

# NORTH SYDNEY GIRLS HIGH SCHOOL



## 2012 TRIAL HSC EXAMINATION

# Mathematics Extension 2

### General Instructions

- Reading Time – 5 minutes
- Working Time – 3 hours
- Write using black or blue pen
- Board approved calculators may be used
- A table of standard integrals is provided at the back of this page
- Show all necessary working in questions 11 – 16

### Total Marks – 100

Section I            10 Marks

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section.

Section II            90 marks

- Attempt Questions 11 – 16

NAME: \_\_\_\_\_

TEACHER: \_\_\_\_\_

NUMBER: \_\_\_\_\_

QUESTION	MARK
1 – 10	/10
11	/15
12	/15
13	/15
14	/15
15	/15
16	/15
TOTAL	/100

**Section I**

**Objective Response Questions**

**Total marks – 10**

**Attempt Questions 1 – 10**

Answer each question on the multiple choice answer sheet provided.

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1.  $\frac{2-i}{-2-i} = ?$

(A)  $-\frac{3}{5} + \frac{4}{5}i$

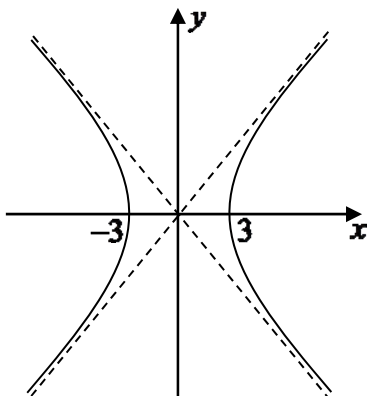
(B)  $-1$

(C)  $-1 + \frac{4}{3}i$

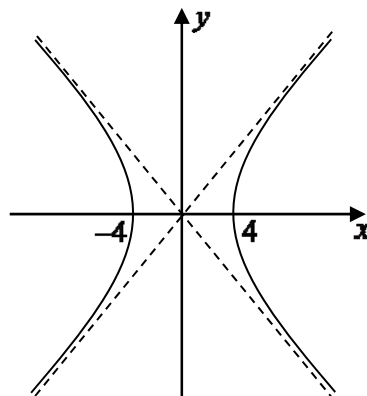
(D)  $-\frac{5}{3}$

2. Which of the following is the graph of  $9x^2 - 16y^2 = 144$  ?

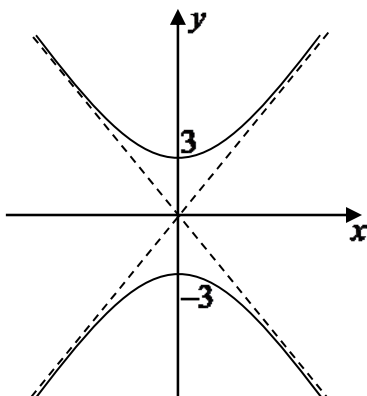
(A)



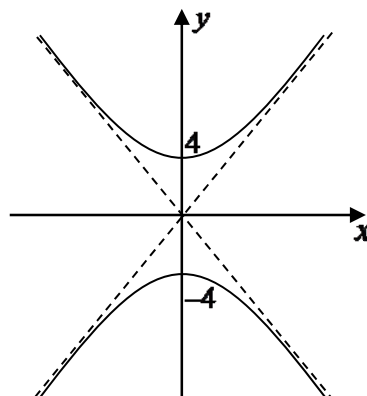
(B)



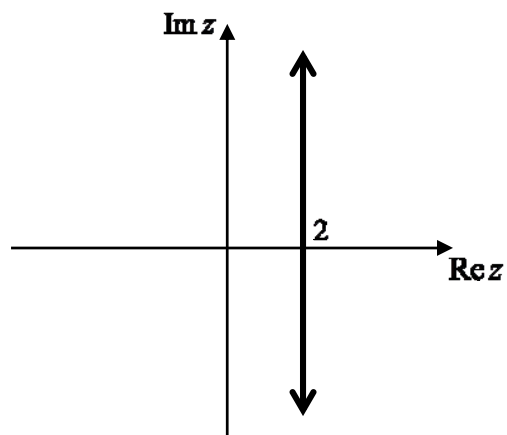
(C)



(D)



3.



