Student's Name……………………..…….. Student Number:………..…………….

PRESBYTERIAN LADIES' COLLEGE SYDNEY 1888

2010
TRIAL
HIGHER SCHOOL CERTIFICATE
EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total Marks – 120

- Attempt questions 1-8
- All questions are of equal value

1	2	3	4	5	6	7	8	Total	Total
								/120	**%**

Question 1 (15 marks) Start a new sheet of writing paper. **Marks**

a) Find

$$\int_0^{\frac{\pi}{4}} \tan^4 x \sec^2 x \, dx .$$ **2**

b) Find $\int \frac{dx}{\sqrt{x^2 - 6x + 8}} dx$ **2**

c) Use partial fractions to show **3**

$$\int_2^5 \frac{2x+2}{(x-1)(2x-1)} dx = \log_e \left(\frac{256}{27} \right).$$

d) Find $\int \sin(\log_e x) \, dx$. **4**

e) Use the substitution $t = \tan \frac{x}{2}$ to find **4**

$$\int_0^{\frac{\pi}{2}} \frac{dx}{(3\cos x + 4\sin x + 5)} .$$

End of Question 1

Question 2 **(15 marks)** **Start a new sheet of writing paper.** **Marks**

a) If $z=\sqrt{3}+i$ and $w=1-i$

i) Write $\dfrac{z}{w}$ in the form $a+ib$ where a and b are real. **1**

ii) Write $\dfrac{z}{w}$ in mod-arg form. **2**

iii) Hence, or otherwise, show that $\cos\dfrac{5\pi}{12}=\dfrac{\sqrt{6}-\sqrt{2}}{4}$ **2**

iv) Express $\left(\dfrac{z}{w}\right)^{12}$ in the form $a+ib$ where a and b are real. **2**

b) The points *Z*, *W* and *O* on the Argand diagram represent the complex numbers z, w and o respectively. If $z=3+i$ and $o=0+0i$. Find the complex number w, in $a+ib$ form where a and b are real, if ΔOZW in anti-clockwise order, is right-angled at *Z* and the distance from *Z* to *W* is twice the distance from *O* to *Z*. **2**

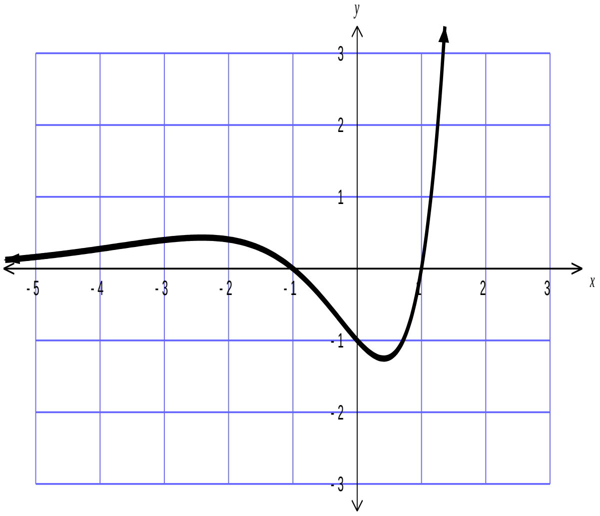
c) The point P on the Argand diagram represents the complex number $z=x+iy$ which satisfies $(z)^2=2-(\overline{z})^2$. Find the equation of the locus of P in terms of *x* and *y*. What type of curve is this locus? **3**

d) If *z* is a complex number such that $z=r(\cos\theta+i\sin\theta)$, where *r* is real, show that $\arg(z+r)=\dfrac{1}{2}\theta$. **3**

End of Question 2

Question 3 **(15 marks)** **Start a new sheet of writing paper.** **Marks**

a) The diagram shows the graph of $y = f(x)$.



