - i\sqrt{3}$, find a) $|z|$ b) $\arg z$

[4] ii) ABCD is a quadrilateral whose equal diagonals bisect each other at the origin. A if is represented by $z = 1 + i\sqrt{3}$ and $\angle AOB = 60°$,

a) find the co-ordinates of B,C,D.

b) what type of quadrilateral is ABCD

[4] iii) If $w = \frac{z+2i}{z-4}$ and w is purely imaginery, find the locus of z

[5] iv) a) Find the 5 complex solutions of $z^5 = -1$

b) Factorize $z^5 + 1$ over the real field

c) Factorize $z^5 + 1$ over the complex field

d) Show that $\cos\frac{\pi}{5} + \cos\frac{3\pi}{5} - \frac{1}{2} = 0$

Question Three

[3] i) a) Determine all the zeros of $8x^4 - 25x^3 + 27x^2 - 11x + 1$, given that it has a root of multiplicity 3

[1] b) Sketch the curve $y = 8x^4 - 25x^3 + 27x^2 - 11x + 1$ showing the of the roots

ii) If α, β, γ are the roots of $x^3 + 2x^2 - 3x + 4 = 0$

[2] a) find $\alpha^2 + \beta^2 + \gamma^2$

[2] b) find $\alpha^3 + \beta^3 + \gamma^3$

[1] c) find the equation whose roots are $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$.

[2] d) find the equation whose roots are $\frac{\alpha\beta}{\gamma}, \frac{\alpha\gamma}{\beta}, \frac{\beta\gamma}{\alpha}$.

[4] iii) If $x^3 + 3mx + n = 0$ has a double root, prove that $n^2 = -4m^3$.

Question Four

[4] i) Find the volume of the solid of revolution formed when the circle $x^2 + y^2 = 9$ is rotated about the line $x = 6$.

[4] ii) The base of a solid is the area enclosed by the curves $y = x^2$ and $x^2 + y = 8$. If each cross section perpendicular to the x axis is a semicircle, find the volume of the solid.

[3] iii) Evaluate $\int_3^4 \frac{4}{x^2 - 3x + 2} \, dx$

[4] iv) Lola is obsessed by the colour of her hair. On any given day there is an 80% chance she will change the colour of her hair for the next day. Her hair is blond 40% of the time, black 30%, red 20% and purple for the remainder.

If, today, Lola has red hair, what is the probability that

a) tomorrow her hair is red

b) tomorrow her hair is black

c) tomorrow her hair is black and on the next day it is blond.

Question Five

i) Find the constants A, B, C, D if

[6] $\sin^7\theta = A\sin 7\theta + B\sin 5\theta + C\sin 3\theta + D\sin\theta$

Hence evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \sin^7\theta . d\theta$

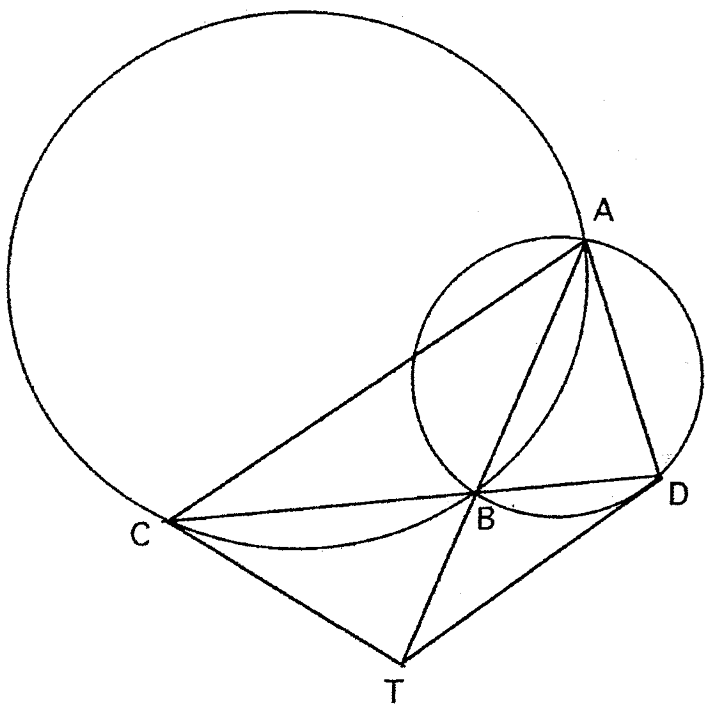
ii) Given that $(2 - i)$ is a zero of $2x^3 + mx^2 + nx + 15$, determine m and n, where m and n are real

iii) BAC, BAD are two circles such that the tangents at C and D meet at T on AB produced. If CBD is a straight line prove that:

[5] a) TCAD is a cyclic quadrilateral

b) $<TAC = <TAD$

c) $TC = TD$



Question Six

i) If $f(x) = (x+3)(x-3)^2$, sketch

a) $y = f(x)$

b) $y = |f(x)|$

[8] c) $y = \dfrac{1}{f(x)}$

d) $y^2 = f(x)$

e) $y = e^{f(x)}$

ii) Gas is escaping from a spherical balloon. Find the radius of the balloon when the rate of decrease in the volume and rate of decrease in the surface area are numerically equal.