Which of the following is NOT a valid algebraic description of this locus?

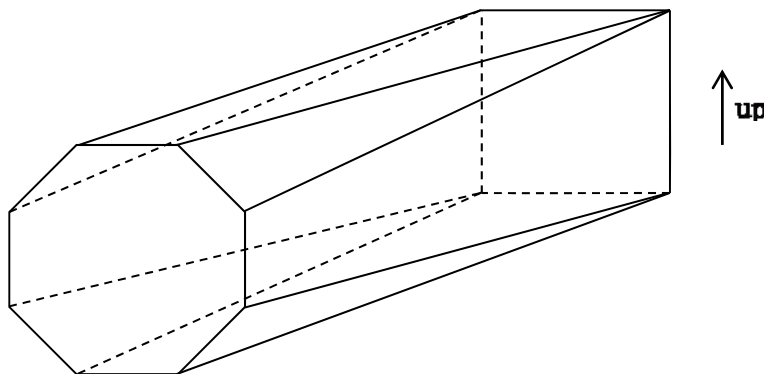
(A)  $\text{Re } z = 2$

(B)  $|z| = |z - 4|$

(C)  $\arg(z - 4) + \arg z = \pi$

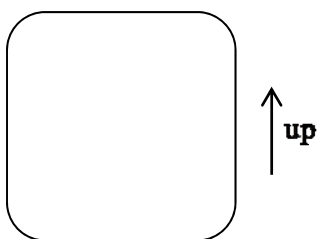
(D)  $z + \bar{z} = 4$

4. The solid shown in the diagram has a pair of parallel faces, one a regular octagon, and one a square, with vertices of each end joined by straight lines.

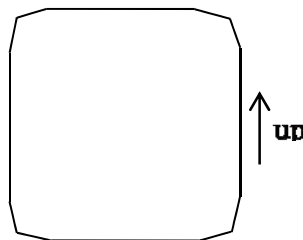


Which of the following diagrams shows a typical cross-section taken parallel to the two end faces?

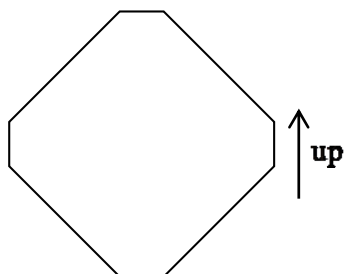
(A)



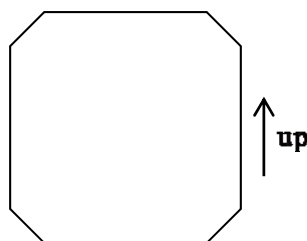
(B)



(C)



(D)



5. The polynomial equation  $P(x) = 0$  has roots  $\alpha$ ,  $\beta$  and  $\gamma$ .

What are the roots of the polynomial equation  $P(3x+2) = 0$ ?

- (A)  $\frac{\alpha}{3} - 2, \frac{\beta}{3} - 2, \frac{\gamma}{3} - 2$                       (B)  $\frac{\alpha-2}{3}, \frac{\beta-2}{3}, \frac{\gamma-2}{3}$ ,
- (C)  $3\alpha + 2, 3\beta + 2, 3\gamma + 2$                       (D)  $\alpha + \frac{2}{3}, \beta + \frac{2}{3}, \gamma + \frac{2}{3}$

6. Consider a polynomial  $P(x)$  of degree 3.

You are given 2 numbers  $a$  and  $b$  such that

- $a < b$
- $P(a) > P(b) > 0$
- $P'(a) = P'(b) = 0$

The polynomial has

- (A) 3 real zeros                      (B) 1 real zero  $\gamma$  such that  $\gamma < a$
- (C) 1 real zero  $\gamma$  such that  $a < \gamma < b$                       (D) 1 real zero  $\alpha$  such that  $\gamma > b$

