SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2012

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 180 minutes
- Write using black pen.
- Board approved calculators may be used
- Show all necessary working in Questions 11–16
- A table of standard integrals is on the back of the multiple choice answer sheet

Total Marks - 100 Marks

Section I **10 Marks**

- Attempt Questions 1–10
- Allow about 15 minutes for this section.

Section II **90 Marks**

- Attempt Questions 11–16
- Allow about 2 hour 45 minutes for this section.

Examiner: *External Examiner*

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate.

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Section I **Objective-response Questions**
Total marks – 10
Attempt Questions 1 – 10

Answer each question on the multiple choice answer sheet provided.

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1 Let $u = 7\cos\frac{\pi}{4} + 7i\sin\frac{\pi}{4}$ and $v = a\cos b + ai\sin b$, where a and b are real constants.

If $uv = 42\cos\frac{\pi}{20} + 42i\sin\frac{\pi}{20}$, then

(A) $a = 35$ and $b = \frac{\pi}{5}$

(B) $a = 6$ and $b = \frac{\pi}{5}$

(C) $a = 35$ and $b = -\frac{\pi}{5}$

(D) $a = 6$ and $b = -\frac{\pi}{5}$

2 If $z^2 = 4\text{cis}\left(\frac{4\pi}{3}\right)$, then z is equal to

(A) $\sqrt{3} + i$ or $-\sqrt{3} - i$

(B) $1 - \sqrt{3}i$ or $-1 + \sqrt{3}i$

(C) $\sqrt{3} - i$ or $\sqrt{3} + i$

(D) $1 - \sqrt{3}i$ or $1 + \sqrt{3}i$

3 Let $z = \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}$.

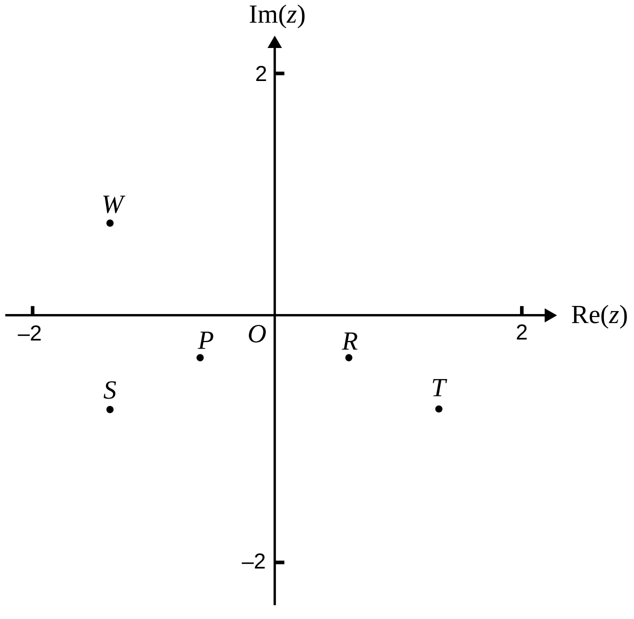
The imaginary part of $z - i$ is

(A) $-\frac{i}{2}$ (B) $-\frac{3i}{2}$ (C) $-\frac{1}{2}$ (D) $-\frac{3}{2}$

4 The point W on the Argand diagram below represents a number w where $|w| = 1\cdot 5$.

The number w^{-1} is best represented by the point

(A) P (B) R (C) S (D) T



5 $P(z)$ is a polynomial in z of degree 4 with real coefficients
Which one of the following statements must be false?

(A) $P(z)$ has four real roots.
(B) $P(z)$ has two real roots and two non-real roots.
(C) $P(z)$ has one real root and three non-real roots.
(D) $P(z)$ has no real roots.

6 The graph of $f(x)=\dfrac{1}{x^2+mx-n}$, where m and n are real constants, has no vertical asymptotes if

(A) $m^2<-4n$ (B) $m^2>-4n$ (C) $m^2<4n$ (D) $m^2>4n$

7 Consider the graph of $f(x)=\sin^3 x$ for $-\pi\le x\le 2\pi$.
The area bounded by the graph of $f(x)$ and the x-axis could be found by evaluating

