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Name: ....

Teacher: ....

PYMBLE LADIES' COLLEGE
ALL' ULTIMO LAVORO
SYDNEY

2011

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total Marks – 120

- Attempt Questions 1–8
- All questions are of equal value

Mark		**/120**
Rank		**/18**
Highest Mark		**/120**

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Total marks – 120
Attempt Questions 1–8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) If $z = 3 + 2i$ and $\omega = 1 + i$, find in the form $a + ib$ where a and b are real

(i) $2z - i\omega$. **1**

(ii) $z\overline{w}$. **1**

(iii) $\dfrac{3}{\omega}$. **1**

(b) On the Argand diagram, sketch the locus of z described by the inequality

$|z - 2 - 3i| \geq |z + i|$. **2**

(c) Let $\alpha = -\sqrt{3} + i$.

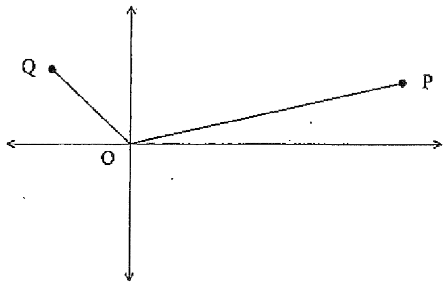
(i) Express α in modulus-argument form. **2**

(ii) Show that α is a root of the equation $z^6 + 64 = 0$. **1**

(iii) Hence, find a real quadratic factor of the polynomial $P(z) = z^6 + 64$. **2**

Question 1 continues on page 3

(d) The diagram shows a complex plane with origin O.



Points P and Q represent non-zero complex numbers z and w respectively.

(i) Write down the length of PQ in terms of z and w. **1**

(ii) Copy the diagram into your booklet. Construct point R that represents $z+w$. **2**

What type of quadrilateral is OPQR?

(iii) Prove that if $|z+w|=|z-w|$, the complex number $\dfrac{w}{z}$ is imaginary. **2**

End of Question 1

Question 2 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) (i) Find the value of a and b such that **2**

$$\frac{1}{(x-1)(2x+3)} = \frac{a}{x-1} + \frac{b}{2x+3}.$$

(ii) Hence find $\int \frac{dx}{(x-1)(2x+3)}$. **2**

(b) Use the substitution $t = \tan x$ to find $\int \operatorname{cosec} 2x \, dx$. **3**

(c) Evaluate $\int_{\frac{-1}{2}}^{0} \frac{dx}{2+4x+4x^2}$. **2**

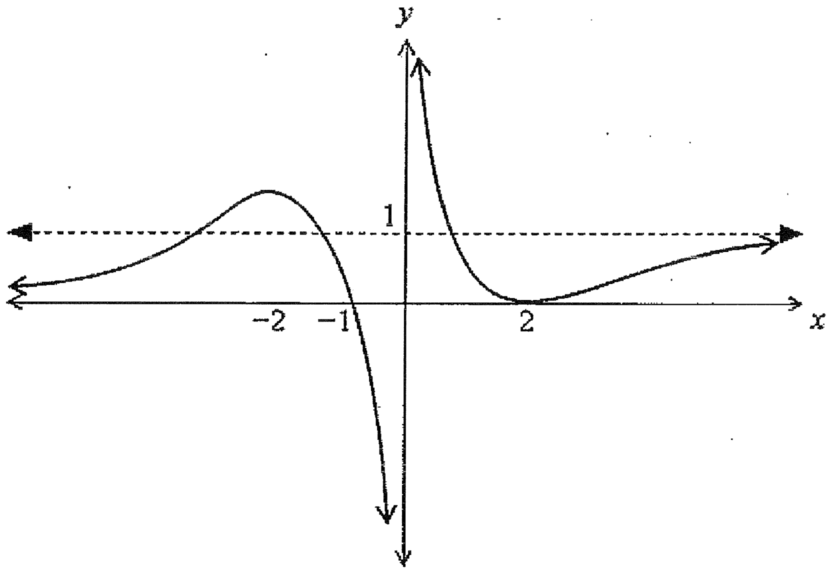
(d) Evaluate $\int x(3^x)\, dx$. **3**

(e) Use the substitution $x = u^6$ to find $\int \frac{dx}{x^{\frac{1}{2}} - x^{\frac{1}{3}}}$. **3**

End of Question 2

Question 3 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) The diagram below is a sketch of the function $y = f(x)$.
The lines $x = 0$, $y = 0$ and $y = 1$ are asymptotes.



Using the answer sheets provided, sketch each of the graphs below.

In each case, clearly label any maxima or minima, intercepts and the equations of any asymptotes.

(i) $y = f(|x|)$. **1**

(ii) $y = \dfrac{1}{f(x)}$. **2**

(iii) $y^2 = f(x)$. **2**

(iv) $y = [f(x)]^2$. **2**

(v) $y = \sin^{-1}[f(x)]$. **2**

Question 3 continues on page 6

Question 3 (continued) **Marks**

(b) Consider the curve $y = \dfrac{x^3 + 4}{x^2}$.