[2]

iii) Given the curve $y = \dfrac{x-1}{x^2}$

a) State the domain of the curve

[5] b) Find any stationary points and determine their nature

c) State the range of the curve

d) Sketch the curve showing the essential features

Question Seven

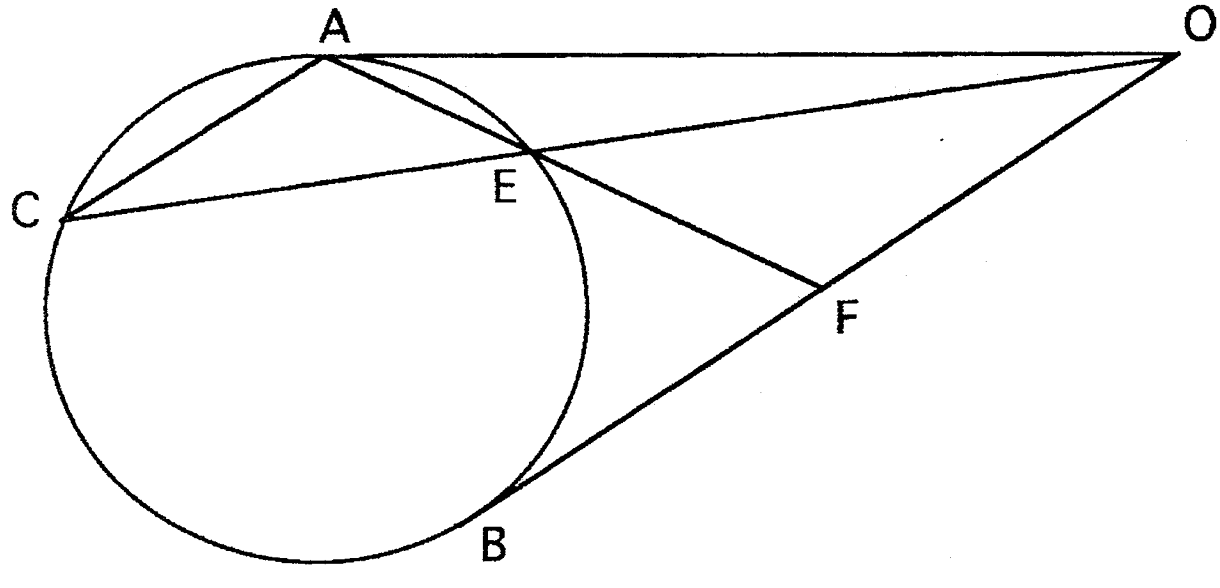
i) Two tangents OA, OB are drawn from a point O to a given circle. Through A a chord AC is drawn parallel to the other tangent OB

OC meets the circle at E

[9] a) Prove that triangles AFO, EFO are similar

b) Hence, show that $OF^2 = AF \times EF$

c) Hence, or otherwise prove that AE extended bisects OB



ii) A smooth circular disc, diameter PQ (0.26m) rotates in a horizontal plane with angular velocity 10 rad / sec. A 5 kg mass at R is connected to P

6] and Q by light inextensible strings where PR is 0.24m and QR is 0.10m.

Find the tension in each string (you may use $g = 10 \text{m.sec}^{-2}$)

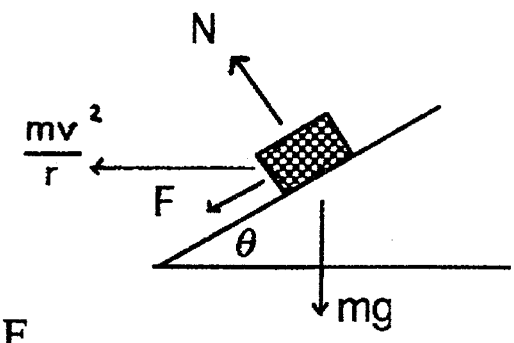
Question Eight

i) Given that tidal motion is simple harmonic, use the below information to solve the following problem

a) On a certain day, low water for a harbour occurs at 1.30am and high water at 7.45am. The corresponding depths of the water being 4m and 14m. If a fully laden ship entering the harbour requires a minimum depth of 11.5m of water, what is the earliest time the ship may enter the harbour [4]

b) If the same ship, (after being unloaded in 2 hours) floats 2.5m higher in the water, what is the latest time the same morning the ship may leave the harbour? [3]

ii) A train line is banked at an angle θ as shown on the diagram



a) If the force of circular motion is given as $\frac{mv^2}{r}$, and the force due to gravity is mg, determine the components of the frictional force F and reaction force N in terms of m, g, v, r, θ. [4]

b) In France, a very fast train turning a corner of radius 10 km at 360 km / hr causes the same frictional force up the slope as it does down the slope if it is travelling at 180 km km / hr. [4]

Find: i) the angle to nearest minute at which the rail line is banked

ii) the velocity in km / hr at which the frictional force is negligible

(You may approximate $g = 10\,m.sec^{-2}$

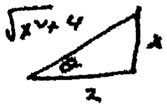
SGHS 4U 1998

1a) $\int \frac{2}{x^2-2x+4} = \int \frac{2}{(x-1)^2+(\sqrt{3})^2}$

$\frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{x-1}{\sqrt{3}}\right) + c$

b) $\int \frac{dx}{(4+x^2)^{3/2}}$ Let $x = 2\tan\theta$ $\therefore dx = 2\sec^2\theta\, d\theta$

$= \int \frac{2\sec^2\theta . d\theta}{8\sec^3\theta}$



$= \int \frac{2}{8\sec\theta} d\theta$

$\int \frac{1}{4} \cos\theta \, d\theta$

$= \frac{1}{4} \sin\theta + c$

$= \frac{1}{4} \frac{x}{\sqrt{x^2+4}} + c$

c) $\int \tan^3 x \, dx = \int \tan x(\sec^2 x - 1)\, dx$

$= \int \tan x \sec^2 x - \int \tan x \, dx$

$= \frac{1}{2}\tan^2 x + \ln(\cos x) + c$

ii) a $\int_{\pi/3}^{\pi/2} \frac{d\theta}{\sin\theta - \cos\theta + 1}$ Let $t = \tan\frac{\theta}{2}$