(A) $\int_{-1}^{1}(1-u^2)\,du$ (B) $3\int_{-1}^{1}(1-u^2)\,du$

(C) $-\int_{-1}^{1}(1-u^2)\,du$ (D) $-3\int_{-1}^{1}(1-u^2)\,du$

8 Given that $\dfrac{dy}{dx}=y^2+1$, and that $y = 1$ at $x = 0$, then

(A) $y=y^2x+x+1$ (B) $y=\tan\left(x+\dfrac{\pi}{4}\right)$

(C) $y=\tan\left(x-\dfrac{\pi}{4}\right)$ (D) $x=\log_e\left(\dfrac{y^2+1}{2}\right)$

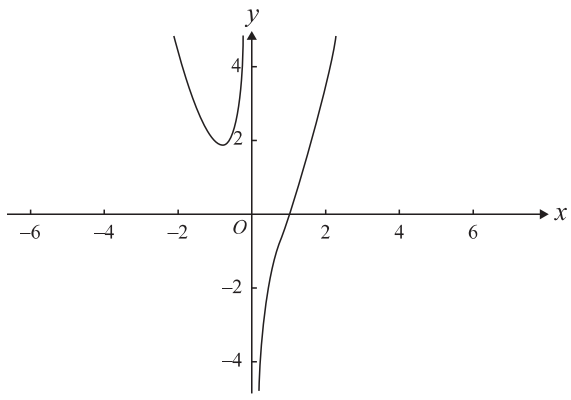
9 The velocity v m/s of a body which is moving in a straight line, when it is x m from the origin, is given by $v=\sin^{-1}x$.
The acceleration of the body in m/s^2 is given by

(A) $-\cos^{-1}x$ (B) $\cos^{-1}x$ (C) $-\dfrac{\sin^{-1}x}{\sqrt{1-x^2}}$ (D) $\dfrac{\sin^{-1}x}{\sqrt{1-x^2}}$

10 Let $f(x) = \dfrac{x^k + a}{x}$, where k and a are real constants.

If k is an odd integer which is greater than 1 and $a < 0$, a possible graph of f could be

(A)



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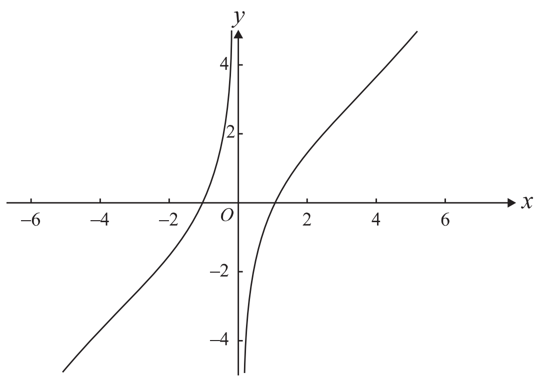
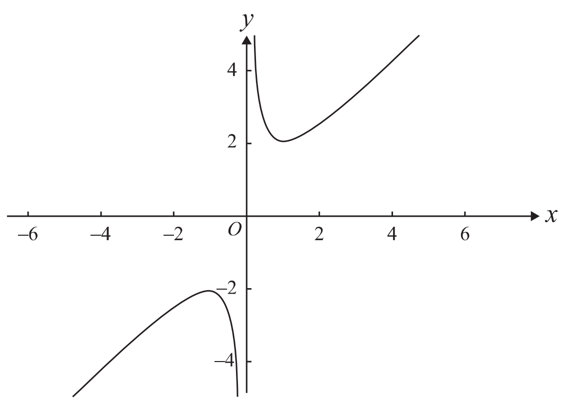


(C)



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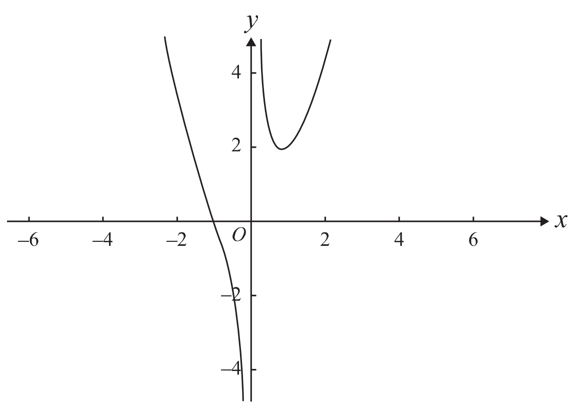


End of Section I

Section II **Free response Questions**
Total marks – 90
Attempt Questions 11 – 16

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Use the substitution $x=\sin^2\theta$ to evaluate $\displaystyle\int_0^{\frac{1}{2}} \frac{\sqrt{x}}{(1-x)^{\frac{3}{2}}}\,dx$ **3**

