THE SCOTS COLLEGE

YEAR 12 MATHEMATICS EXTENSION 2

HSC TRIAL

AUGUST 2008

General Instructions

- All questions are of equal value
- Working time - 3 hours + 5 minutes reading time.
- Write using blue or black pen
- Board approved calculators may be used
- Start a new booklet for each question
- All necessary working should be shown in every question
- A Standard Integrals Table is attached

TOTAL MARKS: 120

WEIGHTING: 40 %

QUESTION 1 **[15 MARKS]** **MARKS**

a. If $z = 1 + i$, find:

(i) $|z|$ **[1]**

(ii) $\arg z$ **[1]**

(iii) z^{-6} in the form $x + iy$ **[2]**

b. Solve the equation for z **[3]**

$$z\overline{z} + 2iz = 12 + 6i$$

c. What is the locus in the Argand Diagram of the point Z which represents the complex number z where: **[2]**

$$z\overline{z} - 2(z + \overline{z}) = 5$$

d. The origin and the points representing the complex numbers z, $\frac{1}{z}$ and $z + \frac{1}{z}$ are joined to form a quadrilateral. Write down the conditions for z so that the quadrilateral will be a:

(i) rhombus **[1]**

(ii) square **[2]**

e. Prove by induction that, for all integers $n \geq 1$, **[3]**

$$(\cos\theta - i\sin\theta)^n = \cos(n\theta) - i\sin(n\theta)$$

QUESTION 2 **[15 MARKS]** **START A NEW BOOKLET** **MARKS**

a. If $f(x)=(x+2)(x-1)$, sketch the graphs of the following functions on separate diagrams.

(i) $y=f(x)$ **[1]**

(ii) $y=\dfrac{1}{|f(x)|}$ **[2]**

(iii) $y=\log_e(f(x))$ **[2]**

(iv) $y^2=f(x)$ **[2]**

b. **(i)** By using implicit differentiation, state where $\dfrac{dy}{dx}$ is undefined for $y^2=-x^2(x+2)(x-1)$. **[2]**

(ii) Hence or otherwise, sketch the curve. **[2]**

c. Let $f(x)=x-2+\dfrac{3}{x+2}$

(i) Find the points for which $f(x)=0$. **[1]**

(ii) Find the asymptotes. **[2]**

(iii) Sketch the curve. Show **all** asymptotes and the x and y intercepts. (There is no need to find or label stationary points.) **[1]**

QUESTION 3 **[15 MARKS]** **START A NEW BOOKLET** **MARKS**

a. Evaluate $\int_0^{\frac{\pi}{4}} x\sin 2x\, dx$ **[3]**

b. Find $\int x\sqrt{1-x}\, dx$ **[2]**

c. Find $\int \frac{1}{x\left(1+x^2\right)}\, dx$ **[3]**

d. By completing the square and using the table of Standard Integrals, find $\int \frac{dx}{\sqrt{x^2-4x+1}}$ **[2]**

e. Explain why the following integral cannot be evaluated. **[1]**

$$\int_0^5 \frac{1}{3-x}\, \mathrm{dx}$$

f. Evaluate $\int_0^{\pi/3} \frac{\tan x}{1+\cos x}\, dx$, using the substitution $t=\tan\frac{x}{2}$. **[4]**

QUESTION 4 [15 MARKS] **START A NEW BOOKLET** **MARKS**

a. For the ellipse $x^2+4y^2=100$

(i) Write down the eccentricity, the co-ordinates of the foci and the equations of the directrices. **[3]**

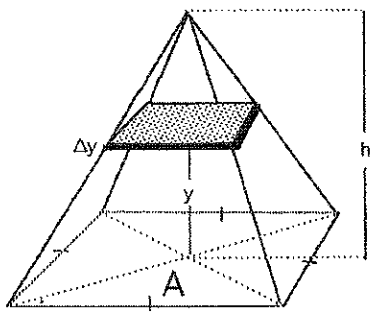
(ii) Sketch the graph of the ellipse showing the above features. **[1]**

(iii) Find the equation of the tangent and normal to the ellipse at the point P(8,3). **[3]**

(iv) The normal at P meets the major axis at G. A point K lies on the tangent which passes through the point P(8,3). A perpendicular from K passes through the origin O. Prove that PG × OK is equal to the square of the length of the semi-minor axis. **[2]**

b. A particle is projected from a point on a straight line with velocity $u\ ms^{-1}$ and moves in such a way that when it has travelled a distance of x metres it has a velocity of $v=\dfrac{u}{4+ux}ms^{-1}$. Prove that the acceleration of the particle is $-v^3 ms^{-2}$. **[2]**

c. One of the largest pyramids in Egypt is approximately 150m high and has a square base with a base area of approximately 50,000m². The diagram below shows a square based pyramid with a base area *A* and height *h*. The thickness of the cross section at height *y* is Δy.



