PENRITH HIGH SCHOOL

MATHEMATICS EXTENSION 2
2012

HSC Trial

Assessor: Mr Ferguson

General Instructions:

- Reading time – 5 minutes
- Working time – **3 hours**
- Write using black or blue pen. Black pen is preferred
- Board-approved calculators may be used.
- A table of standard integrals is provided at the back of this paper.
- A multiple choice answer sheet is provided at the back of this paper.
- Show all necessary working in Questions 11 – 16.
- Work on this question paper will not be marked.

Total marks – 100

SECTION 1 – Pages 2 – 5
10 marks
- Attempt Questions 1 – 10
- Allow about 15minutes for this section.

SECTION 2 – Pages 6 – 12
90 marks
- Attempt Questions 11 – 16
- Allow about 2 hours 45 minutes for this section.

Section1

Question	Mark
1	
2	
3	
4	
5	

Question	Mark
6	
7	
8	
9	
10	
Total	/10

Section 2

Question	Mark
11	/15
12	/15
13	/15
14	/15
15	/15
16	/15

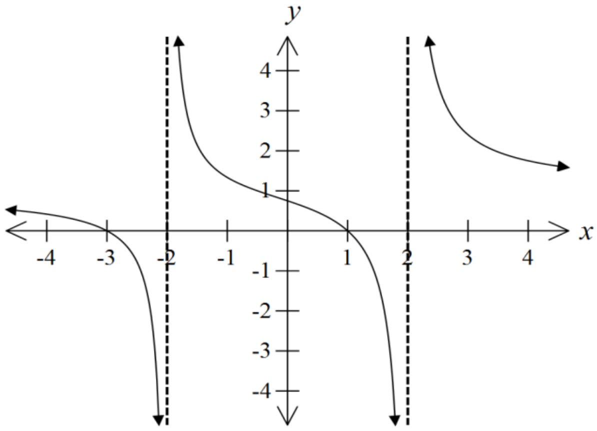
Total	/100
%	

This paper MUST NOT be removed from the examination room

***Student Number:* ………………………………**

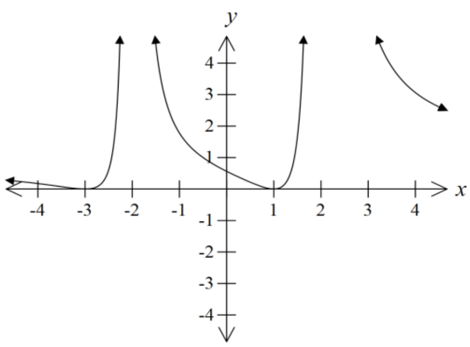
SECTION 1: Circle the correct answer on the multiple choice answer sheet

1 The diagram shows the graph of the function $y = f(x)$.

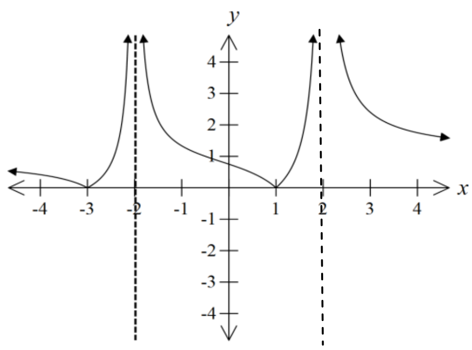


Which of the following is the graph of $y = |f(x)|$?

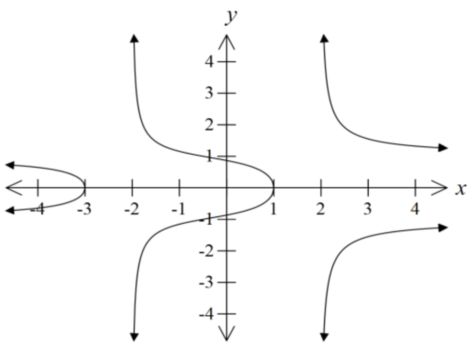
(A)



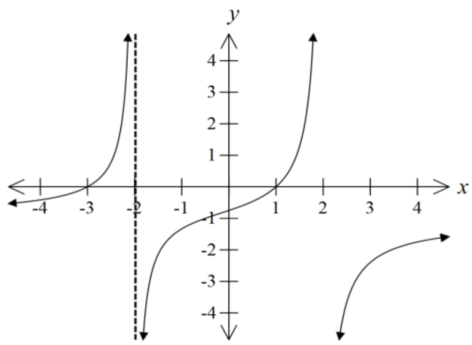
(B)



(C)



(D)