(b) Find $\displaystyle\int x\sqrt{3-x}\,dx$. **2**

(c) (i) By completing the square, find the exact value of $\displaystyle\int_{\frac{1}{8}}^{\frac{1}{4}} \frac{1}{\sqrt{2x(1-2x)}}\,dx$ **2**

(ii) Hence, evaluate $\displaystyle\int_{\frac{1}{8}}^{\frac{1}{4}} \frac{1-x}{\sqrt{2x(1-2x)}}\,dx$ **2**

(d) Find the value of the discriminant for the quadratic equation **2**

$$(1+i)z^2+4iz-2(1-i)=0$$

(e) (i) Find the value of $\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)^6$. **1**

(ii) Show that $(\cos\theta+i\sin\theta)(1+\cos\theta-i\sin\theta)=1+\cos\theta+i\sin\theta$. **1**

(iii) Hence show that $\left(1+\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)^6+\left(1+\cos\frac{\pi}{6}-i\sin\frac{\pi}{6}\right)^6=0$. **2**

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) The line $x = 8$ is a directrix of the ellipse with equation **2**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad a > b > 0$$

and (2, 0) is the corresponding focus.
Find the value of a and b.

(b) (i) Show that $2 - i$ is a solution of the equation $z^3 - (2-i)z^2 + z - 2 + i = 0$. **2**

(ii) Hence find all the solutions of the equation $z^3 - (2-i)z^2 + z - 2 + i = 0$. **2**

(c) Consider the function $f(x) = \log_e(4 - x^2)$.

(i) By first sketching $y = 4 - x^2$, sketch $y = f(x)$. **2**

Let A be the magnitude of the area enclosed by the graph of $y = f(x)$, the coordinate axes and the line $x = 1$.

(ii) Without evaluating A, use (i) to show that $\log_e 3 < A < \log_e 4$. **1**

(iii) Find $\displaystyle\int \frac{x^2}{4 - x^2}\, dx$. **3**

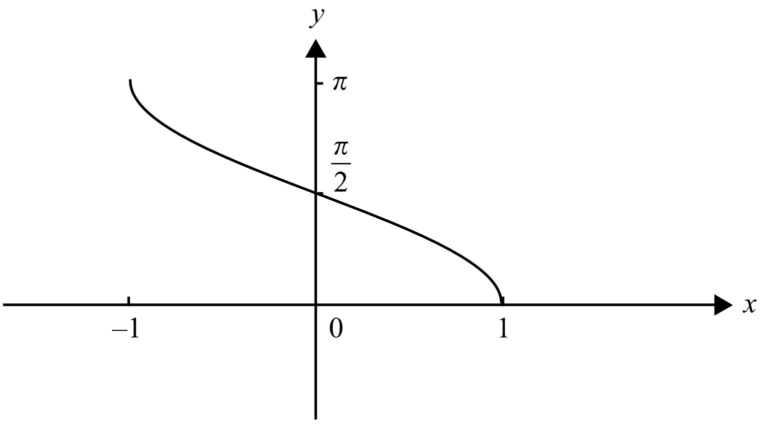
(iv) Hence find the exact value of A in the form $a + b\log_e c$, where a, b and c are integers. **3**

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) Prove using induction for integers $n \geq 2$. **3**

$$n^{n+1} > n(n+1)^{n-1}$$

(b) The diagram below shows the graph of $y = \cos^{-1} x$.

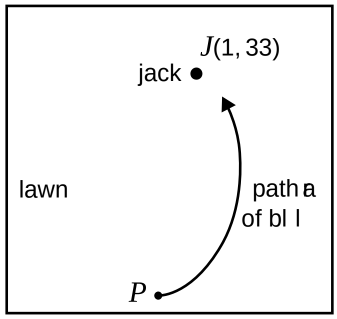


Using the method of cylindrical shells, find the exact volume formed if the graph above is rotated about the y-axis. **3**

Question 13 continues on the next page

(c) The game of lawn bowls is played on a horizontal lawn.
The aim is to roll a ball (usually called a 'bowl') to come to rest as close as possible to a target ball called the 'jack'.

Bowler



View from above

All displacements are in metres.
At one stage during the game, the jack is at the point $J(1, 33)$.
The path of a particular ball in this game is modelled by:

$$x = 2\sin\left(\frac{2t}{15}\right) \text{ and } y = 2 + \frac{5}{3}t - \frac{5}{3}\sin\left(\frac{t}{3}\right),\ 0 \le t \le \frac{15\pi}{2}$$

where t is the time in seconds after the ball is released from the point P.

