SCEGGS Darlinghurst

2010

HIGHER SCHOOL CERTIFICATE
TRIAL EXAMINATION

Mathematics Extension 2

This is a TRIAL PAPER only and does not necessarily reflect the content or format of the Higher School Certificate Examination for this subject.

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 120

- Attempt Questions 1–8
- All questions are of equal value

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Attempt Questions 1–8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (15 marks)

(a) Find $\int \cos^3 x \sin x \, dx$ **2**

(b) Find $\int \frac{1}{1+e^{-x}} \, dx$ **2**

(c) Evaluate $\int_0^1 \frac{x^2}{\sqrt{1-x^2}} \, dx$ **3**

(d) Use the substitution $x = \sec\theta$ to evaluate $\int_1^{\sqrt{2}} \frac{1}{x\sqrt{x^2-1}} \, dx$ **3**

(e) (i) Express $\frac{3}{(x+1)(x^2+2)}$ in the form $\frac{a}{x+1} + \frac{bx+c}{x^2+2}$, where a, b and c are constants. **3**

(ii) Hence find $\int \frac{3}{(x+1)(x^2+2)} \, dx$. **2**

End of Question 1

Marks

Question 2 (15 marks) Use a SEPARATE writing booklet.

(a) Let $z = 2 + 3i$ and $w = \bar{z}$. Find, in the form $x + iy$, where x and y are real:

(i) wz **1**

(ii) $\dfrac{z}{w}$. **1**

(b) (i) Find the square roots of $-3 + 4i$ in the form $a + ib$ where a and b are real. **3**

(ii) Hence, solve the equation $(1 + i)z^2 - z - i = 0$. **3**

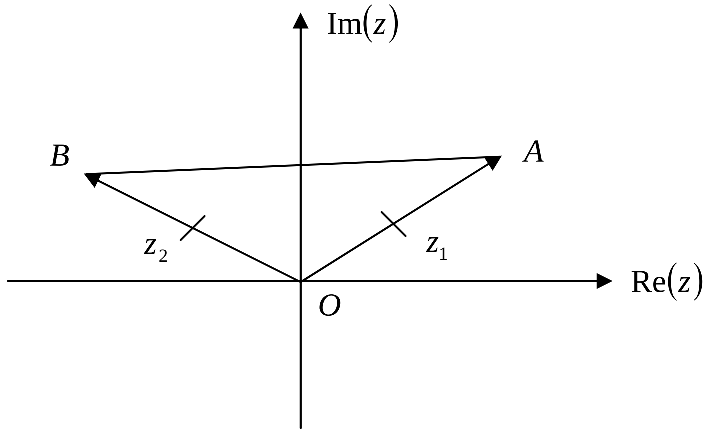
(c) Sketch the region in the complex plane where the inequalities $z + \bar{z} < 8$, $|z| \geq 4$ and $|\arg z| < \dfrac{\pi}{3}$ hold simultaneously. **3**

Question 2 continues on page 4

Marks

Question 2 (continued)

(d) In the Argand diagram, vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ represent the complex numbers z_1 and z_2 respectively.



Given that ΔAOB is isosceles and $\angle BOA = \frac{2\pi}{3}$:

(i) find an expression for z_2 in terms of z_1 **1**

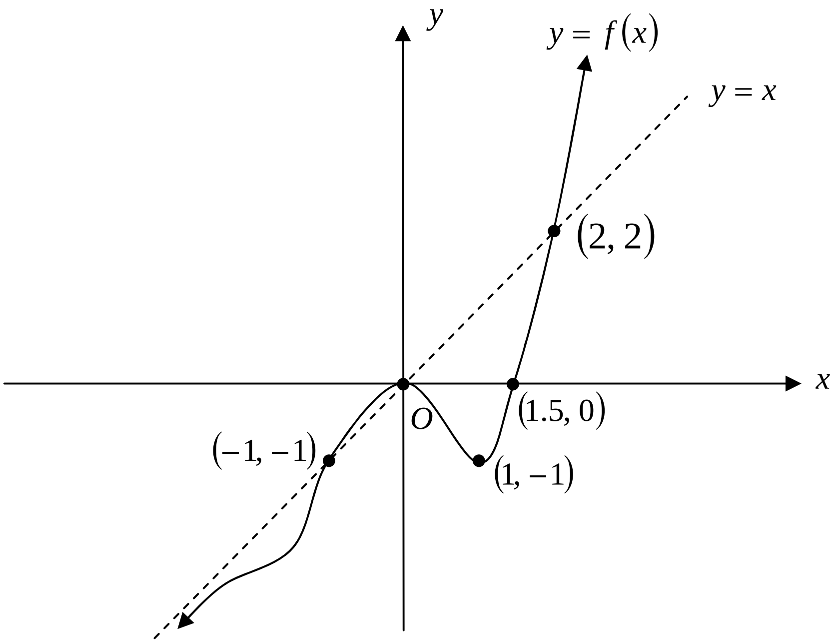
(ii) show that $(z_1 + z_2)^2 = z_1 z_2$. **3**

End of Question 2

Marks

Question 3 (15 marks) Use a SEPARATE writing booklet.