7. Consider the following two statements:

I: 
$$\int_0^1 \frac{dx}{1+x^n} < \int_0^1 \frac{dx}{1+x^{n+1}}$$

II: 
$$\int_0^{\frac{\pi}{2}} \sqrt{\sin x} dx = \int_0^{\frac{\pi}{2}} \sqrt{\cos x} dx$$

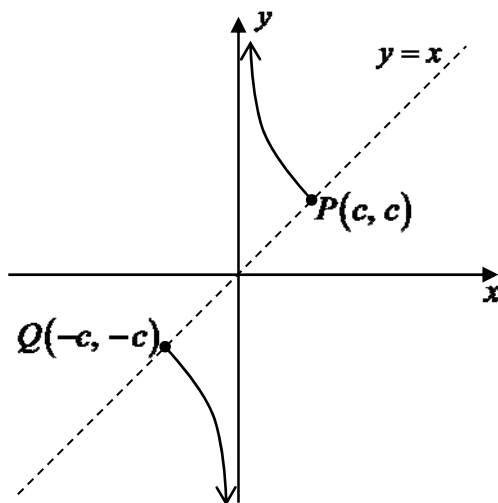
Which of these statements are correct?

- (A) Neither statement                      (B) Statement I only
- (C) Statement II only                      (D) Both statements

8. A particle is moving along the  $x$ -axis, initially moving to the left from the origin. Its velocity and acceleration are given by  $v^2 = 2 \ln(3 + \cos x)$  and  $\ddot{x} = \frac{-\sin x}{3 + \cos x}$ .

Which of the following describe its subsequent motion?

- (A) Heads only to the left, alternately speeding up and slowing down, without becoming stationary.
- (B) Heads only to the left, alternately slowing to a stop then speeding up.
- (C) Slows to a stop, then heads to the right forever.
- (D) Oscillates between two points.
9. The graph shows a part of the hyperbola  $x = ct, y = \frac{c}{t}$ .

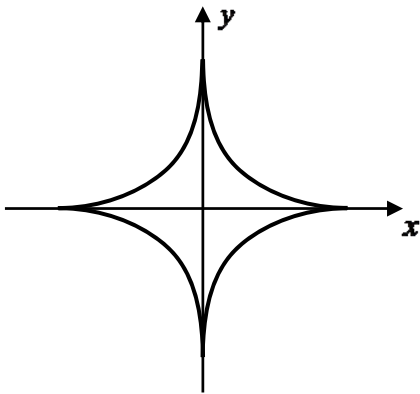


Which pair of parametric equations precisely describe the sections of the hyperbola shown?

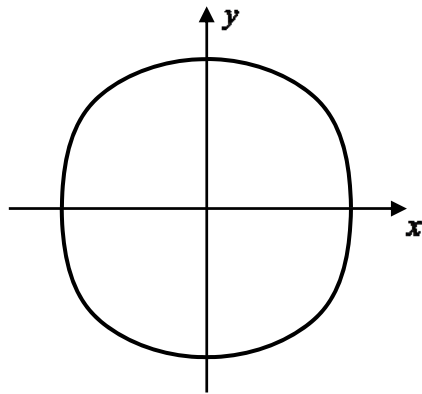
- (A)  $x = c(t^2 + 1), y = \frac{c}{t^2 + 1}$
- (B)  $x = c(1 - t^2), y = \frac{c}{1 - t^2}$
- (C)  $x = c\sqrt{1 - t^2}, y = \frac{c}{\sqrt{1 - t^2}}$
- (D)  $x = c \sin t, y = \frac{c}{\sin t}$

10. After differentiating a relation implicitly, we find that  $\frac{dy}{dx} = \frac{y}{x}$ .  
Which of the following could be a graph of this relation?