(i) Write down the coordinates of P. **1**

(ii) Find expressions for the components of velocity, in metres per second, of the ball at time t seconds after the ball is released. **2**

(iii) At the instant the ball is released, what angle does its path make with the forward direction?
Give your answer correct to 1 decimal place. **2**

(iv) At what time, correct to the nearest tenth of a second, does the ball begin to swing left towards the jack? **2**

(v) Determine whether the path of the ball passes through J. **2**

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

A ‘parasailing’ water-skier i.e. a water-skier with a parachute attached of mass 90 kg is towed by a boat in a straight line from rest.
The boat exerts a constant force of 410 N acting horizontally on the skier.
At this stage the resistance acting on the skier is a constant 50 N, which acts horizontally.

(a) By use of a force diagram, show that the acceleration of the skier is 4 m/s^2. **2**

(b) By starting with $a = 4$, show that the speed of the skier, is given by $v^2 = 8x$, where x is the horizontal distance travelled by the skier.
Hence show that having been towed a distance of 32 m, his speed is 16 m/s. **2**

After the skier has been towed 32 m across the water the drag of the parachute becomes significant. The drag of the parachute produces an *additional* resistance of $6v$ N to the horizontal motion of the skier, where v m/s is the velocity of the skier.
Let a m/s^2 is the acceleration of the skier.

(c) Show that $a = \dfrac{1}{15}(60 - v)$ **1**

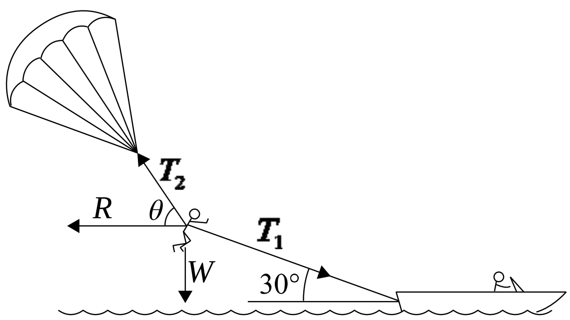
(d) Find the time required to reach a speed of 20 m/s from a speed of 16 m/s.
Give your answer in seconds, correct to one decimal place. **3**

After some time, the parasailing skier is being towed horizontally at a *constant speed* and at a fixed distance above the water.

The tow rope from the boat makes an angle of $30°$ to the horizontal, and the parachute cord makes an angle of θ to the horizontal.

The diagram below shows all the forces that are now acting on the parasailing water skier:

The tow rope now exerts a force, T_1, of 500 N on the skier.
The skier is experiencing a horizontal resistance, R, of 100 N.
Let the tension exerted by the parachute cord on the skier be T_2,
and the force due to gravity on the skier be W.
Take $g = 10$, where g is the magnitude of the acceleration due to gravity.



(e) By resolving in the horizontal and vertical directions, show that **3**

$$\begin{cases} 500\cos 30° - T_2\cos\theta - 100 = 0 \\ T_2\sin\theta - 500\sin 30° - 90g = 0 \end{cases}$$

Question 14 continues on the next page

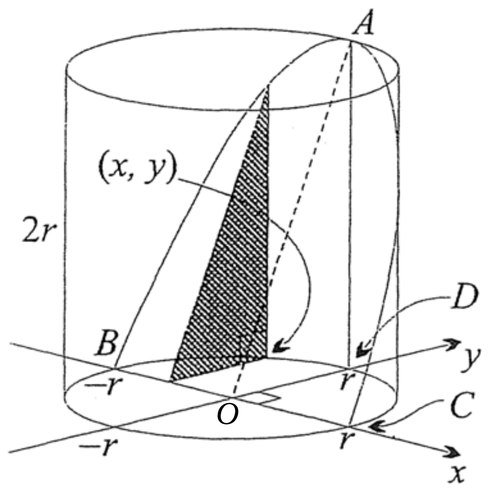
(f) Show that $\tan\theta = \dfrac{115}{25\sqrt{3}-10}$. **2**

(g) Hence, find the value of T_2 correct to the nearest integer. **2**

End of Question 14

Question 15 (15 marks) Use a SEPARATE writing booklet.

(a) The diagram below shows a cylindrical wedge *ABCD*, the cross sections of which are all right triangles.
Each cross section is similar to triangle *AOD*.
The base of each cross section is parallel to *OD*.
The height of the cylinder is equal to the diameter of its base.
Let the radius of the base be *r* units.