(a) The diagram shows $y = f(x)$. The line $y = x$ is an asymptote.



Draw separate one-third page sketches of the graphs of the following.
Clearly label important features.

(i) $y = f(-x)$ **1**

(ii) $y = f(x+2)$ **1**

(iii) $y = \sqrt{f(x)}$ **2**

(iv) $y = x.\,f(x)$ **2**

Question 3 continues on page 6

Marks

Question 3 (continued)

(b) The equation $x^3 + 4x^2 + 2x - 1 = 0$ has roots α, β and γ.

(i) Evaluate $\alpha^2 + \beta^2 + \gamma^2$ **2**

(ii) Evaluate $\alpha^3 + \beta^3 + \gamma^3$ **2**

(iii) Find a cubic polynomial with integer coefficients whose roots are α^2, β^2 and γ^2. **2**

(c) Find the equation of the tangent to the curve defined by $x^2 - xy + y^2 = 5$ at the point $(2, -1)$. **3**

End of Question 3

Marks

Question 4 (15 marks) Use a SEPARATE writing booklet.

(a) The equation $x^3 - 3x^2 - 9x + k = 0$ has a double root. **2**

Find the possible values of k.

(b) Use the substitution $t = \tan\dfrac{x}{2}$ to find $\displaystyle\int \frac{1}{5 + 4\cos x + 3\sin x}\,dx$ **3**

(c) (i) By completing the square, show that $4x^2 + 9y^2 + 24x - 36y + 36 = 0$ represents an ellipse in the form $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$ **1**

(ii) Find the eccentricity, e. **1**

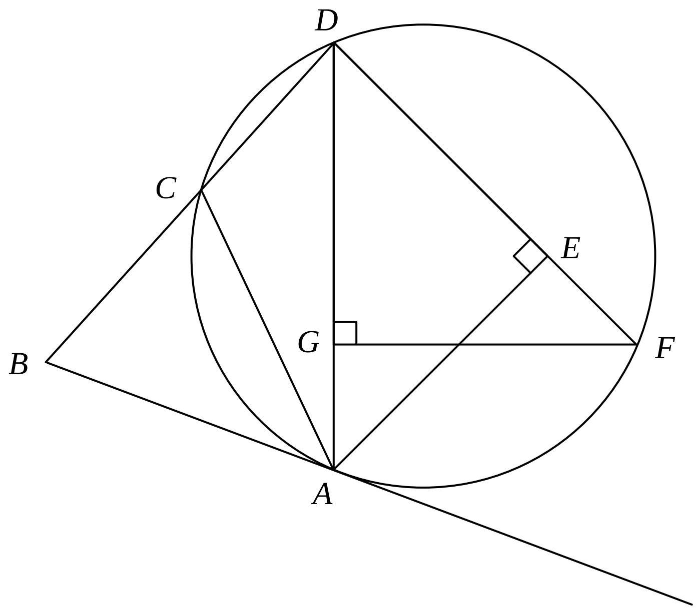
(iii) Sketch the ellipse showing the centre, the foci and the directrices. **3**

Question 4 continues on page 8

Marks

Question 4 (continued)

(d)



In the diagram given, *BA* is a tangent to the circle at *A* and the secant *BD* cuts the circle at *C*.

DA and *DF* are two chords such that *FG* and *AE* are perpendicular to *DA* and *DF* respectively.

Copy the diagram.