(i) Find the coordinates of the stationary point and show that this curve is always concave up. **2**

(ii) Find the equations of any asymptotes. **1**

(iii) Sketch the curve. **2**

(iv) Find the values of k for which the equation $x^3 - kx^2 + 4 = 0$ has 3 distinct real roots. **1**

End of Question 3

Question 4 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Given that $(x+1)$ is a factor of the polynomial $P(x) = x^3 + 2x^2 + 2x + 1$, factorise $P(x)$ over the field of complex numbers. **2**

(b) The polynomial equation $x^3 - 3x^2 + 5x - 1 = 0$ has roots α, β and γ.

(i) Find the value of $\alpha^2 + \beta^2 + \gamma^2$. **2**

(ii) Hence explain why only one root of the equation is real. **1**

(c) ω and ω^2 are the two complex cube roots of unity. **3**

If ω and ω^2 are also the roots of the equation $x^3 + px^2 + qx + r = 0$, show that $p = q$.

(d) Given $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ and using the substitution $x = \cos\theta$,

(i) Solve $8x^3 - 6x + 1 = 0$. **2**

(ii) Hence prove that $\sec\frac{2\pi}{9} + \sec\frac{4\pi}{9} + \sec\frac{8\pi}{9} = 6$. **2**

(e) Let the roots of $x^3 - x - 1 = 0$ be α, β and γ. **3**

Find the polynomial whose roots are $\frac{1+\alpha}{1-\alpha}$, $\frac{1+\beta}{1-\beta}$ and $\frac{1+\gamma}{1-\gamma}$.

Question 5 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) Consider the ellipse $\frac{x^2}{9}+\frac{y^2}{25}=1$

(i) Find the coordinates of the foci and the equations of the directrices of the ellipse. **2**

(ii) Sketch the ellipse, showing all key features, including intercepts. **2**

(b) $P\left(2p,\frac{2}{p}\right)$ and $Q\left(2q,\frac{2}{q}\right)$ are points on the rectangular hyperbola $xy=4$.

P and Q move on the hyperbola so that PQ always passes through $(6,4)$.

(i) Show that $pq=\frac{p+q-3}{2}$. **3**

(ii) If M is the midpoint of PQ, find the equation of the locus of M. **2**

(c) The tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $(a>b>0)$ at $P(a\cos\theta, b\sin\theta)$ passes through a focus of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ with eccentricity e.

(i) Show that the tangent to the ellipse at P has equation $bx\cos\theta+ay\sin\theta=ab$. **2**

(ii) Show that P lies on the directrix of the hyperbola. **2**

(iii) Find the possible values of the gradient of the tangent at P. **2**

Question 6 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) On the Argand diagram $P(z)$ is a point in the first quadrant of the circle $|z|=3$.

If $arg(z)=\theta$, find in terms of θ, expressions for:

(i) $\arg z^4$ **1**

(ii) $\arg(z-3)$. **2**

(b) A stone is projected from a point on the ground and it just clears a fence d metres away. The height of the fence is h metres.
The angle of projection to the horizontal is θ and the speed of projection is v m/s. The displacement equations, measured from the point of projection are

$$x = vt\cos\theta \text{ and } y = \frac{-1}{2}gt^2 + vt\sin\theta.$$

(i) Show that $v^2 = \dfrac{gd^2\sec^2\theta}{2(d\tan\theta - h)}$. **2**

(ii) Show that the maximum height reached by the stone is **3**

$$\frac{d^2\tan^2\theta}{4(d\tan\theta - h)}.$$

(iii) Show that the stone will just clear the fence at its highest point if **3**

$$\tan\theta = \frac{2h}{d}.$$

(c) (i) If $I_n = \displaystyle\int_0^{\frac{1}{2}} \frac{x^n}{1-x^2}\,dx$ for $n = 0,1,2,3,.......$ show that **2**

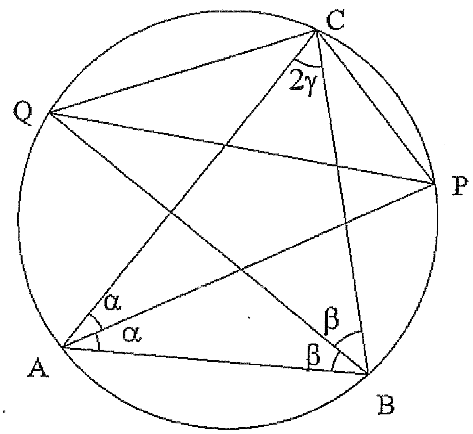
$$I_{n-2} - I_n = \frac{1}{(n-1)2^{n-1}} \quad \text{for } n = 2,3,4,............$$

(ii) Given that $\displaystyle\int_0^{\frac{1}{2}} \frac{1}{1-x^2}\,dx = \frac{1}{2}\log_e 3$, find the exact value of **2**

$$\int_0^{\frac{1}{2}} \frac{x^2}{1-x^2}\,dx.$$

Question 7 (15 marks) Use a SEPARATE writing booklet **Marks**

(a)



In the diagram above, AB is a **fixed** chord of a circle and C is a **variable** point on the major arc AB.