(i) Show that the typical triangular cross-section shaded has area $\left(r^2 - x^2\right)$ square units. **2**

(ii) Hence find the volume of the wedge. **2**

(b) For positive real numbers *x* and *y*

(i) Prove that $\dfrac{x+y}{2} \geq \sqrt{xy}$ **2**

When is there equality?

(ii) Hence by considering $\dfrac{1}{a} + \dfrac{1}{b}$, or otherwise, prove that $\dfrac{2ab}{a+b} \leq \sqrt{ab}$ **1**

for positive real numbers *a*, *b*.

(iii) Hence, or otherwise prove that $\dfrac{1}{x-1} + \dfrac{1}{x} + \dfrac{1}{x+1} > \dfrac{3}{x}$ for any $x > 1$ **2**

(iv) If $H = 1 + \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} + \dfrac{1}{5} + \dfrac{1}{6} + \dfrac{1}{7} + \ldots + \dfrac{1}{n}$, where *n* is an integer $n > 1$, **2**

use (iii) to show that $\lim\limits_{n \to \infty} H = \infty$.

Question 15 continues on the next page

(c) (i) Given that ω is one of the non-real roots of $z^3 = 1$, show that $1 + \omega + \omega^2 = 0$. **1**

(ii) Using (i), or otherwise, show that **3**

$$\left(\frac{\omega}{1+\omega}\right)^k + \left(\frac{\omega^2}{1+\omega^2}\right)^k = (-1)^k \, 2\cos\tfrac{2}{3}k\pi \text{, where } k \in \mathbb{Z}.$$

End of Question 15

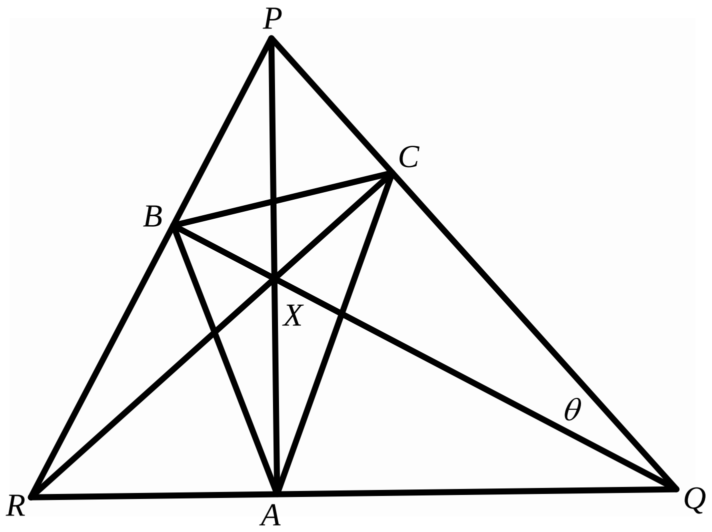
Question 16 (15 marks) Use a SEPARATE writing booklet.

(a) $I_n = \int_0^a (a-x)^n \cos x \, dx$, $a > 0$ and n is an integer with $n \geq 0$.

(i) Show that, for $n \geq 2$, $I_n = na^{n-1} - n(n-1)I_{n-2}$. **3**

(ii) Hence evaluate $\int_0^{\frac{\pi}{2}} \left(\frac{\pi}{2} - x\right)^3 \cos x \, dx$ **3**

(b) In the figure below, ΔPQR is acute angled and AP, BQ and CR are altitudes concurrent at X. Also $\angle XQC = \theta$.
ΔABC is called the *pedal triangle* of ΔPQR.



(i) Prove that $\angle XRB = \theta$. **2**

(ii) Prove that X, A, Q and C are concyclic. **1**

(iii) Deduce that $\angle XAC = \theta$. **1**

(iv) Hence deduce that in an acute angled triangle the altitudes bisect the angles of the pedal triangle through which they pass. **2**

Question 16 continues on the next page

Question 16 continued

(c) (i) A binary string is a sequence of **1**s and **0**s,
e.g. **1 1 0 1 1 1 1 0 0 1 0 1** is a binary string of length 12.

In a binary string of length 50, how many ways are there to have a string with exactly 9 **1**s and that no two **1**s are adjacent? Justify your answer. **2**

(ii) Given 50 cards with the integers 1, 2, 3, … 50 printed on them, how many ways are there to select 9 distinct cards, such that no two cards have consecutive numbers printed on them? **1**
(An answer with no reasoning will get no credit.)

End of paper

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