(i) Prove that $\angle ACB = \angle BAD$. **2**

(ii) Explain why *AGEF* is a cyclic quadrilateral with diameter *AF*. **1**

(iii) Prove that $\angle AGE = \angle ACD$. **2**

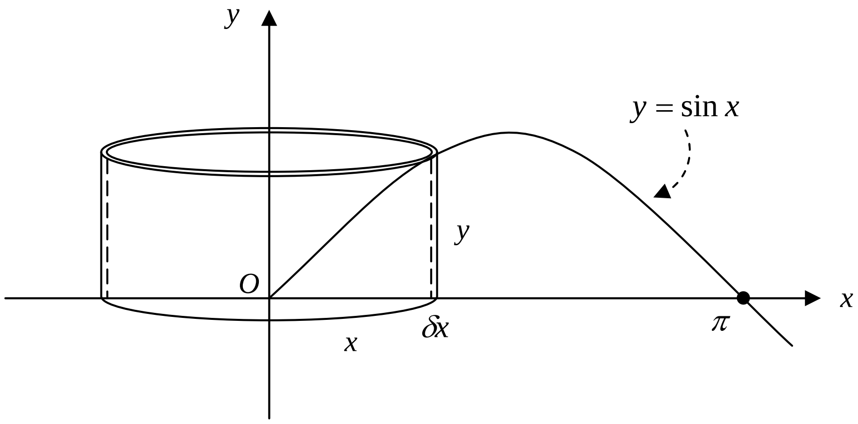
End of Question 4

Marks

Question 5 (15 marks) Use a SEPARATE writing booklet.

(a) Sebastian, a mathematically-minded sculptor, decided to make a series of pieces modelled on volumes formed by mathematical curves.

His first piece entitled "Give me a Sine" was formed by taking the area under the curve $y = \sin x$ between $x = 0$ to $x = \pi$ and rotating it about the y-axis.



(i) Using the method of cylindrical shells, show that the volume, V, of the resulting solid of revolution is given by **2**

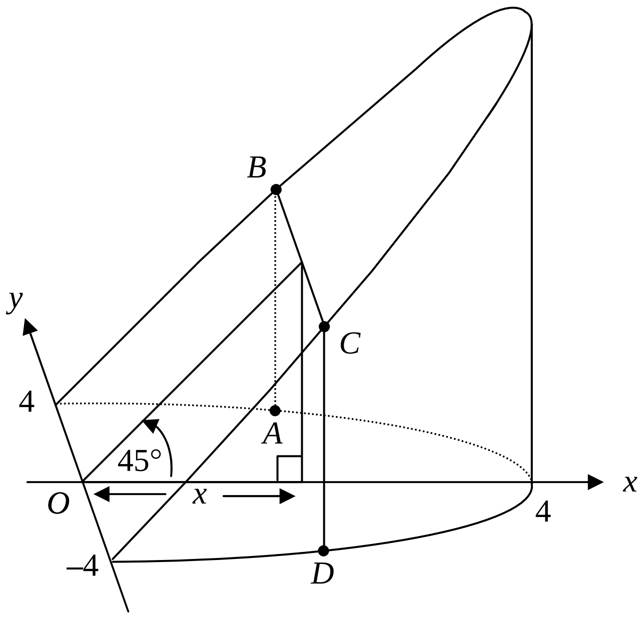
$$V = 2\pi \int_0^{\pi} x \sin x \ \ dx$$

(ii) Use integration by parts to find the exact volume. **3**

Question 5 continues on page 10

Question 5 (continued)

(b) Sebastian's second sculpture, "The Wedge" was obtained by cutting a right cylinder of radius 4 units at $45°$ through a diameter of its base.



A rectangular slice *ABCD*, of thickness δx, is taken perpendicular to the base of the wedge at a distance *x* from the *y*-axis.

(i) Show that the area of *ABCD* is given by $2x\sqrt{16 - x^2}$. **2**

(ii) Find the exact volume of the wedge. **3**

(c) (i) Prove that $\cos(A - B)x - \cos(A + B)x = 2\sin Ax \sin Bx$. **1**

(ii) Using the above result, show that the equation $\sin 3x \sin x = 2\cos 2x + 1$ can be written as a quadratic equation in terms of $\cos 2x$. **2**

(iii) Hence find the general solution of $\sin 3x \sin x = 2\cos 2x + 1$. **2**

End of Question 5

Marks

Question 6 (15 marks) Use a SEPARATE writing booklet.

(a) In preparation for a school formal, a committee of three is to be chosen from four Year 12 Prefects and n Year 12 non-Prefects $(n \geq 2)$.

(i) Show that the number of possible committees containing exactly one Prefect is $2n(n-1)$. **1**

(ii) Find the number of possible committees containing exactly two Prefects. **1**

(iii) Deduce that the probability P of the committee containing either one or two Prefects is **1**

$$P = \frac{12n}{(n+4)(n+3)}$$

(b) For each integer $n \geq 0$, let $I_n = \int_1^2 x(\ln x)^n \, dx$.