$dt = \frac{1}{2}\sec^2\frac{\theta}{2} d\theta$

$\theta = \frac{\pi}{3}, \ t = \sqrt{3}/2$

$\theta = \pi/2, \ t = 1$

$= \int_{\sqrt{3}/2}^{1} \frac{\frac{2dt}{1+t^2}}{\frac{2t}{1+t^2} - \frac{1-t^2}{1+t^2} + 1}$

$= \int_{\sqrt{3}/2}^{1} \frac{2dt}{2t - 1 + t^2 + 1 + t^2}$

$= \int_{\sqrt{3}/2}^{1} \frac{dt}{t+t^2}$ Let $\frac{1}{t(1+t)} = \frac{A}{t} + \frac{B}{1+t}$

$1 = A(1+t) + B(t)$

at $t=0$, $A = +1$

$t=-1$, $B = -1$

$= \int_{\sqrt{3}/2}^{1} \left(\frac{1}{t} - \frac{1}{1+t}\right) dt$

$= \left[\ln t - \ln(t+1)\right]$

$= \ln 1 - \ln 2 - \ln\frac{\sqrt{3}}{2} + \ln\left(\frac{\sqrt{3}}{2}+1\right) = \ln\left(\frac{2+\sqrt{3}}{2\sqrt{3}}\right)$

iii) $\int_0^{\pi/2} \sin^n x \, dx = \int_0^{\pi/2} \sin^{n-1}x . \sin x \, dx$

Let $u = \sin^{n-1}x$ $\quad dv = \sin x \, dx$

$du = (n-1)\cos x \sin^{n-2}x \, dx$ $\quad v = -\cos x$

$\therefore I_n = \left[-\cos x \sin^{n-1}x\right]_0^{\pi/2} + (n-1)\int_0^{\pi/2} \cos^2 x \sin^{n-2}x \, dx$

$= 0 + (n-1)\int_0^{\pi/2} (1-\sin^2 x)\sin^{n-2}x \, dx$

$= (n-1)\int_0^{\pi/2} \sin^{n-2}x \, dx - (n-1)\int_0^{\pi/2} \sin^n x \, dx$

$\therefore n\int_0^{\pi/2} \sin^n x \, dx = (n-1)\int_0^{\pi/2} \sin^{n-2}x \, dx$

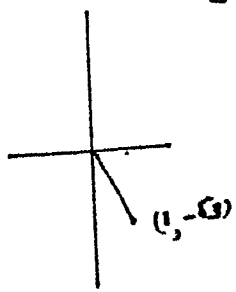
$\therefore I_n = \frac{n-1}{n} . I_{n-2}$

$\therefore \int_0^{\pi/2} \sin^7 x \, dx = \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} \int_0^{\pi/2} \sin x \, dx$

$= \frac{16}{35}\left[-\cos x\right]_0^{\pi/2}$

$= \frac{16}{35}.$

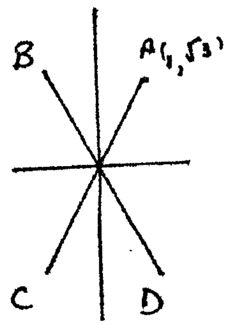
Q2.



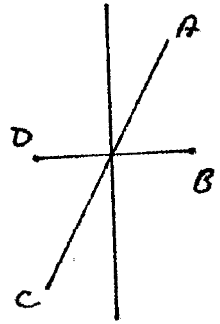
$|z| = \sqrt{1+3} = 2$

$\arg z = \tan^{-1}(-\sqrt{3}) = -\pi/3$

ii)



OR



B is a rotation of 60°

$\therefore B$ is $(1+i\sqrt{3})(\cos\pi/3 + i\sin\pi/3)$

$= (1+i\sqrt{3})(1/2 + i\sqrt{3}/2)$

$= \frac{1}{2}(1+i\sqrt{3})^2$

$= \frac{1}{2}(1-3+2i\sqrt{3})$

$= -1+i\sqrt{3}$

C is $-1-i\sqrt{3}$, D is $1-i\sqrt{3}$

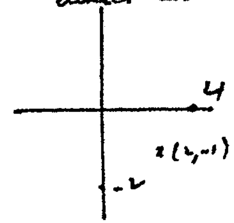
Quad is a rectangle, diag's equal & bisect each other

iii) $w = \frac{z+2i}{z-4}$

If purely imaginary

$\arg(w) = \pm\pi/2$

$\therefore$ Circle, centre $(2,-1)$ radius $\sqrt{5}$



$\therefore$ Eqn is $(x-2)^2 + (y+1)^2 = 5$

excluding $(0,-2)$ & $(4,0)$

iv) a) $z^5 = -1$

Let $z = \cos\theta + i\sin\theta$, $\therefore z^5 = \cos 5\theta + i\sin 5\theta = -1$