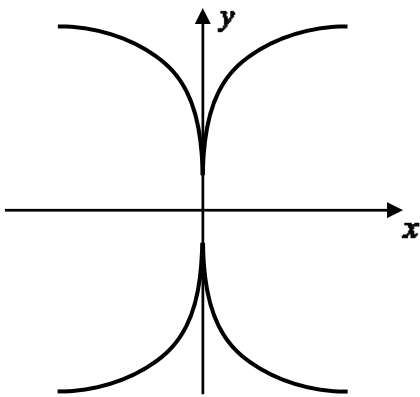
(A)



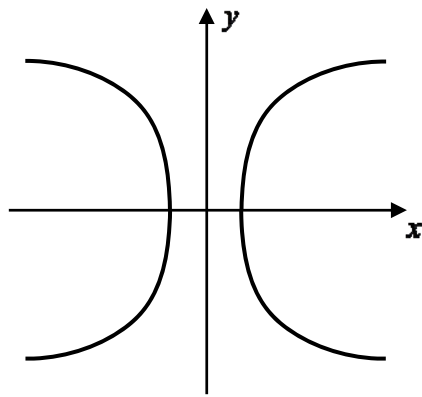
(B)



(C)



(D)



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**Section II****Free Response Questions****Total marks – 90****Attempt Questions 11 – 16**Answer each question on the multiple choice answer sheet provided.

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**Question 11** (15 marks)

(a) Find the exact value of  $\int_0^1 xe^{-x^2} dx$ . **2**

(b) Find  $\int \frac{dx}{x^2 + 6x + 10}$ . **1**

(c) Evaluate  $\int_0^1 \sin^{-1} x dx$ . **3**

(d) (i) Show that  $\int_0^1 \frac{5 - 5x^2}{(1 + 2x)(1 + x^2)} dx = \frac{1}{2} \left( \pi + \ln \frac{27}{16} \right)$ . **3**

(ii) Hence evaluate  $\int_0^{\frac{\pi}{4}} \frac{\cos 2x dx}{1 + 2 \sin 2x + \cos 2x}$  using the substitution  $t = \tan x$ . **3**

(e) (i) Show that  $\int_{-a}^a f(x) dx = \int_{-a}^a f(-x) dx$ . **1**

(ii) Hence, or otherwise, evaluate  $\int_{-4}^4 (e^x - e^{-x}) \cos x dx$ . **2**



**Question 12** (15 marks) Start a new booklet

(a) Given that  $z = 6i - 8$ , find the square roots of  $z$  in the form  $a + ib$ . 3

(b) (i) Write  $2 + 2\sqrt{3}i$  in modulus-argument form. 1

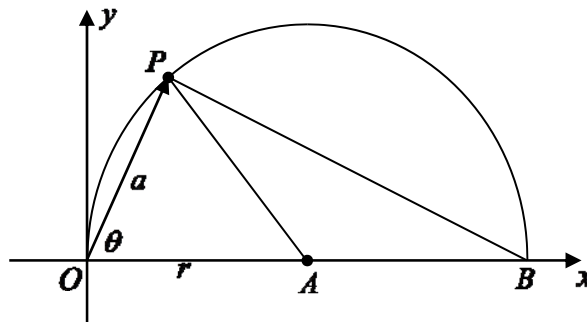
Hence:

(ii) Express  $(2 + 2\sqrt{3}i)^3$  in the form  $x + iy$ . 1

(iii) Find all unique solutions to the equation  $z^4 = 2 + 2\sqrt{3}i$ , giving answers in modulus-argument form. 2

(c) Given  $z$  is a complex number, sketch on a number plane the locus of a point  $P$  representing  $z$  such that  $\arg z = \arg [z - (1 + i)]$  2

(d) In the diagram, a semi-circle has diameter  $OB$  and centre  $A$ , with  $OA = r$ .  $P$  is a point on the semicircle, and the vector  $OP$  represents the complex number  $a \operatorname{cis} \theta$ .