Draw separate one-third page sketches of the graphs of the following:

i) $y = \dfrac{1}{f(x)}$ **1**

ii) $|y| = f(x)$ **1**

iii) $y = [f(x)]^2$ **2**

iv) $y = \sqrt{f(x)}$ **2**

v) $y = x(f(x))$ **2**

b) i) Express the complex number $1+i$ in the form $r(\cos\theta + i\sin\theta)$. **1**

ii) Hence prove that $(1+i)^n + (1-i)^n = 2(2^{\frac{n}{2}}\cos\frac{n\pi}{4})$ where n is a positive integer. **3**

iii) If $(1+x)^n = p_0 + p_1x + p_2x^2 + ... + p_nx^n$, prove that **3**

$$p_0 - p_2 + p_4 - ..... = 2^{\frac{n}{2}}\cos\frac{n\pi}{4} \text{ and } p_1 - p_3 + p_5 - .... = 2^{\frac{n}{2}}\sin\frac{n\pi}{4}.$$

End of Question 3

Question 4 **(15 marks)** **Start a new sheet of writing paper.**

Marks

a) Consider the hyperbola H with equation $\frac{(x-1)^2}{16}-\frac{(y+3)^2}{25}=1$

i) Find the centre, the eccentricity and the co-ordinates of the foci of *H*. **3**

ii) Write down the equations of the directrices and the asymptotes of *H*. **3**

iii) Sketch *H* showing all of the above features. **1**

b) Using the focus-directrix definition of the ellipse, centred at the origin, prove that the sum of the focal lengths is constant. **2**

c) Given the hyperbola $x^2-y^2=a^2$

i) Show that $(a\sec\theta,\ a\tan\theta)$ are the parametric coordinates of a point on the hyperbola. **1**

ii) Show that the equation of the tangent to $x^2-y^2=a^2$ at $(a\sec\theta,\ a\tan\theta)$ is $x-y\sin\theta=a\cos\theta$. **2**

iii) Prove that the area of the triangle bounded by a tangent and the asymptotes is a constant. **3**

End of Question 4

Question 5 **(15 marks)** **Start a new sheet of writing paper.**

Marks

a) Consider $P(x) = x^4 - 6x^2 - 8x - 3$.

i) Given that $P(x)$ has a zero of multiplicity 3, express $P(x)$ as a product of linear factors. **2**

ii) Sketch the graph of $P(x)$. **2**

b) $P(x)$ is a polynomial of the form $P(x) = ax^3 + bx^2 + cx + d$, where a, b, c and d are real. **4**
$P(x)$ has roots of 5 and i and when divided by $(x-2)$ the remainder is 15.
Find $P(x)$.

c) i) Show that $(1-\sqrt{x})^{n-1} - (1-\sqrt{x})^n = (1-\sqrt{x})^{n-1}\sqrt{x}$. **1**

ii) If $I_n = \int_0^1 (1-\sqrt{x})^n \, dx$ for $n \geq 0$ show that $I_n = \dfrac{n}{n+2} I_{n-1}$ for $n \geq 1$. **2**

iii) Hence show that $\dfrac{1}{I_n} = {}^{n+2}C_n$ for $n \geq 0$. **2**

d) If $a > 0,\ b > 0,\ c > 0,\ d > 0,$ show that $\dfrac{a}{b} + \dfrac{b}{c} + \dfrac{c}{d} + \dfrac{d}{a} \geq 4$ **2**

End of Question 5

Question 6 **(15 marks)** **Start a new sheet of writing paper.**

Marks

a) i) Prove that $\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$ **3**

ii) Show that roots of the equation $16x^4 - 20x^2 + 5 = 0$ are $x = \sin\dfrac{k\pi}{5}$, **2**

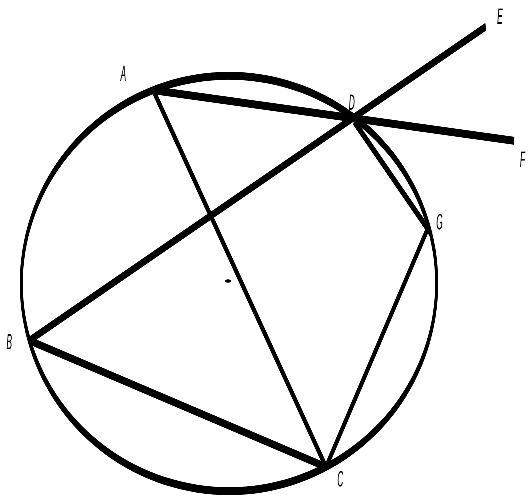
where $k = 1, 2, 3, 4$.

iii) Construct an equation whose roots are each 1 greater than those of $16x^4 - 20x^2 + 5 = 0$ **2**

iv) Hence or otherwise find the exact value of $\displaystyle\sum_{k=1}^{4} \frac{1}{1 + \sin\frac{k\pi}{5}}$ **2**

b) Find the equation of the tangent to the curve $5x^2 - 6xy + y^2 - 2x + 4y - 3 = 0$ at the point $(1, 2)$. **3**

c)



AC bisects $\angle BCG$ **3**
ADF and BDE are straight lines.