(i) Show that for $n \geq 1$, **2**

$$I_n = 2(\ln 2)^n - \frac{n}{2} I_{n-1}$$

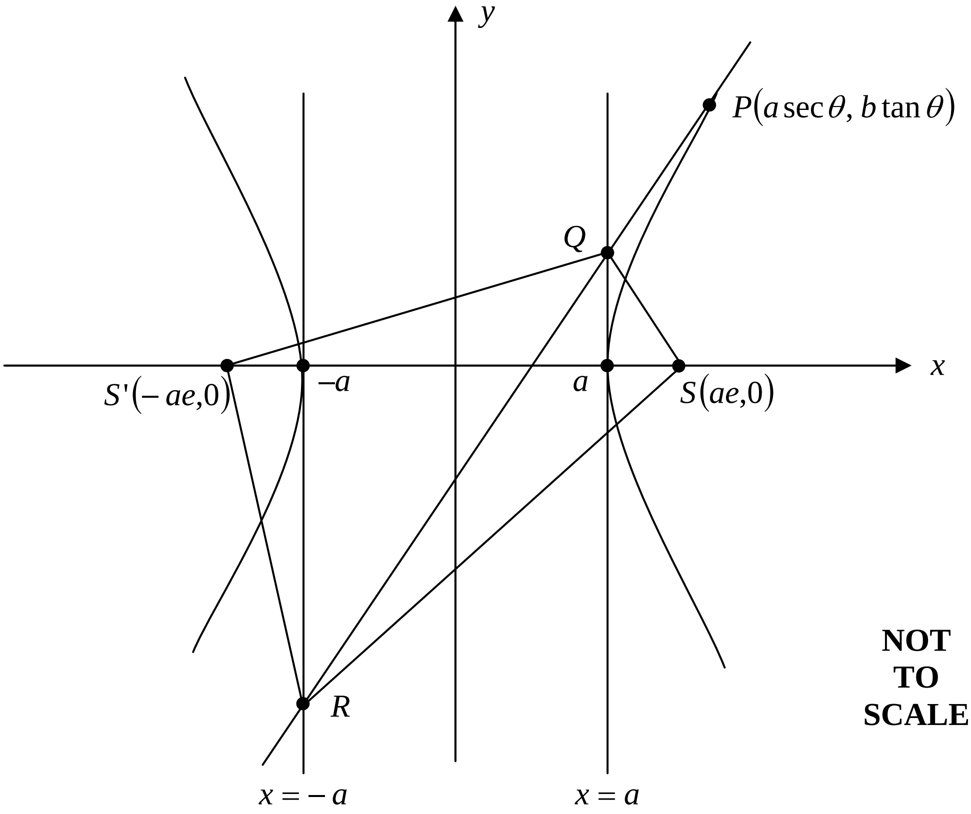
(ii) Hence evaluate I_3. **3**
(Leave your answer in exact form.)

Question 6 continues on page 12

Marks

Question 6 (continued)

(c)



$P(a\sec\theta, b\tan\theta)$ lies on the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2} = 1$.

The tangent at P meets the lines $x=a$ and $x=-a$ at Q and R respectively.

(i) Show that the equation of the tangent is given by $\dfrac{x\sec\theta}{a} - \dfrac{y\tan\theta}{b} = 1$. **2**

(ii) Find the coordinates of Q and R. **1**

(iii) Show that QR subtends a right angle at the focus $S(ae,0)$. **2**

(iv) Deduce that Q, S, R, S' are concyclic. **2**

End of Question 6

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Marks

Question 7 (15 marks) Use a SEPARATE writing booklet.

(a) A family of six are sitting at a round table. **2**
In how many ways can they be arranged so that Mum and Dad sit together and the youngest daughter, Chelsea, does not sit opposite Dad?

(b) (i) Factorise the cubic polynomial $z^3 - 64$:

(A) over the real numbers. **1**

(B) over the complex numbers. **1**

(ii) Let ω be one of the complex roots of the equation $z^3 - 64 = 0$.

(A) Show that $\omega^2 = -4(\omega + 4)$. **1**

(B) Hence evaluate $(4\omega + 16)^3$. **1**

(c) A sequence of numbers T_1, T_2, T_3, $\ldots$ is defined by $T_1 = 1$, $T_2 = 5$ and $T_k = 5T_{k-1} - 6T_{k-2}$.

(i) Show that the statement $T_n = 3^n - 2^n$ is true for $n = 1, 2, 3$. **2**

(ii) Prove by induction that $T_n = 3^n - 2^n$ for all integers $n \geq 1$. **2**

(d) (i) Using the substitution $u = a - x$, show that **1**

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

(ii) Hence evaluate $\int_0^\pi x \sin^2 x \, dx$. **4**

End of Question 7

Marks

Question 8 (15 marks) Use a SEPARATE writing booklet.