2 Let $z = 4 + i$. What is the value of $\overline{iz}$?

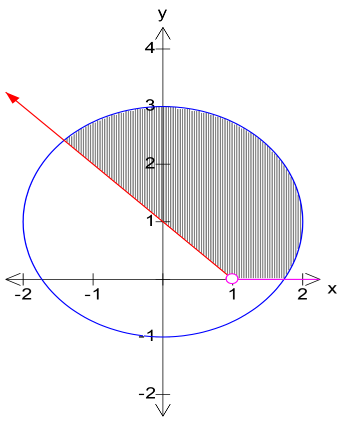
(A) $-1-4i$

(B) $-1+4i$

(C) $1-4i$

(D) $1+4i$

3 Consider the Argand diagram below.



Which inequality could define the shaded area?

(A) $|z-i| \le 2$ and $0 \le \arg(z-1) \le \dfrac{3\pi}{4}$

(B) $|z+i| \le 2$ and $0 \le \arg(z-1) \le \dfrac{3\pi}{4}$

(C) $|z-i| \le 2$ and $0 \le \arg(z-1) \le \dfrac{\pi}{4}$

(D) $|z+i| \le 2$ and $0 \le \arg(z-1) \le \dfrac{\pi}{4}$

4 Consider the hyperbola with the equation $\dfrac{x^2}{9} - \dfrac{y^2}{5} = 1$.

What are the coordinates of the vertex of the hyperbola?

(A) $(\pm 3, 0)$

(B) $(0, \pm 3)$

(C) $(0, \pm 9)$

(D) $(\pm 9, 0)$

5 The points $P(cp, \dfrac{c}{p})$ and $Q(cq, \dfrac{c}{q})$ lie on the same branch of the hyperbola $xy = c^2$ ($p \neq q$). The tangents at P and Q meet at the point T. What is the equation of the normal to the hyperbola at P?

(A) $p^2x - py + c - cp^4 = 0$

(B) $p^3x - py + c - cp^4 = 0$

(C) $x + p^2y - 2c = 0$

(D) $x + p^2y - 2cp = 0$

6 What is the value of $\int \sec x dx$? Use the substitution $t = \tan\frac{x}{2}$.

(A) $\ln|(t+1)(t-1)|+c$ (B) $\ln\left|\frac{1+t}{1-t}\right|+c$

(C) $\ln|(1+t)(1-t)|+c$ (D) $\ln\left|\frac{t+1}{t-1}\right|+c$

7 Let $I_n = \int_0^x \sin^n t dt$, where $0 \le x \le \frac{\pi}{2}$.

Which of the following is the correct expression for I_n?

(A) $I_n = \left(\frac{n-1}{n}\right) I_{n-2}$ with $n \ge 2$.

(B) $I_n = \left(\frac{n+1}{n}\right) I_{n-2}$ with $n \ge 2$.

(C) $I_n = n(n-1) I_{n-2}$ with $n \ge 2$.

(D) $I_n = n(n+1) I_{n-2}$ with $n \ge 2$.

8 The region enclosed by $y = x^3$, $y = 0$ and $x = 2$ is rotated around the *y*-axis to produce a solid. What is the volume of this solid?

(A) $\frac{8\pi}{5}$ units^3

(B) $\frac{32\pi}{5}$ units^3

(C) $\frac{64\pi}{5}$ units^3

(D) $\frac{16\pi}{5}$ units^3

9 What is the angle at which a road must be banked so that a car may round a curve with a radius of 100 metres at 90 km/h without sliding? Assume that the road is smooth and gravity to be 9.8 ms^{-2}.

(A) $83^\circ 10'$ (B) $32^\circ 32'$

(C) $83^\circ 6'$ (D) $32^\circ 53'$

10 The polynomial equation $x^3+4x^2-2x-5=0$ has roots α, β and γ. Which of the following polynomial equations have roots α^2, β^2 and γ^2?

(A) $x^3-20x^2-44x-25=0$

(B) $x^3-20x^2+44x-25=0$

(C) $x^3-4x^2+5x-1=0$

(D) $x^3+4x^2+5x-1=0$

SECTION 2

Question 11 (15 marks) (Use a new page to write your answers)

(a) Find (i) $\int \frac{t^2 - 1}{t^3} dt$. 4

(ii) $\int \frac{dx}{\sqrt{6 - x - x^2}}$

(b) Evaluate (i) $\int_0^1 \frac{x}{(x+1)(2x+1)} dx$ 3

(ii) $\int_0^{\frac{\pi}{4}} x \tan^2 x dx$ 3

(c) (i) If $I_n = \int_0^{\frac{\pi}{2}} x^n \cos x dx$, show that for $n > 1$, 3

$I_n = (\frac{\pi}{2})^n - n(n-1)I_{n-2}$

(ii) Hence find the area of the finite region bounded by the curve 2

$y = x^4 \cos x$ and the x axis for $0 \le x \le \frac{\pi}{2}$.