$\therefore \cos 5\theta = -1$, $5\theta = \pi, 3\pi, 5\pi, 7\pi, 9\pi$

& $\theta = \pi/5, 3\pi/5, \pi, 7\pi/5, 9\pi/5$

$\therefore z_1 = \cos\frac{\pi}{5} + i\sin\pi/5$

$z_2 = \cos 3\pi/5 + i\sin 3\pi/5$

$z_3 = \cos\pi + i\sin\pi$

$z_4 = \cos 3\pi/5 - i\sin 3\pi/5$

$z_5 = \cos\pi/5 - i\sin\pi/5$

c) over complex

$z^5+1 = (z+1)(z - \cos\pi/5 - i\sin\pi/5)(z - \cos\pi/5 + i\sin\pi/5).$

$(z - \cos 3\pi/5 - i\sin 3\pi/5)(z - \cos 3\pi/5 + i\sin 3\pi/5)$

b) $z^5+1 = (z+1)(z^2 - 2z\cos\pi/5 + 1)(z^2 - 2z\cos 3\pi/5 + 1)$

d) adding roots

$2\cos\pi/5 + 2\cos 3\pi/5 + \cos\pi = 0 \quad (-b/a = 0)$

$\therefore \cos\pi/5 + \cos 3\pi/5 - 1/2 = 0.$

2. i a) $f(x) = 8x^4 - 25x^3 + 27x^2 - 11x + 1$

$f'(x) = 32x^3 - 75x^2 + 54x - 11$

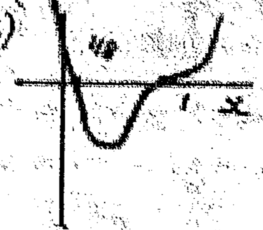
$f''(x) = 96x^2 - 150x + 54$

$= 6(16x^2 - 25x + 9)$

$= 6(16x - 9)(x - 1)$

∴ $(x-1)$ is a factor since constant term is 1.

∴ $f'(x) = (x-1)^3(8x-1)$



ii) $x^3 + 2x^2 - 3x + 4 = 0$

a) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2\Sigma\alpha\beta$

$= 4 + 6$

$= 10$

b) $\alpha^3 + 2\alpha^2 - 3\alpha + 4 = 0$

$\beta^3 + 2\beta^2 - 3\beta + 4 = 0$

$\gamma^3 + 2\gamma^2 - 3\gamma + 4 = 0$

∴ $\Sigma\alpha^3 + 2\Sigma\alpha^2 - 3\Sigma\alpha + 12 = 0$

∴ $\Sigma\alpha^3 = -2\Sigma\alpha^2 + 3\Sigma\alpha - 12$

$= -20 - 6 - 12$

$= -38$

c) $(x - \frac{1}{\alpha})(x - \frac{1}{\beta})(x - \frac{1}{\gamma}) = 0$

↓ $(\frac{1}{x} - \alpha)(\frac{1}{x} - \beta)(\frac{1}{x} - \gamma) = 0$

∴ Eqn is $\frac{1}{x^3} + \frac{2}{x^2} - \frac{3}{x} + 4 = 0$

i.e. $1 + 2x - 3x^2 + 4x^3 = 0$

i.e. Eqn is $4x^3 - 3x^2 + 2x + 1 = 0$

d) Roots are $\frac{\alpha\beta}{\gamma}, \frac{\alpha\gamma}{\beta}, \frac{\beta\gamma}{\alpha}$

or $\frac{\alpha\beta\gamma}{\gamma^2}, \frac{\alpha\beta\gamma}{\beta^2}, \frac{\alpha\beta\gamma}{\alpha^2}$

$\alpha\beta\gamma = -4$

∴ Roots are $-\frac{4}{\alpha^2}, -\frac{4}{\beta^2}, -\frac{4}{\gamma^2}$

∴ Eqn is $(x + \frac{4}{\alpha^2})(x + \frac{4}{\beta^2})(x + \frac{4}{\gamma^2}) = 0$

∴ $(\alpha^2 x + 4)(\beta^2 x + 4)(\gamma^2 x + 4) = 0$

∴ $(\sqrt{\frac{-4}{x}} - \alpha)(\sqrt{\frac{-4}{x}} - \beta)(\sqrt{\frac{-4}{x}} - \gamma) = 0$

∴ Eqn becomes

$-\frac{4}{x}\sqrt{-\frac{4}{x}} + 2(-\frac{4}{x}) - 3\sqrt{-\frac{4}{x}} + 4 = 0$

$\sqrt{-\frac{4}{x}}(-\frac{4}{x} - 3) = \frac{8}{x} - 4$

∴ $-\frac{4}{x}(\frac{16}{x^2} + \frac{24}{x} + 9) = \frac{64}{x^2} - \frac{64}{x} + 16$

∴ $-64 - 96x - 36x^2 = 64x - 64x^2 + 16x^3$

∴ $16x^3 - 28x^2 + 160x + 64 = 0$

or $4x^3 - 7x^2 + 40x + 16 = 0$.

iii) $x^3 + 3mx + n = 0$

$f'(x) = 3x^2 + 3m$

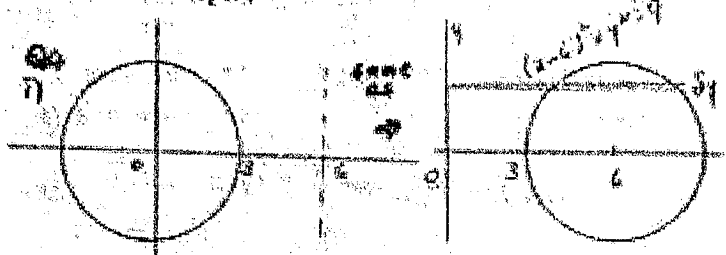
If $f'(x) = 0$, $x^2 + m = 0$, $x = \pm\sqrt{-m}$

∴ $-m\sqrt{-m} + 3m\sqrt{-m} + n = 0$

∴ $2m\sqrt{-m} = -n$

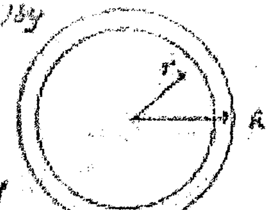
∴ $-4m^3 = n^2$

Q4 i)