Write in simplest modulus-argument form the complex number represented by the vector

(i)  $AP$  1

(ii)  $BP$  2

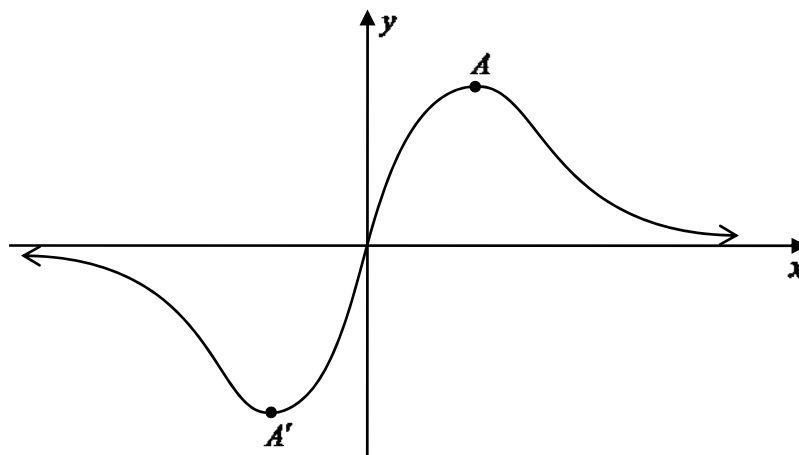
(e) In a bank of 12 switches, each switch can be set to one of three positions.

(i) Write down the total number of ways all the switches in the bank can be arranged. 1

(ii) Find the probability that if all the switches are set randomly, there will be equal numbers in each position. 2

**Question 13** (15 marks) Start a new booklet

(a) Drawn below is the graph of  $y = \frac{2x}{1+x^2}$ .



(i) Find the coordinates of the turning points  $A$  and  $A'$ . (There is no need to test their nature.) 2

(ii) On separate diagrams draw graphs of the following functions:

1.  $y = \frac{|2x|}{1+x^2}$  1

2.  $y = \frac{1+x^2}{2x}$  2

3.  $y^2 = \frac{2x}{1+x^2}$  2

4.  $y = \ln\left(\frac{2x}{1+x^2}\right)$  2

(b) (i) The polynomial equation  $P(x) = 0$  has a double root  $x = \alpha$ . Show that  $x = \alpha$  is also a root of the equation  $P'(x) = 0$ . 2

(ii) You are given that  $y = mx$  is a tangent to the curve  $y = 3 - \frac{1}{x^2}$ . Show that the equation  $mx^3 - 3x^2 + 1 = 0$  has a double root. 1

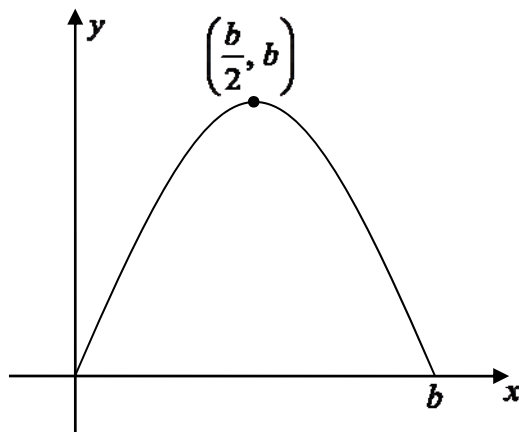
(iii) Hence find the equations of any such tangents. 3

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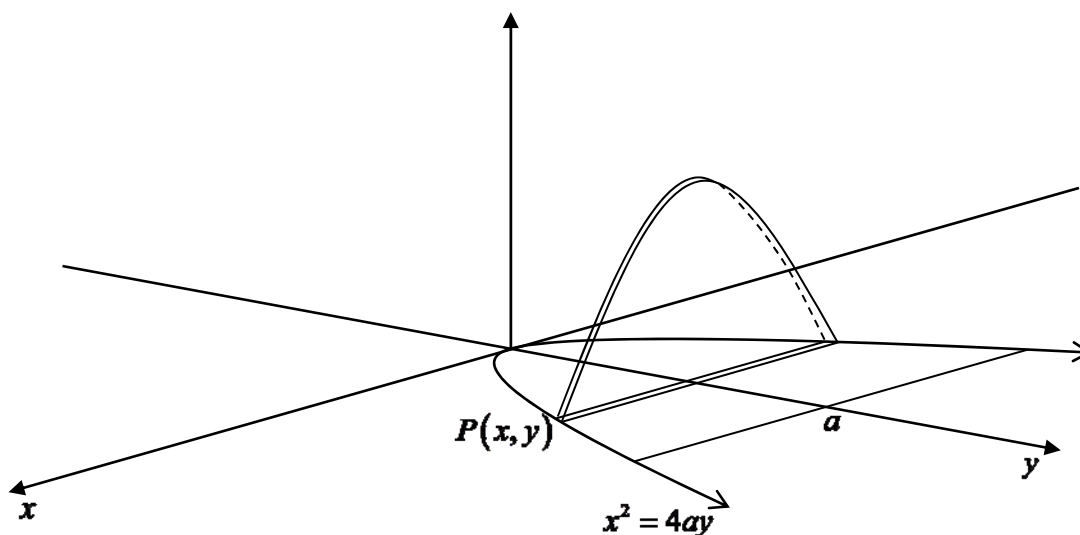
**Question 14** (15 marks) Start a new booklet