(a) Let $f(x) = x - \frac{1}{2}\tan x$.

(i) Show that $f(x)$ is an odd function. **1**

(ii) Find the value of any stationary points in the domain $-\frac{\pi}{2} < x < \frac{\pi}{2}$ and determine their nature. **3**

(iii) Sketch the curve $y = f(x)$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$. **2**
(You do not need to find the values of all x-intercepts.)

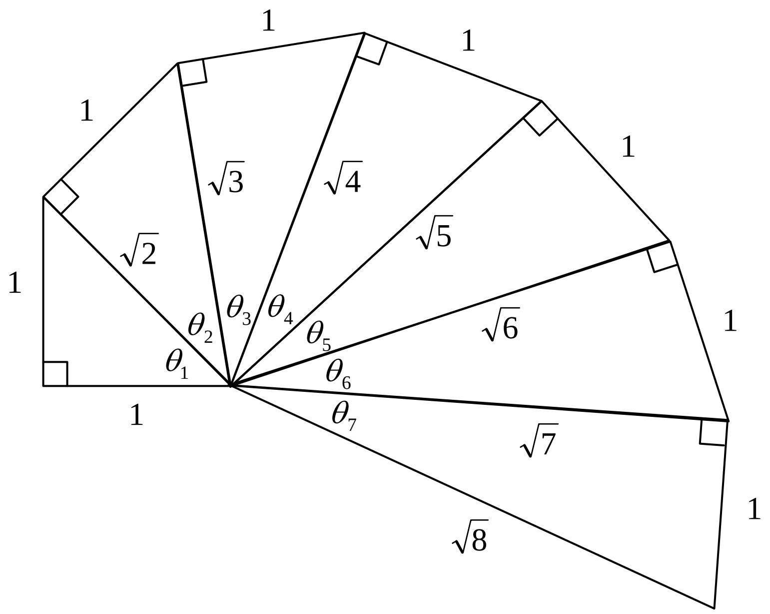
(iv) Hence, or otherwise, show that $x \geq \frac{1}{2}\tan x$ for $0 \leq x \leq \frac{\pi}{4}$ and state when equality holds in the given domain. **2**

Question 8 continues on page 16

Marks

Question 8 (continued)

(b) A spiral is created by constructing a right-angled triangle on the hypotenuse of the previous triangle as shown in the diagram.



Each triangle has an altitude of 1 unit and the hypotenuse lengths form a sequence $1, \sqrt{2}, \sqrt{3}, \sqrt{4}, \sqrt{5}, \ \ldots$

Let the angle, θ, in each triangle be $\theta_1, \theta_2, \theta_3, \ \ldots$ as shown in the diagram. The angle in the *n*th triangle is given by θ_n.

(i) Write down expressions for $\tan\theta_1$, $\tan\theta_2$, $\tan\theta_3$ and $\tan\theta_n$. **1**

(ii) Within what range of values does θ lie? **1**

(iii) Using the result from part (a) (iv), show that $\displaystyle\sum_{n=1}^{k}\theta_n \geq \frac{1}{2}\sum_{n=1}^{k}\frac{1}{n}$. **3**

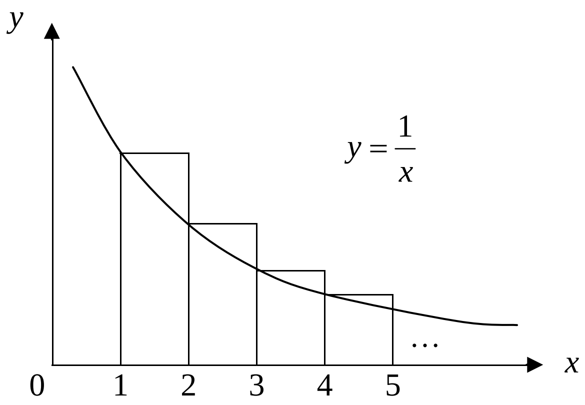
Question 8 continues on page 17

Marks

Question 8 (continued)

(b) (continued)

(iv) The curve $y = \dfrac{1}{x}$ is drawn in the first quadrant and upper rectangles are drawn as shown in the diagram. **1**



Show that $\displaystyle\sum_{n=1}^{k} \frac{1}{n} > \int_{1}^{k+1} \frac{1}{n}\, dn$.

(v) Hence, deduce that $\displaystyle\sum_{n=1}^{k} \theta_n > \ln\sqrt{k+1}$. **1**

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