$\delta V = [\pi R^2 - \pi r^2]\delta y$

$R = x_1$, $r = x_2$



∴ $\delta V = \pi(x_1^2 - x_2^2)\delta y$

$(x-6)^2 = 9 - y^2$

$x - 6 = \pm\sqrt{9 - y^2}$

∴ $x_1 = 6 + \sqrt{9 - y^2}$, $x_2 = 6 - \sqrt{9 - y^2}$

∴ $\delta V = \pi\{12\}\{2\sqrt{9 - y^2}\}\,dy$

$= 24\pi\sqrt{9 - y^2}\,dy$

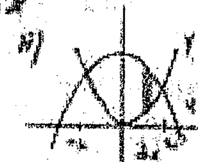
$V = \lim_{\delta y \to 0}\sum_{-3}^{3} 24\pi\sqrt{9 - y^2}\,dy$

$= \int_{-3}^{3} 24\pi\sqrt{9 - y^2}\,dy$

$= 24\pi \cdot \frac{1}{2} \cdot \pi \cdot 9$

∴ Volume $= 108\pi^2$ u³.

ii)



$\delta V = \frac{1}{2}\pi r^2 \delta x$

$r = \frac{1}{2}(x^2 - 8 + x^2)$

$= x^2 - 4$

$V = \lim_{\delta x \to 0}\sum_{0}^{2} \frac{1}{2}\pi(x^2 - 4)^2 \delta x$

$= \frac{1}{2}\int_0^2 \pi(x^4 - 8x^2 + 16)\,dx$

$= \frac{1}{2}\pi\left[\frac{x^5}{5} - \frac{8x^3}{3} + 16x\right]_0^2$

$= \frac{\pi}{2}\left[\frac{32}{5} - \frac{64}{3} + 32\right]$

$= 32\pi\left(\frac{[illegible]}{15}\right)$

Volume $= \frac{256\pi}{15}$ u³.

iii) $\int_3^4 \frac{4\,dx}{x^2 - 3x + 2}$

$\frac{4}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$

∴ $4 = A(x-2) + B(x-1)$

at $x = 2$, $B = 4$, $x = 1$, $A = -4$

$I = 4\int_3^4 \left(\frac{1}{x-2} - \frac{1}{x-1}\right)dx$

$= 4\left[\ln\left(\frac{x-2}{x-1}\right)\right]_3^4$

$= 4\left\{\ln\frac{2}{3} - \ln\frac{1}{2}\right\} = 4\ln 4/3$

v) P(Red) = 0.2

P(Black) = P(change) × P(Black)

$= 0.8 \times 3/8 = 0.3$

P(Black then Blond) $= 0.3 \times 0.8 \times 4/7$

$= \frac{24}{175}$.

i) $z^n - \left(\frac{1}{z}\right)^n \quad 2i\sin n\theta$

∴ $z - \frac{1}{z} = 2i\sin\theta$, $z^5 - \frac{1}{z^5} = 2i\sin 5\theta$

$z^3 - \frac{1}{z^3} = 2i\sin 3\theta$, $z^7 - \frac{1}{z^7} = 2i\sin 7\theta$

$\left(z - \frac{1}{z}\right)^7 = z^7 - 7z^5 + 21z^3 - 35z + \frac{35}{z} - \frac{21}{z^3} + \frac{7}{z^5} - \frac{1}{z^7}$

$(2i\sin\theta)^7 = 2i\sin 7\theta - 14i\sin 5\theta + 42i\sin 3\theta - 70i\sin\theta$

$-i128\sin^7\theta = 2i\sin 7\theta - 14i\sin 5\theta + 42i\sin 3\theta - 70i\sin\theta$

$\sin^7\theta = -\frac{1}{64}\sin 7\theta + \frac{7}{64}\sin 5\theta - \frac{21}{64}\sin 3\theta + \frac{35}{64}\sin\theta$

$\int_0^{\pi/2} \sin^7\theta = \left[\frac{1}{448}\cos 7\theta - \frac{7}{320}\cos 5\theta + \frac{7}{64}\cos 3\theta - \frac{35}{64}\cos\theta\right]_0^{\pi/2}$

$= -\frac{1}{448} + \frac{7}{320} - \frac{7}{64} + \frac{35}{64}$

$= \frac{16}{35}$

i) If $(2-i)$ is a zero of $2x^3 + mx^2 + nx + 15$

$(2+i)$ is also a zero

∴ $P(x) = (x-2+i)(x-2-i)(ax+b)$

$= (x^2 - 4x + 5)(ax+b)$

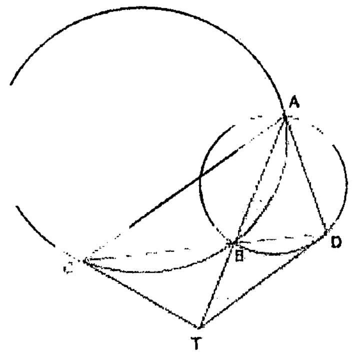
Equate x^2, $a = 2$, Equate constants $b = 3$

∴ $P(x) = (x^2 - 4x + 5)(2x+3)$

$= 2x^3 - 5x^2 - 2x + 15$

$m = -5,\ n = -2$

iii)



$TC^2 = TA.TB$ &

$TD^2 = TA.TB$

(segments of chord multiply to the square of the tangent)

∴ TC = TD

∴ ∠DCT = ∠CDT

(base angles of isos Δ)

∠TAD = ∠CDT (angle in alt segment

& ∠TAC = ∠DCT (angle in alt segment)

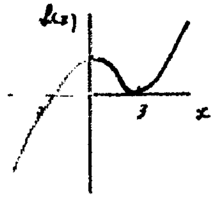
∴ ∠TAD = ∠CDT = ∠DCT = ∠TAC (shown above)

Since ∠TAD = ∠TCD & both angles lie on the line TD, these represent equal angles standing on a segment

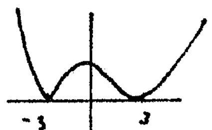
∴ TCAD is a cyclic quad'l.