- (a) Use the method of cylindrical shells to find the volume of the solid formed when the region bounded by the curve  $y = \frac{1}{2x+1}$ , the  $x$ -axis, the  $y$ -axis and the line  $x = 1$  is rotated about the line  $x = 1$ . 4

- (b) The diagram shows a part of the graph of a function of the form  $y = b \sin nx$ .



- (i) Express  $n$  in terms of  $b$ . 1
- (ii) Show that the area bounded by the curve and the  $x$ -axis is  $\frac{2b^2}{\pi}$  units<sup>2</sup>. 2
- (iii) The base of a solid is the region bounded by the parabola  $x^2 = 4ay$  and its latus rectum. 2

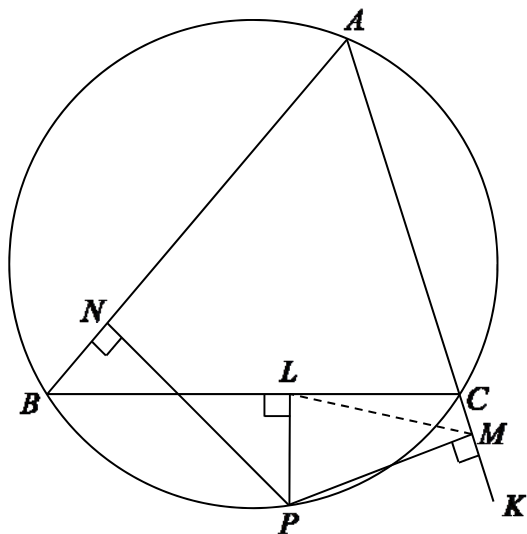


Each slice of width  $\delta y$  taken perpendicular to both the base and the axis of symmetry is half of a sine curve, whose amplitude is equal to its base length.

Find the volume of this solid in terms of  $a$ .

**Question 14 continues on the following page**

- (c) In the diagram,  $A$ ,  $B$  and  $C$  are three points on a circle.  
 $P$  is another point on the circle, lying on the minor arc  $BC$ .  
 Points  $L$ ,  $M$  and  $N$  are the feet of the perpendiculars from  $P$  to the sides  $BC$ ,  $CA$  and  $AB$  respectively.



(i) Explain why  $P$ ,  $L$ ,  $N$  and  $B$  are concyclic. 1

(ii) Explain why  $P$ ,  $L$ ,  $C$  and  $M$  are concyclic. 1

Let  $\angle PLM = \alpha$ .