Question 12 (15 marks) (Use a new page to write your answers)

(a) Given that $z = \sqrt{2} - \sqrt{2}i$ and $w = -\sqrt{2}$, find, in the form $x + iy$:

(i) wz^2 1

(ii) $\arg z$ 1

(iii) $\dfrac{z}{z+w}$ 2

(iv) $|z|$ 1

(v) z^{10} 2

(b) Find the values of real numbers a and b such that $(a+ib)^2 = 5 - 12i$ 2

(c) Draw Argand diagrams to represent the following regions 2

(i) $1 \le |z + 4 - 3i| \le 3$

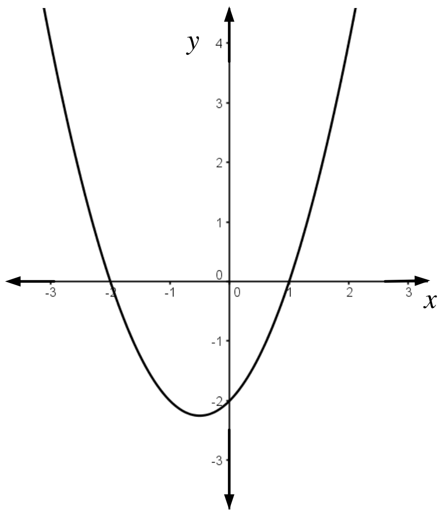
(ii) $\dfrac{\pi}{6} \le \arg z \le \dfrac{\pi}{3}$

(d) (i) Show that $\dfrac{1+\cos\theta + i\sin\theta}{1-\cos\theta - i\sin\theta} = i\cot\dfrac{\theta}{2}$ 2

(ii) Hence solve $\left(\dfrac{z-1}{z+1}\right)^8 = -1$ 2

Question 13 (15 marks) (Use a new page to write your answers)

(a) The diagram shows the graph of the function $f(x) = x^2 + x - 2$. On separate diagrams sketch the following graphs, showing clearly any intercepts on the coordinate axes and the equations of any asymptotes.



(i) $y = |f(x)|$ 1

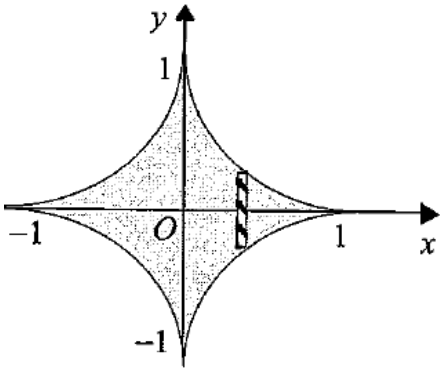
(ii) $y = [f(x)]^2$ 1

(iii) $y = \dfrac{1}{f(x)}$ 2

(iv) $y = \log_e f(x)$ 2

(b) The horizontal base of a solid is the area enclosed by the curve $|x|^{\frac{1}{2}}+|y|^{\frac{1}{2}}=1$.

Vertcial cross sections taken perpendicular to the x-axis are equilateral triangles with one side in the base.



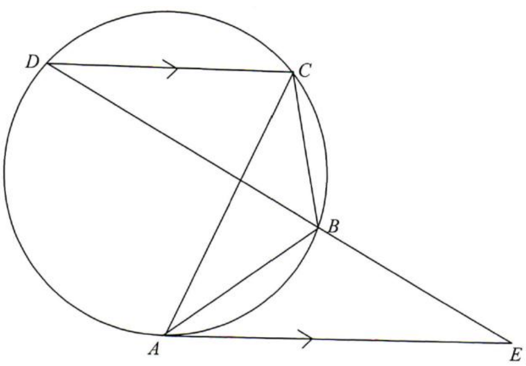
(i) Show that the volume of the solid is given by $V=2\sqrt{3}\int_0^1(1-\sqrt{x})^4\,dx$ 2

(ii) Use the substitution of $u=1-\sqrt{x}$ to evaluate this integral. 3

(c) The tangent AE is parallel to the chord DC.

(i) Prove that $(AB)^2=BC.BE$ 3

(ii) Hence or otherwise prove that $\dfrac{AC}{AE}=\sqrt{\dfrac{BC}{BE}}$ 1



Question 14 (15 marks) (Use a new page to write your answers)

(a) The equation of an ellipse is given by $4x^2 + 9y^2 = 36$.