Q6 i) $f(x) = (x+3)(x-3)^2$

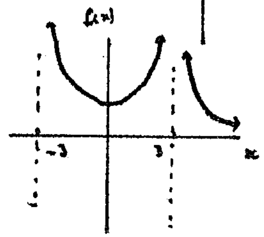
a)



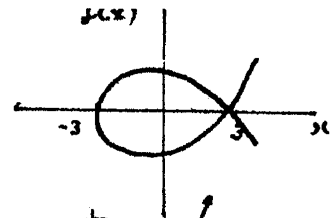
b) $y = |f(x)|$



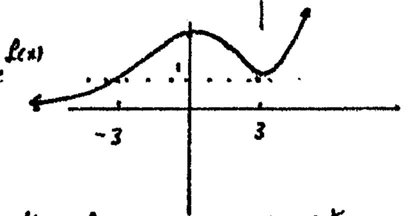
c) $y = \frac{1}{f(x)}$



d) $y^2 = f(x)$



e) $y = e^{f(x)}$



ii) $V = \frac{4}{3}\pi r^3$, $A = 4\pi r^2$

$\frac{dV}{dt} = 4\pi r^2 \cdot \frac{dr}{dt}$ & $\frac{dA}{dt} = 8\pi r \cdot \frac{dr}{dt}$

If $\frac{dV}{dt} = \frac{dA}{dt}$ $\quad 4\pi r^2 \cdot \frac{dr}{dt} = 8\pi r \cdot \frac{dr}{dt}$

∴ r = 2 or 0.

iii) $y = \frac{x-1}{x^2} = \frac{1}{x} - \frac{1}{x^2}$

a) D: all real x, $x \neq 0$

b) $y' = -\frac{1}{x^2} + \frac{2}{x^3} = \frac{-x+2}{x^3}$

$y'' = \frac{2}{x^3} - \frac{6}{x^4} = \frac{2x-6}{x^4}$

Stat pts at $y' = 0$, ∴ $x = 2$, $y = 1/4$

& $y'' = -2/16 < 0$, ∴ max at $(2, 1/4)$

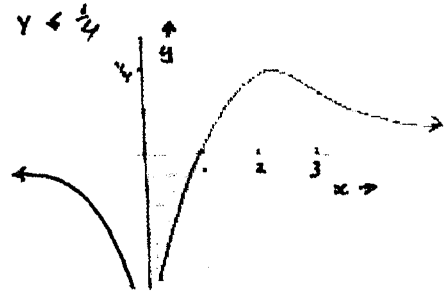
Inflex pts at $y'' = 0$ ∴ $x = 3$, $y = 2/9$

If $x < 3$, say 2, $y'' < 0$

$x > 3$, say 4, $y'' > 0$

Since y'' changes sign $(3, 2/9)$ is an inflex pt.

c) R: $y \leq \frac{1}{4}$



Q3. i(a) $f(x) = 8x^4 - 25x^3 + 27x^2 - 11x + 1$

$f'(x) = 32x^3 - 75x^2 + 54x - 11$

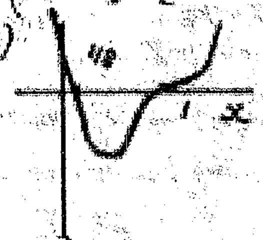
$f''(x) = 96x^2 - 150x + 54$

$= 6(16x^2 - 25x + 9)$

$= 6(16x - 9)(x - 1)$

∴ $(x-1)$ is a factor since constant term is 1.

∴ $f'(x) = (x-1)^3(8x-1)$



ii) $x^3 + 2x^2 - 3x + 4 = 0$

a) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2\Sigma\alpha\beta$

$= 4 + 6$

$= 10$

b) $\alpha^3 + 2\alpha^2 - 3\alpha + 4 = 0$

$\beta^3 + 2\beta^2 - 3\beta + 4 = 0$

$\gamma^3 + 2\gamma^2 - 3\gamma + 4 = 0$

∴ $\Sigma\alpha^3 + 2\Sigma\alpha^2 - 3\Sigma\alpha + 12 = 0$

∴ $\Sigma\alpha^3 = -2\Sigma\alpha^2 + 3\Sigma\alpha - 12$

$= -20 - 6 - 12$

$= -38$

c) $(x - \frac{1}{\alpha})(x - \frac{1}{\beta})(x - \frac{1}{\gamma}) = 0$

↓ $(\frac{1}{x} - \alpha)(\frac{1}{x} - \beta)(\frac{1}{x} - \gamma) = 0$

∴ Eqn is $\frac{1}{x^3} + \frac{2}{x^2} - \frac{3}{x} + 4 = 0$

∴ $1 + 2x - 3x^2 + 4x^3 = 0$

i.e. Eqn is $4x^3 - 3x^2 + 2x + 1 = 0$

d) Roots are $\frac{\alpha\beta}{\gamma}, \frac{\alpha\gamma}{\beta}, \frac{\beta\gamma}{\alpha}$

or $\frac{\alpha\beta\gamma}{\gamma^2}, \frac{\alpha\beta\gamma}{\beta^2}, \frac{\alpha\beta\gamma}{\alpha^2}$