(i) Show that the area of the cross section at height *y* can be represented as:

$$A\times\left(\frac{h-y}{h}\right)^2$$ **[1]**

(ii) Find the volume of the pyramid by using the slicing technique. **[3]**

QUESTION 5 **[15 MARKS]** **START A NEW BOOKLET** **MARKS**

a. Let $I_n = \int_0^{\pi} x^n \sin x \, dx$, where n is a positive integer.

(i) Show that $I_n = \pi^n - n(n-1)I_{n-2}$, for $n \geq 2$ **[3]**

(ii) Hence evaluate I_5 **[3]**

b. Find, by the method of cylindrical shells, the volume of the solid generated when the region bounded by the curve $y = x^2 + 1$, the line $x = 2$ and the co-ordinate axes is rotated about the line $x = 3$. **[5]**

c. Let θ be a real number and consider $(\cos\theta + i\sin\theta)^3$

(i) Prove $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$ **[3]**

(ii) Find a similar expression for $\sin 3\theta$ **[1]**

QUESTION 6 **[15 MARKS]** **START A NEW BOOKLET**

a. Let α, β, δ be the roots of the equation $x^3 + qx + r = 0$, where q and r are integers. Write down, in terms of q and r, the cubic equation whose roots are:

(i) $\alpha^{-1},\ \beta^{-1},\ \delta^{-1}$ **[2]**

(ii) $\alpha^2,\ \beta^2,\ \delta^2$ **[2]**

b. Consider the following statements about a polynomial $P(x)$.

Indicate whether each of the following statements is true or false. Give reasons for your answer.

(i) If $P(x)$ is even, then $P'(x)$ is odd. **[2]**

(ii) If $P'(x)$ is even, then $P(x)$ is odd. **[2]**

c. **(i)** Evaluate $\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}\,dx$ **[2]**

(ii) Show that for $n \geq 2$ and $0 \leq x \leq \frac{1}{2}$, then $1 \geq 1 - x^n \geq 1 - x^2$ **[2]**

(iii) If $n \geq 2$, explain carefully why $\frac{1}{2} \leq \int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^n}} \leq \frac{\pi}{6}$ **[3]**

QUESTION 7 **[15 MARKS]** **START A NEW BOOKLET** **MARKS**

a. Consider a sequence of numbers a_1, a_2, a_3 ... where $a_1 = 2$, $a_2 = 3$ and $a_n = 3a_{n-1} - 2a_{n-2}$ for all $n \geq 3$. Use mathematical induction to prove that $a_n = 2^{n-1} + 1$ for all $n \geq 1$. **[5]**

b. A particle is projected from a height H above a horizontal plane with speed V at an angle of elevation θ to the horizontal.

(i) If the range of the particle in the horizontal plane is R, show that $gR^2 \sec^2\theta = 2V^2(R\tan\theta + H)$. **[4]**

(ii) If R_1 is the maximum value of R and θ_1 is the corresponding value of θ, prove that $R_1 = \frac{v}{g}\sqrt{v^2 + 2gH}$ and $\theta_1 = \tan^{-1}\left(\frac{v^2}{gR}\right)$ **[4]**

(iii) Show that $\tan 2\theta_1 = \frac{R_1}{H}$ **[2]**

QUESTION 8 **[15 MARKS]** **START A NEW BOOKLET**

a. Let $m = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$

(i) Prove that $1 + m + m^2 + \ldots + m^6 = 0$ **[2]**

(ii) The complex number $\alpha = m + m^2 + m^4$ is a root of the quadratic equation $x^2 + ax + b = 0$, where a and b are real. The second root of the quadratic equation $x^2 + ax + b = 0$ is β. Express β in terms of positive powers of m. Justify your answer. **[2]**

(iii) Find the values of the coefficients a and b. **[2]**

(iv) Deduce that $\sin\frac{2\pi}{7} + \sin\frac{3\pi}{7} - \sin\frac{\pi}{7} = \frac{\sqrt{7}}{2}$ **[2]**

b. Given that $p + q + r = 1$ and $p + q + r \geq 3\sqrt[3]{pqr}$ (where p, q, r are positive real numbers):

(i) Prove that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \geq 9$ **[4]**

(ii) Hence, or otherwise, show $\left(\frac{1}{p} - 1\right)\left(\frac{1}{q} - 1\right)\left(\frac{1}{r} - 1\right) \geq 8$ **[3]**

END OF EXAMINATION