(i) Find S and S' the foci of the ellipse **2**

(ii) Find the equations of the directrices M and M' **1**

(iii) Sketch the ellipse showing foci, directrices and axial intercepts. **2**

(iv) Let P be any point on the ellipse. **2**
Show $SP + S'P = 6$

(v) Find the equation of the chord of contact from an external point $(3, 2)$ **1**

(b) (i) Sketch the rectangular hyperbola $xy = c^2$, labelling the **1**
point $P\left(ct, \frac{c}{t}\right)$ on it.

(ii) Show that the equations of the tangent and normal to the hyperbola **3**
at P are $x + t^2 y = 2ct$ and $ty + ct^4 = t^3 x + c$ respectively.

(iii) If the tangent at P meets the coordinate axes at X and Y respectively **3**
and the normal at P meets the lines $y = x$ and $y = -x$ at R and S respectively,
prove that the quadrilateral $RYSX$ is a rhombus.

Question 15 (15 marks) (Use a new page to write your answers)

(a) When a certain polynomial is divided by $x+1$, $x-3$ the respective remainders are 6 and -2. Find the remainder when this polynomial is divided by x^2-2x-3. 3

(b) The cubic equation $x^3+px+q=0$ has 3 non-zero roots $\alpha,\ \beta,\ \gamma$. 3

Find, in terms of the constants p,q the values of

(i) $\alpha^2+\beta^2+\gamma^2$

(ii) $\alpha^3+\beta^3+\gamma^3$.

(c) If $\alpha,\ \beta,\ \gamma$ are the roots of the equation $3x^3-5x^2-4x+3=0$, find the cubic equation with roots $\alpha-1, \beta-1, \gamma-1$. 3

(d) A polynomial of degree n is given by $P(x)=x^n+ax-b$. It is given that the polynomial has a double root at $x=\alpha$.

(i) Find the derived polynomial $P'(x)$ and show that $\alpha^{n-1}=-\dfrac{a}{n}$. 3

(ii) Show that $\left(\dfrac{a}{n}\right)^n+\left(\dfrac{b}{n-1}\right)^{n-1}=0$. 2

(iii) Hence deduce that the double root is $\dfrac{bn}{a(n-1)}$. 1

Question 16 (15 marks) (Use a new page to write your answers)

(a) For $a>0$, $b>0$, $c>0$ and $d>0$ and given that $\frac{a+b}{2} \geq \sqrt{ab}$, show that 2

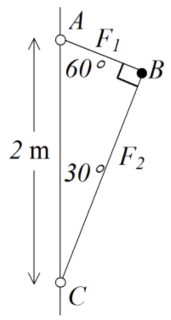
$$\frac{a+b+c+d}{4} \geq \sqrt[4]{abcd}$$

(b) (i) Use De Moivre's theorem to express $\tan 5\theta$ in terms of powers of $\tan\theta$. 3

(ii) Hence show that $x^4 - 10x^2 + 5 = 0$ has roots $\pm\tan\frac{\pi}{5}$ and $\pm\tan\frac{2\pi}{5}$. 2

(iii) Deduce that $\tan\frac{\pi}{5}.\tan\frac{2\pi}{5}.\tan\frac{3\pi}{5}.\tan\frac{4\pi}{5} = 5$ 1

(c) A mass 10 kg, centre B is connected by light rods to sleeves A and C which revolve freely about the vertical axis AC but do not move vertically.



(i) Given $AC = 2$ metres, show that the radius of the circular path of rotation of B is $\frac{\sqrt{3}}{2}$ metres. 1

(ii) Find the tensions in the rods AB, BC when the mass makes 90 revolutions per minute about the vertical axis. 3

(d) Given that $a_n = \sqrt{2 + a_{n-1}}$ for integers $n \geq 1$ and $a_0 = 1$, by mathematical induction prove that for $n \geq 1$: 3

$$\sqrt{2} < a_n < 2$$

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1;\ x \neq 0, \ \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x + \sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x + \sqrt{x^2+a^2}\right)$$

Note: $\ln x = \log_e x, \quad x > 0$

Student Number:____

Multiple Choice Answer Sheet

1.	A	B	C	D
2.	A	B	C	D
3.	A	B	C	D
4.	A	B	C	D
5.	A	B	C	D
6.	A	B	C	D
7.	A	B	C	D
8.	A	B	C	D
9.	A	B	C	D
10.	A	B	C	D