$\alpha\beta\gamma = -4$

∴ Roots are $\frac{-4}{\alpha^2}, \frac{-4}{\beta^2}, \frac{-4}{\gamma^2}$

∴ Eqn is $(x + \frac{4}{\alpha^2})(x + \frac{4}{\beta^2})(x + \frac{4}{\gamma^2}) = 0$

∴ $(\alpha^2 x + 4)(\beta^2 x + 4)(\gamma^2 x + 4) = 0$

∴ $(\sqrt{\frac{-4}{x}} - \alpha)(\sqrt{\frac{-4}{x}} - \beta)(\sqrt{\frac{-4}{x}} - \gamma) = 0$

∴ Eqn becomes

$-\frac{4}{x}\sqrt{-\frac{4}{x}} + 2(-\frac{4}{x}) - 3\sqrt{-\frac{4}{x}} + 4 = 0$

$\sqrt{-\frac{4}{x}}(-\frac{4}{x} - 3) = \frac{8}{x} - 4$

∴ $-\frac{4}{x}(\frac{16}{x^2} + \frac{24}{x} + 9) = \frac{64}{x^2} - \frac{64}{x} + 16$

∴ $-64 - 96x - 36x^2 = 64x - 64x^2 + 16x^3$

∴ $16x^3 - 28x^2 + 160x + 64 = 0$

∴ $16x^3 - 28x^2 + 160x + 16 = 0$.

or $4x^3 - 7x^2 + 40x + 16 = 0$

iii) $x^3 + 3mx + n = 0$

$f'(x) = 3x^2 + 3m$

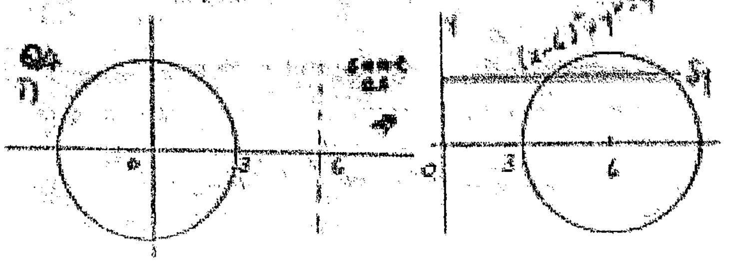
If $f'(x) = 0$ $x^2 + m = 0$, $x = \pm\sqrt{-m}$

∴ $-m\sqrt{-m} + 3m\sqrt{-m} + n = 0$

∴ $2m\sqrt{-m} = -n$

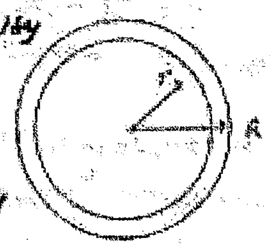
∴ $-4m^3 = n^2$

Q4 i)



$\delta V = (\pi R^2 - \pi r^2)\delta y$

$R = x_1$ $r = x_2$



∴ $\delta V = \pi(x_1^2 - x_2^2)\delta y$

$(x-6)^2 = 9 - y^2$

$x - 6 = \pm\sqrt{9 - y^2}$

∴ $x_1 = 6 + \sqrt{9 - y^2}$, $x_2 = 6 - \sqrt{9 - y^2}$

∴ $\delta V = \pi\{12\}\{2\sqrt{9 - y^2}\}\delta y$

$= 24\pi\sqrt{9 - y^2}\,\delta y$

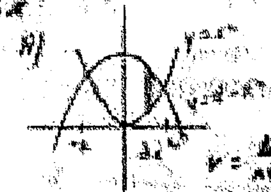
$V = \lim_{\delta y \to 0} \sum 24\pi\sqrt{9 - y^2}\,\delta y$

$= \int_{-3}^{3} 24\pi\sqrt{9 - y^2}\,dy$

$= 24\pi \cdot \frac{1}{2}\pi \cdot 9$

∴ Volume $= 108\pi^2$ u^3.

ii)



$\delta V = \frac{1}{2}\pi r^2 \delta x$

$r = \frac{1}{2}(x^2 - 8 + x^2)$

$= x^2 - 4$

$V = \lim_{\delta x \to 0} \sum_{x=-2}^{2} \pi(x^2 - 4)^2 \delta x$

$= \int_{-2}^{2} \pi(x^4 - 8x^2 + 16)\,dx$

$= 2\pi\left[\frac{x^5}{5} - \frac{8x^3}{3} + 16x\right]_0^2$

$= \pi\left[\frac{32}{5} - \frac{64}{3} + 32\right]$

$= 32\pi\left(\frac{3 - 10 + 15}{15}\right)$

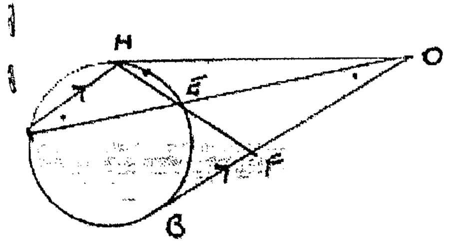
Volume $= \frac{256\pi}{15}$ u^3.

iii) $\int_3^4 \frac{4\,dx}{x^2 - 3x + 2}$

$\frac{4}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$

∴ $4 = A(x-2) + B(x-1)$

when $x = 2$, $B = 4$, $x = 1$, $A = -4$



a) $\angle OAF = \angle ACO$ ($\angle$ in alt. segment)

$\angle ACO = \angle COB$ (alt $\angle$'s, $AC \parallel BO$)

$\therefore \angle OAF = \angle COF$

in Δ's AOF, EOF

$\angle AFO = \angle EFO$ (common)

$\angle OAE = \angle EOF$ (proven above)

$\therefore \angle AOF = \angle FEO$ (3rd angle of Δ)

$\therefore \Delta FAO \ ||| \ \Delta FOE$ (equiangular)

$\therefore \frac{FA}{FO} = \frac{FO}{FE} = \frac{AO}{OE}$ (sides in prop'n)

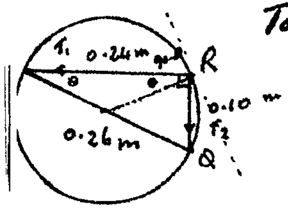
$\therefore OF^2 = EF.AF$

Now $AF.FE = FB^2$ (products of secant & tangent)

$\therefore OF^2 = FB^2$

i.e. $OF = FB$

$\therefore AE$ extended, bisects OB.