Prove that FD bisects $\angle EDG$.

End of Question 6

Question 7 **(15 marks)** **Start a new sheet of writing paper.** **Marks**

a) The n^{th} derivative of $f(x)$ is $\frac{d^n}{dx^n} f(x) = \frac{d^{n-1}}{dx^{n-1}}\left[\frac{d}{dx} f(x)\right]$.

i) Show that $\frac{d^n}{dx^n}(x^n) = n!$ **2**

ii) Prove, by mathematical induction, that for all positive integers, n **3**

$$\frac{d^n}{dx^n}(x^n \ln x) = n!(\ln x + \frac{1}{1} + \frac{1}{2} + ... + \frac{1}{n})$$

b) A railway line has been constructed around a circular curve of radius 500m. The distance between the rails is 1.5m and the outside rail is 0.1m above the inside rail.

i) Draw a diagram showing all forces on the train. **1**

ii) Show that $tan\,\theta = \frac{v^2}{gr}$, given that there is no sideways force on the wheels for a train on this curve. **2**

iii) Find the optimal speed for the train around this curve. (Take $g = 9.8\ m/s^2$) **1**

c) A particle of mass m projected vertically upwards with initial speed u m/s experiences a resistance of magnitude Kmv Newtons when the speed is v m/s where K is a positive constant. After T seconds the particle attains its maximum height h. Let the acceleration due to gravity be $g\ m/s^2$.

i) Show that the acceleration of the particle is given by $\ddot{x} = -(g + Kv)$ where x is the height of the particle t seconds after the launch. **1**

ii) Prove that T is given by $T = \frac{1}{K}\log_e\left(\frac{g + Ku}{g}\right)$ seconds. **2**

iii) Prove that h is given by $h = \frac{u - gT}{K}$ metres. **3**

End of Question 7

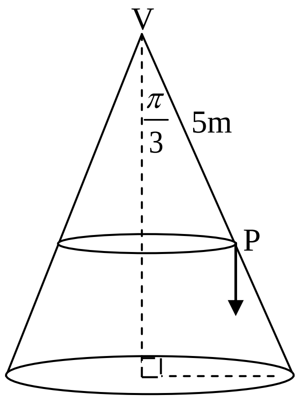
Question 8 **(15 marks)** **Start a new sheet of writing paper.** **Marks**

a) A particle is projected from the origin with initial velocity U to pass through a point (a,b).

i) Show that the Cartesian equation of the motion of the particle is given by $y = \frac{-gx^2}{2U^2}\sec^2\alpha + x\tan\alpha$. You must DERIVE all equations of motion. **3**

ii) Prove that there are two possible trajectories if: **3**
$\left(U^2 - gb\right)^2 > g^2\left(a^2 + b^2\right)$

b) A circular cone of semi-vertical angle $\frac{\pi}{3}$ is fixed with its vertex upwards. A particle P of mass *m* kg is attached to the vertex at *V* by a light inextensible string of length 5m. The particle *P* rotates with uniform angular velocity ω rad/sec in a horizontal circle whose centre is vertically below *V*, on the outside surface of the cone and in contact with it. Let *T* be the tension in the string, *N* the normal reaction force and *mg* the gravitational force at *P*.



i) Resolve the forces on *P* in the horizontal and vertical directions. **3**

ii) Show that $T = \frac{m}{4}\left(2g + 15\omega^2\right)$ and find a similar expression for *N*. **4**

iii) Show that for the particle to remain in uniform circular motion on the surface of the cone, then $\omega^2 < \frac{2g}{5}$, where *g* is the acceleration due to gravity. **2**

End of Examination

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STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \; x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \; x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \; a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \; a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \; a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \; a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \; a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \; a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \; a > 0, \; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2} \right), \; x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2} \right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

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