The angle bisectors of $\angle CAB$ and $\angle ABC$ meet the circle again at P and Q respectively.

Let $\angle CAB = 2\alpha$, $\angle ABC = 2\beta$ and $\angle BCA = 2\gamma$.

(i) Show that $\angle PCQ = \alpha + \beta + 2\gamma$. **1**

(ii) Hence explain why the distance PQ is constant. **2**

(iii) Use the sine rule to show that $\dfrac{AB}{PQ} = 2\sin\gamma$. **2**

(b) (i) Use DeMoivre's Theorem to show that when n is a positive integer, **2**

$$(1+i\tan\theta)^n + (1-i\tan\theta)^n = \frac{2\cos n\theta}{\cos^n\theta} \quad (\cos\theta \neq 0\ ).$$

(ii) Hence show that for the equation $(1+z)^4 + (1-z)^4 = 0$ (where $\mathrm{Re}(z) = 0$) the roots are **2**

$$z = \pm\, i\tan\frac{\pi}{8} \text{ and } z = \pm\, i\tan\frac{3\pi}{8}.$$

Question 7 continues on page 11

Question 7 (continued) **Marks**

(c) (i) Given that $\sin x \geq \frac{2x}{\pi}$ for $0 < x < \frac{\pi}{2}$, explain why

$$\int_0^{\frac{\pi}{2}} e^{-\sin x} dx < \int_0^{\frac{\pi}{2}} e^{\frac{-2x}{\pi}} dx .$$ **2**

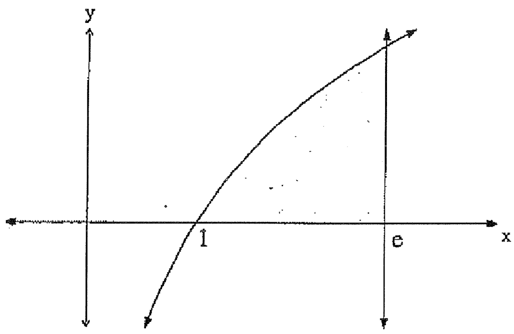
(ii) Show that $\int_{\frac{\pi}{2}}^{\pi} e^{-\sin x} dx = \int_0^{\frac{\pi}{2}} e^{-\sin x} dx$. **2**

(iii) Hence, show that $\int_0^{\pi} e^{-\sin x} dx < \frac{\pi}{e}(e-1)$. **2**

End of Question 7

Question 8 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) The diagram below shows the area bounded by the curve $y = \log_e x$, the x-axis and the line $x = e$.



This area is rotated about the y-axis to form a solid. By considering slices perpendicular to the y-axis, find the volume of the solid of revolution formed. **2**

(b) (i) Show that $\tan^{-1}\left(\dfrac{x}{x+1}\right) + \tan^{-1}\left(\dfrac{1}{2x+1}\right)$ is a constant for $2x+1>0$. **2**

(ii) Hence, find the exact value of the constant. **1**

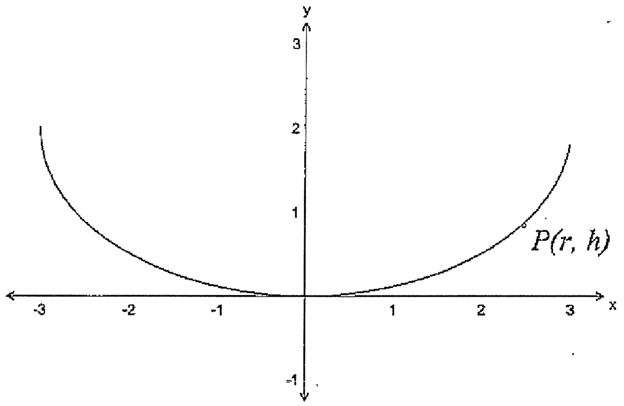
(c) (i) Prove that $\dfrac{\cos\theta - \cos(\theta+2\alpha)}{2\sin\alpha} = \sin(\theta+\alpha)$. **2**

(ii) Hence use mathematical induction to prove that **3**

$$\sin\theta + \sin 3\theta + \sin 5\theta + \ldots\ldots + \sin(2n-1)\theta = \frac{1-\cos 2n\theta}{2\sin\theta}.$$

Question 8 continues on page 13

(d) The semi-ellipse given by $\frac{x^2}{9}+\frac{(y-2)^2}{4}=1$ where $0 \le y \le 2$ is shown below:



The point (r,h) lies on the ellipse where $r>0$ and $0<h<2$.
The tangent at P makes an angle α with the positive direction of the x-axis.

(i) Show that $\tan\alpha = \frac{4r}{9(2-h)}$. **2**

(ii) Hence, show that $\tan\alpha = \frac{2\sqrt{4-(2-h)^2}}{3(2-h)}$. **2**

(iii) Show that the acute angle between the normal at the point P and the vertical line $x=r$ is equal to the angle between the tangent at P and the positive direction of the x-axis. **1**

END OF PAPER

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$