$\sin\theta = \frac{10}{26} = \frac{5}{13}$

$\cos\theta = \frac{12}{13}$

Tangentially:

$T_1\cos(90-\theta) = T_2\cos\theta$

$\therefore T_1\sin\theta = T_2\cos\theta$

$\therefore T_1 \cdot \frac{5}{13} = T_2 \cdot \frac{12}{13}$

& $5T_1 = 12T_2$

Normally:

$T_1\cos\theta + T_2\cos(90-\theta) = mr\omega^2$

$\therefore T_1 \cdot \frac{12}{13} + T_2 \cdot \frac{5}{13} = 5(\cdot 13)(10)^2$

$\therefore 12T_1 + 5T_2 = 65 \times 13$

sub $T_1 = \frac{12T_2}{5}$

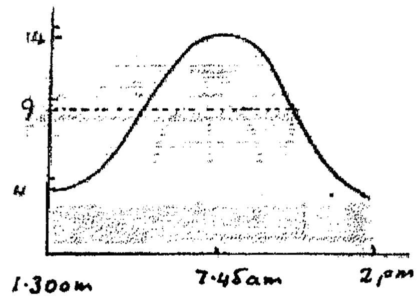
$\therefore \frac{144T_2}{5} + 5T_2 = 65 \times 13$

$169T_2 = 5^2 \times 13^2$

$\therefore T_2 = 25\,N$

& $T_1 = 60\,N$

8 i)



$T = 12{\cdot}5$ hrs $\quad \frac{2\pi}{n}$, $\therefore n = \frac{4}{2}$

From graph $\quad X = 9 - 5\cos\frac{4\pi t}{25}$

a) at $X = 11{\cdot}5$,

$11{\cdot}5 = 9 - 5\cos\frac{4\pi t}{25}$

$-\frac{1}{2} = \cos\frac{4\pi t}{25}$

$\therefore \frac{4\pi t}{25} = \frac{2\pi}{3}, \frac{4\pi}{3}$ but 1st only required

$\therefore t = \frac{2}{3} \times \frac{25}{4} = \frac{25}{6}$

$\therefore$ Earliest entry is 1·30 + 4·10 hrs

$\therefore$ time is 5·40 am

b) Requires only 9m to leave

$9 = 9 - 5\cos\frac{4\pi t}{25}$

$\therefore 5\cos\frac{4\pi t}{25} = 0$

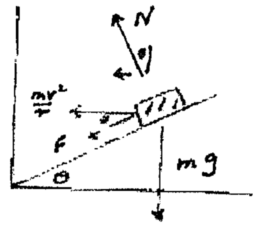
$\frac{4\pi t}{25} = \frac{3\pi}{2}$ ($\neq \frac{\pi}{2}$ but 2nd req'd)

& $t = \frac{3}{2} \times \frac{25}{4} = \frac{75}{8} = 9\frac{3}{8}$ hrs

Latest time is

1·30 + 9 hrs $22\frac{1}{2}$ mins

$= 10{\cdot}52\frac{1}{2}$ am



Vertically

$N\cos\theta - F\sin\theta = mg$ (1)

Horizontally

$N\sin\theta + F\cos\theta = \frac{mv^2}{r}$ (2)

From (1) $N\cos^2\theta - F\sin\theta\cos\theta = mg\cos\theta$

" (2) $N\sin^2\theta + F\sin\theta\cos\theta = \frac{mv^2}{r}\sin\theta$

adding $N = mg\cos\theta + \frac{mv^2}{r}\sin\theta$

From (1) $-N\cos\theta\sin\theta + F\sin^2\theta = -mg\sin\theta$

" (2) $N\sin\theta\cos\theta + F\cos^2\theta = \frac{mv^2}{r}\cos\theta$

adding $F = \frac{mv^2}{r}\cos\theta - mg\sin\theta$

i) $V = 360$ km/hr $\to 100\ ms^{-1}$

$\frac{1}{2}V = 180$ km/hr $\to 50\ ms^{-1}$

$|F_1| = |F_2|$

$\therefore m\left(\frac{100^2}{10\,000}\cos\theta - 10\sin\theta\right) = m\left|\frac{50^2}{10000}\cos\theta - 10\sin\theta\right|$

$\therefore \cos\theta - 10\sin\theta = -\frac{\cos\theta}{4} + 10\sin\theta$

$\therefore \frac{5}{4}\cos\theta = 20\sin\theta$

$\tan\theta = \frac{5}{80} = \frac{1}{16}$ & $\theta \doteqdot 3°35'$

ii) If $F = 0$ $\frac{mv^2}{r}\cos\theta - mg\sin\theta = 0$

$\therefore v^2\cos\theta = rg\sin\theta$

$v = \sqrt{rg\tan\theta}$

$= \sqrt{10000 \cdot 10 \cdot \frac{1}{16}}$

$= 25\sqrt{10}$

& $v = 25\sqrt{10}\ ms^{-1}$

$= 90\sqrt{10}$ km/hr