(iii) Show that  $\angle ABP = \alpha$ . 2

(iv) Hence show that  $M$ ,  $L$  and  $N$  are collinear. 2

**End of question 14**

**Question 15** (15 marks) Start a new booklet

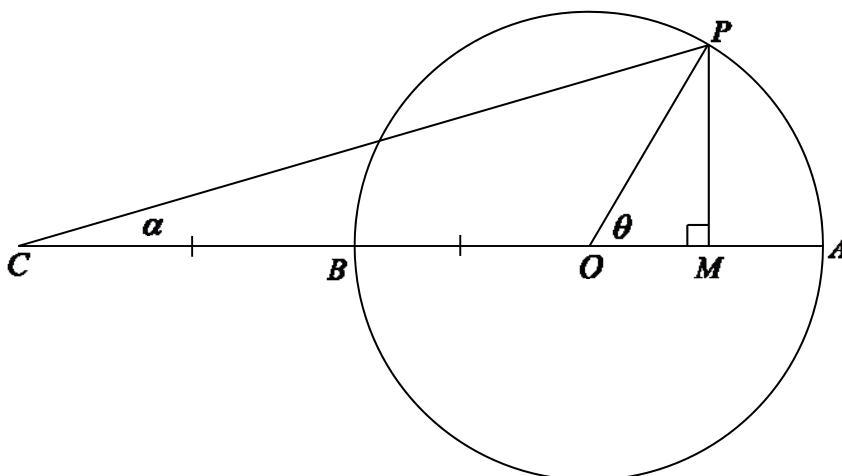
- (a) The point  $P(a \cos \theta, b \sin \theta)$  lies on the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  and the ellipse meets the  $x$ -axis at the points  $A$  and  $A'$ .
- (i) Prove that the tangent at  $P$  has the equation  $\frac{x \cos \theta}{a} + \frac{y \sin \theta}{b} = 1$ . **3**
- (ii) The tangent at  $P$  meets the tangents from  $A$  and  $A'$  at points  $Q$  and  $Q'$  respectively. Find the coordinates of  $Q$  and  $Q'$ . **2**
- (iii) The points  $A$ ,  $A'$ ,  $Q'$  and  $Q$  form a trapezium. Prove that the product of the lengths of the parallel sides is independent of the position of  $P$ . **2**
- (b) Consider the integral  $I_n = \int_0^1 \frac{x^n}{\sqrt{1-x}} dx$ .
- (i) Show that  $I_1 = \frac{4}{3}$ . **2**
- (ii) Show that  $I_{n-1} - I_n = \int_0^1 x^{n-1} \sqrt{1-x} dx$ . (No integration is needed.) **1**
- (iii) Use integration by parts on the result of part (ii) to show that  $I_n = \frac{2n}{2n+1} I_{n-1}$ . **2**
- (c) (i) Show that  $a^2 + b^2 \geq 2ab$  for any values of  $a$  and  $b$ . **1**
- (ii) Hence show that  $\tan^2 \theta + \cos^2 \theta + \operatorname{cosec}^2 \theta \geq \sin \theta + \sec \theta + \cot \theta$  for all values of  $\theta$ . **2**

**Question 16** (15 marks) Start a new booklet

(a) Consider the function  $f(x) = \frac{3 \sin x}{2 + \cos x}$ .

(i) Show that  $\frac{3 \sin x}{2 + \cos x} < x$  for  $x > 0$ . 3

The diagram shows a circle with centre  $O$ , where  $OA = OB = BC$ ,  $\angle POM = \theta$ ,  $\angle PCO = \alpha$ .



(ii) Show that  $\tan \alpha = \frac{\sin \theta}{2 + \cos \theta}$ . 2

(iii) Hence show that  $\alpha < \frac{\theta}{3}$ . 2

(b) The equation  $x^2 + x + 1 = 0$  has roots  $\alpha$  and  $\beta$ .

A series is defined by  $T_n = \alpha^n + \beta^n$  for  $n = 1, 2, 3, \dots$

(i) Show that  $T_1 = -1$  and  $T_2 = -1$ . 2

(ii) Show that  $T_n = -T_{n-1} - T_{n-2}$  for  $n = 3, 4, 5, \dots$  2

(iii) Hence use Mathematical Induction to show that  $T_n = 2 \cos \frac{2n\pi}{3}$  for  $n = 1, 2, 3, \dots$  3

(iv) Hence write down the value of  $\sum_{k=1}^{2012} T_k$ . 1

**End of paper**

## STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln \left( x + \sqrt{x^2 - a^2} \right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln \left( x + \sqrt{x^2 + a^2} \right)$$

NOTE :  $\ln x = \log_e x, \quad x > 0$